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ON THE COLLOCATION METHODS FOR HIGH-ORDER VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS*

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Abstract

We study the numerical solution of high-order Volterra integro-differential equations by means of collocation techniques in certain polynomial spline spaces. The attainable order of global convergence and local superconvergence of these methods is analyzed.

§1. Introduction

In this paper we shall be concerned with the approximate solution of the initial-value problem for a nonlinear (high-order) Volterra integro-differential equation (VIDE)

$$y^{(r)}(t) = f(t, y(t), \cdots, y^{(r-1)}(t)) + \int_0^t k(t, s, y(s), \cdots, y^{(r-1)}(s))ds, \ t \in I := [0, T], \quad (1.1)$$

with initial conditions $y^{(j)}(0) = y_{0j}, 0 \leq j \leq r-1$. Here, $r \geq 1$ is a natural number; the given functions $f : I \times I\!R^r \to I\!R$ and $k : S \times I\!R^r \to I\!R$ (with $S := \{(t, s) : 0 \leq s \leq t \leq T\}$) are assumed to be continuous and such that (1.1) has a unique solution $y \in C^r(I)$ satisfying the given intial conditions.

The analysis of the convergence properties of any numerical method for (1.1) will necessarily involve the linearization of the given equation and lead to a problem of the form

$$y^{(r)}(t) = \sum_{j=0}^{r-1} a_j(t)y^{(j)}(t) + b(t) + \int_0^t (\sum_{j=0}^{r-1} K_j(t, s)y^{(j)}(s))ds, \quad t \in I. \quad (1.2)$$

Equations of the form (1.1) (or (1.2)) have a frequent use in the mathematical modeling of various physical and biological phenomena. Many authors studied the first-order problem using collocation methods. A complete convergence theory of collocation approximations in $S_m^{(0)}(Z_N)$ (see (1.5) for the symbol) for (1.1) when $r = 1$, including local superconvergence results and the discretization of the collocation equations, may be found in Brunner (1984) and Brunner & Houwen (1986). For $r = 2$, Aguilar & Brunner (1986) considered the equations of the form

$$y''(t) = f(t, y(t)) + \int_0^t k(t, s, y(s))ds, \quad t \in I. \quad (1.3)$$

The equations of the form (1.3) arise, for example, in one-dimensional visco-elastic problems, in the construction of a field-theoretical model for electron-beam devices, and in problems of one-dimensional heat flow in materials with memory (see Burton (1983), and Hrusa & Nohel (1984)). For the linear counterpart of (1.3), the attainable order of (local) convergence

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of the numerical method used is analysed in Aguilar & Brunner (1986). Furthermore, the convergence analysis of high-order equations of the form

$$y^{(r)}(t) = p(t)y(t) + q(t) + \int_0^t k(t,s)y(s)ds, \quad t \in I, \tag{1.4}$$

could be found in Aguilar (1986).

In practical applications, one occasionally encounters high-order integro-differential equations which are of the form (1.1) or (1.2) (see also Burton (1983)). Pouzet (1962) introduced a special class of explicit Runge-Kutta methos for the numerical solutions of certain classes of (1.1) when $r = 2$. Wahr (1977) investigated the convergence and application of collocation methods to high-order linear VIDEs. The recent paper by Bellen (1985) and recent book by Bruuner & Houwen (1986) contatin, among other things, a concise survey of recent advances in the numerical solution of VIDEs by collocation and related methods.

In this paper, VIDEs of the form (1.1) will be solved numerically in certain polynomial spline spaces. In order to describe these approximation spaces, let

$$\Pi_N : 0 = t_0 < t_1 < \cdots < t_N = T, \quad \text{where } t_n = t_n^{(N)}$$

be a mesh for the given interval I, and set

$$\begin{aligned} &\sigma_n := [t_n, t_{n+1}], \quad h_n := t_{n+1} - t_n, \quad n = 0, \cdots, N-1, \\ &Z_n := \{t_n : n = 1, \cdots, N-1\} \text{ (interior mesh points, or knots)}, \\ &\bar{Z}_N := Z_N \cup T. \end{aligned}$$

Moreover, let P_k denote the space of (real) polynomials of degree not exceeding k. We then define, for given integers k and d with $0 \le d \le k-1$,

$$\begin{aligned} S_k^{(d)}(Z_N) := \Big\{ u : u = u_n \in P_k \text{ on } \sigma_n, 0 \le n \le N-1; \\ u_{n-1}^{(j)}(t_n) = u_n^{(j)}(t_n) \text{ for } t_n \in Z_N \text{ and } 0 \le j \le d \Big\} \end{aligned} \tag{1.5}$$

to be the space of polynomial splines (or piecewise polynomials) of degree k whose elements possess the knots Z_N, and d times continuously differentiable on I. It is easily seen that the dimension of this linear vector space is equal to $N(k-d) + (d+1)$. In the following we shall deal with the space $S_{m+r-1}^{(r-1)}(Z_N)$, with $r \ge 1$, whose dimension is given by $Nm + \Gamma$.

In order to determine an approximation $u \in S_{m+r-1}^{(r-1)}(Z_N)$ to the solution of the VIDE (1.1), let $\{c_j\}$ be a given set of parameters satisfying

$$0 \le c_1 < c_2 < \cdots < c_m \le 1,$$

and define the points

$$t_{nj} := t_n + c_j h_n, \quad j = 1, \cdots, m; \; n = 0, \cdots, N-1, \tag{1.6a}$$

with

$$X_n := \{t_{nj} : j = 1, \cdots, m\}, \tag{1.6b}$$

and

$$X(N) := \bigcup_{n=0}^{N-1} X_n. \tag{1.6c}$$

We shall refer to $\{c_j\}$ as the collocation parameters; the set $X(N)$ will be called the set of collocation points. We now seek an element $u \in S_{m+r-1}^{(r-1)}(Z_N)$ satisfying the VIDE (1.1) on $X(N)$ and subjected to the given initial conditions, i.e.,

$$u^{(r)}(t) = f(t, u(t), \cdots, u^{(r-1)}(t)) + \int_0^t k(t, s, u(s), \cdots, u^{(r-1)}(s))ds, \quad \text{for } t \in X(N),$$

with

$$u^{(j)}(0) = y_{0j}.$$

This collocation equation may be written in the form

$$\begin{aligned} u_n^{(r)}(t_{nj}) &= f(t_{nj}, u_n(t_{nj}), \cdots, u_n^{(r-1)}(t_{nj})) \\ &\quad + h_n \int_0^{c_j} k(t_{nj}, t_n + sh_n, u_n(t_n + sh_n), \cdots, u_n^{(r-1)}(t_n + sh_n))ds \qquad (1.7a) \\ &\quad + F_n(t_{nj}, u_n, \cdots, u_n^{(r-1)}), \quad j = 1, \cdots, m; n = 0, \cdots, N-1, \end{aligned}$$

where

$$F_n(t, u, \cdots, u^{(r-1)}) := \sum_{i=0}^{n-1} h_i \int_0^1 k(t, t_i + sh_i, u_i(t_i + sh_i), \cdots, u_i^{(r-1)}(t_i + sh_i))ds, t \in \sigma_n \tag{1.7b}$$

denotes the so-called lag term with respect to the subinterval σ_n. Note that if the collocation parameters are chosen so that $c_1 = 0$ and $c_m = 1$, then the approximation u lies in the smoother spline space

$$s_{m+r-1}^{(r-1)}(Z_N) \bigcap C^r(I) = S_{m+r-1}^{(r)}(Z_N).$$

We now rewrite (1.7a) in a form which is more amenable to numerical computations. Since $u_n^{(r)}$ is a polynomial of degree at most $m-1$, we may write

$$u_n^{(r)}(t_n + sh_n) = \sum_{j=1}^{m} L_j(s) Y_{nj}, \quad t_n + sh_n \in \sigma_n', \tag{1.8a}$$

where

$$Y_{nj} := u_n^{(r)}(t_{nj}),$$

$$L_j(s) := \prod_{k \neq j}^{m} (s - c_k)/(c_j - c_k),$$

and $\sigma_0' := \sigma_0, \sigma_n' := (t_n, t_{n+1}], n = 1, \cdots, N-1$. It then follows that

$$u^{(r-q)}(t_n + sh_n) = \sum_{i=1}^{q} w_{n,r-i}(sh_n)^{q-i}/(q-i)! + h_n^q (\sum_{j=1}^{m} a_j^q(s) Y_{nj})/(q-1)!, \tag{1.8b}$$

$$q = 1, \cdots, r;$$

with $w_{n,r-i} := u_n^{(r-i)}(t_n)$ and

$$a_j^q(s) := \int_0^s (s-z)^{q-1} L_j(z)dz, \quad q = 1, \cdots, r. \tag{1.8c}$$

Submitting (1.8) to equation (1.7a), we find that, for each $n = 0, \cdots, N-1$, equation (1.7a) is a system of m nonlinear equations for $Y_n := (Y_{n1}, \cdots, Y_{nm})^T$. Once Y_n has been computed, the approximating spline $u \in S_{m+r-1}^{(r-1)}(Z_N)$ and its derivatives are completely determined on the subinterval σ_n by (1.8). Furthermore, at $t = t_{n+1}$ the values of u and its derivatives are given by

$$u_n^{(r)}(t_{n+1}) = \sum_{j=1}^{m} L_j(1) Y_{nj}, \tag{1.9a}$$

and

$$u_{n+1}^{(r-q)}(t_{n+1}) = u_n^{(r-q)}(t_{n+1}) = \sum_{i=1}^{q} w_{n,r-q} h_n^{q-i}/(q-i)! + h_n^q (\sum_{j=1}^{m} a_j^q(1) Y_{nj})/(q-1)!,$$

$$q = 1, \cdots, r. \tag{1.9b}$$

As mentioned in Aguilar and Brunner (1986), most applications yield linear VIDEs of the form (1.2) (where the kernel is often of convolution type : $K_j(t,s) = G_j(t-s)$). We shall present the global convergence and local superconvergence results to this case in Sections 2 and 3. The analogous convergence results hold for the nonlinear VIDE (1.1) under appropriate assumptions on functions f and k (see Section 2).

§2. Convergence Results: Globle Convergence and Local Superconvergence

For ease of exposition we shall give the convergence results for linear VIDE,

$$y^{(r)}(t) = \sum_{j=0}^{r-1} a_j(t) y^{(j)}(t) + b(t) + \int_0^t (\sum_{j=0}^{r-1} K_j(t,s) y^{(j)}(s)) ds, \quad t \in I. \tag{2.1}$$

with intial conditions $y^{(j)}(0) = y_{0j}, 0 \le j \le r-1$. It can be shown that, under appropriate assumptions on f and k (concerning essentially the boundedness of functions $f, k, \partial f(t, x_1, \cdots, x_{r-1})/\partial x_i$ and $\partial k(t, s, x_1, \cdots, x_{r-1})/\partial x_i, 1 \le i \le r-1$), analogous convergence results hold for the nonlinear VIDE (1.1). Moreover, recall that h denotes the diameter of the mesh Π_N, $h := \max(h_n : 0 \le n \le N-1)$. Let

$$Y(t) := (y(t), y'(t), \cdots, y^{(r-1)}(t))^T, \tag{2.2a}$$

$$P(t) := \begin{bmatrix} O_{r-1,1} & I_{r-1,r-1} \\ a_0 & a_1 \cdots \quad a_{r-1} \end{bmatrix}, \tag{2.2b}$$

$$Q(t) := (0, \cdots, 0, b(t))^T, \tag{2.2c}$$

and

$$K(t,s) := \begin{bmatrix} O_{r-1,r} \\ K_0(t,s) \cdots K_{r-1}(t,s) \end{bmatrix}. \tag{2.2d}$$

Here $O_{m,n}$ denotes the $m \times n$ zero matrix ($m = r-1, n = 1$ or r) and $I_{r-1,r-1}$ the $(r-1)\times(r-1)$ identity matrix. Then Eq. (2.1) is equivalent to

$$Y'(t) = P(t)Y(t) + Q(t) + \int_0^t K(t,s)Y(s)ds, \quad t \in I, \tag{2.2e}$$

with initial condition $Y(0) = Y_0 := (y_{00}, y_{01}, \cdots, y_{0,r-1})^T$.

Theorem 2.1. *In (2.1) let $a_j \in C^m(I), K_j \in C^m(S), 0 \le j \le r-1$, and $b \in C^m(I)(m \ge 1)$. Let $\bar{h} > 0$ be sufficiently small, so that for all $h \in (0, \bar{h})$ the collocation equation (1.7) defines a unique approximation $u \in S_{m+r-1}^{(r-1)}(Z_N)$ to the solution y of (2.1). Then we have*

$$\|y^{(k)} - u^{(k)}\| = O(h^m), \quad k = 0, \cdots, r-1 \tag{2.3}$$

for all collocation parameters $\{c_j\}$ with $0 \le c_1 < c_2 < \cdots < c_m \le 1$. Here $\|y^{(k)} - u^{(k)}\| := \sup(|y^{(k)}(t) - u^{(k)}(t)| : t \in I)$.

Without loss of generality we will restrict our discussion to uniform partitions of I. The generalization to quasi-uniform partitions, where the quantities

$$H'_N := \min\{h_n : 0 \le n \le N-1\},$$
$$H''_N := \max\{h_n : 0 \le n \le N-1\}$$

satisfy $H''_N / H'_N \le$ const. for all N, is straightforward. Furthermore, without loss of generality we will restrict our discussion to $r = 2$.

Proof. Let $U(t) := (u(t), u'(t), \cdots, u^{(r-1)}(t))^T$. It follows from (2.2) (with $t = t_n + c_i h$) and from the linear counterpart of (1.7) that

$$\begin{aligned} e'_n(t_{ni}) &= P(t_{ni})e_n(t_{ni}) + h\int_0^{c_i} K(t_{ni}, t_n + sh)e_n(t_n + sh)ds \\ &+ h\sum_{k=0}^{n-1}\int_0^1 K(t_{ni}, t_k + sh)e_k(t_k + sh)ds, \end{aligned} \tag{2.4}$$

where $e(t) := Y(t) - U(t)$.

When $r = 2$, we have $Y(t) = (y(t), y'(t))^T, U(t) = (u(t), u'(t))^T$. Since $a_0, a_1, b \in C^m(I)$ and $K_0, K_1 \in C^m(S)$, we have $y \in C^{m+2}(I)$. Then we may write

$$e_n^0(t_n + sh) := y(t_n + sh) - u_n(t_n + sh) = h^{m+1}(\sum_{j=0}^{m+1} d_{nj}s^j + hR_n(s)), \quad t_n + sh \in \sigma_n, \tag{2.5}$$

and

$$e_n^1(t_n + sh) := y'(t_n + sh) - u'_n(t_n + sh) = h^m(\sum_{j=1}^{m+1} jd_{nj}s^{j-1} + hR'_n(s)), \quad t_n + sh \in \sigma_n, \tag{2.6}$$

$$\begin{aligned} e_n^2(t_n + sh) &:= y''(t_n + sh) - u''_n(t_n + sh) = h^{m-1}(\sum_{j=2}^{m+1} j(j-1)d_{nj}s^{j-2} + hR''_n(s)), \\ &\qquad t_n + sh \in (t_n, t_{n+1}], \quad n = 0, \cdots, N-1. \end{aligned} \tag{2.7}$$

Here, $h^{m+1}d_{nj} := c_{nj} - \alpha_{nj}(j = 0, \cdots, m)$, with $c_{nj} := h^j y^{(j)}(t_n)/j!$, $u_n(t_n + sh) := \sum_{j=0}^{m+1} \alpha_{nj} s^j$, and

$$R_n(s) := \int_0^s (s-z)^{m+1} y^{(m+2)}(t_n + zh)dz/(m+1)!. \tag{2.8}$$

Since $e(t) := Y(t) - U(t)$, we have

$$e_n(t_n + sh) = h^m(\sum_{j=0}^{m+1} \bar{d}_{nj} s^j + h\bar{R}_n(s)), \tag{2.9}$$

and

$$e_n'(t_n + sh) = h^{m-1}(\sum_{j=1}^{m+1} j\bar{d}_{nj} s^{j-1} + h\bar{R}'(s)), \tag{2.10}$$

where

$$\bar{d}_{nj} := (hd_{nj}, (j+1)d_{n,j+1})^T, \quad 0 \le j \le m,$$
$$\bar{d}_{nj} := (hd_{nj}, 0)^T, \quad j = m+1,$$
$$\bar{R}_n(s) := (hR_n(s), R_n'(s))^T.$$

Along the lines of [5], we have

$$\begin{aligned}
&\sum_{j=1}^{m+1} (jc_i^{j-1} I_2 - hP(t_{ni})c_i^j - h^2 \int_0^{c_i} K(t_{ni}, t_n + sh)s^j ds)\bar{d}_{nj} \\
&= (hP(t_{ni}) + h^2 \int_0^{c_i} K(t_{ni}, t_n + sh)ds)\bar{d}_{n0} + h^2 \sum_{k=1}^{n-1} \int_0^1 K(t_{ni}, t_k + sh)ds\bar{d}_{k0} \\
&\quad + h^2 \sum_{k=0}^{n-1} \sum_{j=1}^{m+1} \int_0^1 K(t_{ni}, t_k + sh)s^j ds\bar{d}_{kj} + \bar{q}_{ni}, \quad i = 1, \cdots, m,
\end{aligned} \tag{2.11a}$$

with

$$\begin{aligned}
\bar{q}_{ni} := &-h\bar{R}_n'(c_i) + h^2 P(t_{ni})\bar{R}_n(c_i) + h^3 \int_0^{c_i} K(t_{ni}, t_n + sh)\bar{R}_n(s)ds \\
&+ h^3 \sum_{k=0}^{n-1} \int_0^1 K(t_{ni}, t_k + sh)\bar{R}_k(s)ds,
\end{aligned} \tag{2.11b}$$

where I_2 on the left-hand side of (2.11a) is the 2×2 identity matrix. We first note that the matrix defined by the coefficients of $\{\bar{d}_{nj}\}$ on the left-hand side of (2.11a) is inversible whenever $h > 0$ is sufficiently small: this follows from the fact that $\|P(t)\|_1 \le$ const., $\|K(t,s)\|_1 \le$ const. for all $t \in I$ and $(t,s) \in S$, and the observation that, for $h = 0$, the determinant of this matrix is $(m! \prod_{i>j}(c_i - c_j))^2$. Furthermore, in complete analogy to the technique used by Brunner (1984), we have

$$\|\bar{d}_n\|_1 := \sum_{j=1}^{m+1} \|\bar{d}_{nj}\|^* = O(h), \tag{2.12a}$$

and

$$\|\bar{d}_{n0}\|^* = O(1), \tag{2.12b}$$

when h is sufficiently small. Here $\|\ \|^*$ denotes the l_1-norm for 2-dimension vectors, i.e., if $V = (v_1, v_2)$, then $\|V\|^* := |v_1| + |v_2|$. Hence, from (2.9) and (2.10), we obtain

$$\|e_n(t_n + sh)\|^* = O(h^m), \quad \|e_n'(t_n + sh)\|^* = O(h^m).$$

The proof of Theorem 2.1 is thereby complete.

Aguilar and Brunner (1986) considered the equations of the form (1.3). They found that there exist some sets of collocation parameters $\{c_j\}$ such that, at the mesh points $\bar{Z}_N$, the convergence rate of u (and u', u'') is superior to the globle convergence rate in Theorem 2.1. In [1], Aguilar considered the equations of the form (1.4) when $r > 2$. The anologous results are obtained in this case. In the following we shall be concerned with the following question: are there some analogous results for the general high-order Volterra integro-differential equation (2.1)? The following theorems concerning this question will be given precisely.

Theorem 2.2. *Assume that $a_j \in C^{2m}(I), K_j \in C^{2m}(S), 0 \le j \le r-1$, and $b \in C^{2m}(I)(m \ge 1)$. Let $u \in S^{(r-1)}_{m+r-1}(Z_N)$ be the collocation approximation determined by (1.7). If the collocation parameters $\{c_j\}$ are the zeros of $P_m(2s-1)$ (i.e., the m Gauss points for $(0,1)$) , then*

$$\max_{t_n \in \bar{Z}_N} |y^{(k)}(t_n) - u^{(k)}(t_n)| = O(h^{2m}), \quad 0 \le k \le r-1, \tag{2.13a}$$

while

$$\max_{t_n \in \bar{Z}_N} |y^{(r)}(t_n) - u^{(r)}(t_n)| = O(h^m), \tag{2.13b}$$

as $h \to 0_+$ (with $Nh \le$ const. T). The values $u^{(k)}(t_n), 0 \le k \le r$, are given by (1.9).

Theorem 2.3. *In (2.1) let $a_j \in C^{2m-1}(I), K_j \in C^{2m-1}(S), 0 \le j \le r-1$, and $b \in C^{2m-1}(S)$ (with $m \ge 1$). If $u \in S^{(r-1)}_{m+r-1}(Z_N)$ denotes the collocation approximation determined by (1.7), and if the collocation parameters $\{c_j\}$ are the zeros of the polynomial $P_m(2s-1) - P_{m-1}(2s-1)$ (i.e., the m Radau II Points for $(0,1]$), then*

$$\max_{t_n \in \bar{Z}_N} |y^{(k)}(t_n) - u^{(k)}(t_n)| = O(h^{2m-1}), \quad 0 \le k \le r, \tag{2.14}$$

as $h \to 0_+$ (with $Nh \le$ const. T).

Note that the collocation approximations occurring in Theorems 2.2 and 2.3 are in $C^{r-1}(I)$ but, in general, not in $C^r(I)$. If u is to be in $C^r(I)$, then the local superconvergence orders of (2.13a) and (2.14) cannot be attained (compare also [2]).

Theorem 2.4. *In (2.1) let $a_j \in C^{2m-2}(I)$, $K_j \in C^{2m-2}(S), 0 \le j \le r-1$, and $b \in C^{2m-2}(I)$(with $m \ge 2$), and let $u \in C^{(r-1)}_{m+r-1}(Z_N)$ be the collocation solution defined by (1.7), with the underlying collocation parameters $\{c_j\}$ being given by the zeros of the* *polynomial* $s(s-1)P'_{m-1}(2s-1)$ *(i.e., the m Lobatto points for* $[0,1]$*; here* $c_1 = 0$ *and* $c_m = 1$*). We then have*

$$\max_{t_n \in Z_n} |y^{(k)}(t_n) - u^{(k)}(t_n)| = O(h^{2m-2}), \quad 0 \le k \le r, \tag{2.15}$$

as $h \to 0$ *(with* $Nh \le$ const. T*).*

For a comprehensive analysis of the quadrature formulas of Gauss, Radau and Lobatto we refer the reader to Ghizzetti & Ossicini (1970) and Brunner & Houwen (1986).

The Proofs of the local superconvergence results given in Theorems 2.2, 2.3 and 2.4 will be presented in the following section.

§3. Local superconvergence : Proofs

Consider linear integro-differential equations

$$X'(t) = H(t) + A(t)X(t) + \int_0^t B(t,s)X(s)ds, \quad t \in I, \tag{3.1}$$

with initial condition $X(0) = X_0 \in I\!R^r$, where $H : I \to I\!R^r$ is continuous, A an $r \times r$ matrix continuous on I and B and $r \times r$ matrix continuous on S.

Lemma 3.1. Assume that $H, A \in C^l(I)$ and $B \in C^l(S)$. Then the (unique) solution $X \in C^{l+1}(I)$ of (3.1) may be expressed in the form

$$X(t) = R(t,0)X_0 + \int_0^t R(t,s)H(s)ds, \quad t \in I. \tag{3.2}$$

Here, $R(t,s)$ denotes the resolvent associated with (3.1); it is defined by the resolvent equations

$$\frac{\partial R(t,s)}{\partial s} = -R(t,s)A(s) - \int_s^t R(t,u)B(u,s)du, \quad (t,s) \in S \tag{3.3a}$$

with

$$R(t,t) = I_r \quad \text{for } t \in I, \tag{3.3b}$$

where I_r is the $r \times r$ identity matrix. Moreover, we have $R \in C^{l+1}(S)$.

The proof of this result may be found in Grossman & Miller (1973) or in Burton (1983). Using Lemma 3.1, we can easily obtain the following result.

lemma 3.2. *Consider equations of the form*

$$Y'(t) = P(t)Y(t) + Q(t) + \int_0^t K(t,s)Y(s)ds, \quad t \in I, \tag{3.4}$$

with $Y(0) = Y_0$*, where* $Y(t), P(t), Q(t)$ *and* $K(t,s)$ *are defined in* (2.2a), (2.2b), (2.2c) *end* (2.2d) *respectively. Assume* $P, Q \in C^l(I)$ *and* $K \in C^l(S)$*. Then the (unique) solution* $Y \in C^{l+1}(I)$ *of* (3.4) *may be expressed in the form*

$$Y(t) = R(t,0)Y_0 + \int_0^t R(t,s)Q(s)ds, \quad t \in I. \tag{3.5}$$

Here, the resolvent $R(t,s)$ *satisfies*

$$\frac{\partial R(t,s)}{\partial s} = -R(t,s)P(s) - \int_s^t R(t,u)K(u,s)du, \quad (t,s) \in S, \tag{3.6a}$$

with

$$R(t,t) = I_r \quad \text{for } t \in I. \tag{3.6b}$$

Moreover, we have $R \in C^{l+1}(S)$.

As a corollary of the above results, we have

Lemma 3.3. *it* (2.1) *let* $a_j \in C^k(I), K_j \in C^k(I), 0 \le j \le r-1$, *and* $b \in C^k(I)$ *(with* $k \ge 1$*). Then the (unique) solution* $y \in C^{k+r}(I)$ *of the initial value problem for the VIDE* (2.1) *is given by*

$$y^{(i-1)}(t) = -\sum_{j=1}^{r} R_{ij}(t,0)y_{j0} - \int_0^t R_{ir}(t,s)b(s)ds, \quad t \in I, \tag{3.7}$$

with $1 \le i \le r$. *Here* R_{ij} *is an element of the* $r \times r$ *matrix* $R = [R_{ij}]$ *which was defined by (3.6).*

The collocation equation defined by the approximation $u \in S^{(r-1)}_{m+r-1}(Z_N)$ to the solution of VIDE (2.1) may be written in the form

$$u^{(r)}(t) = \sum_{j=0}^{r-1} a_j(t)u^{(j)}(t) + b(t) + r(t) + \int_0^t \Big(\sum_{j=0}^{r-1} K_j(t,s)u^{(j)}(s)\Big)ds, \quad t \in I, \tag{3.8}$$

with initial conditions $u^{(j)}(0) = y_{j0}, 0 \le j \le r-1$. Here the residual r vanishes at the collocation points $X(N)$, i.e., $r(t_{nj}) = 0$ for $1 \le j \le m, 0 \le n \le N-1$. Moreover, if the given functions $a_j, K_j, 0 \le j \le r-1$, and b are smooth on their respective domains, then r is smooth on each of the subintervals $\sigma_n, 0 \le n \le N-1$. Note also that $r(t)$ is uniformly bounded on I as $h \to 0_+$ (and $Nh \le$ const. T) since u converges uniformly to y (cf. Theorem 2.1).

Let $e := y - u$ denote the collocation error. Then e satisfies that VIDE

$$e^{(r)}(t) = \sum_{j=0}^{r-1} a_j(t)e^{(j)}(t) - r(t) + \int_0^t \Big(\sum_{j=0}^{r-1} K_j(t,s)e^{(j)}(s)\Big)ds, \quad t \in I, \tag{3.9a}$$

and the initial conditions

$$e^{(j)}(0) = 0, \quad 1 \le j \le r-1. \tag{3.9b}$$

The following result is a corollary of Lemma 3.2 and Lemma 3.3.

Lemma 3.4. *In* (3.9), *assume* $a_j \in C^l(I)$ *and* $K_j \in C^l(S), 0 \le j \le r-1, l \ge 1$. *Then, there exist functions* $R_j \in C^{l+1}(S)(0 \le j \le r-1)$ *satisfying* $R_j(t,t) = 0 (j \ne r-1)$ *and* $R_j(t,t) = 1 (j = r-1)$, *such that*

$$e^{(j)}(t) = \begin{cases} \int_0^t R_j(t,s)r(s)ds, & 0 \le j \le r-1, \\ r(t) + \int_0^t \frac{\partial R_{r-1}(t,s)}{\partial t} r(s)ds, & j = r. \end{cases} \tag{3.10}$$

By use of the above result and a discussion analogous to [2] (see also [5] and [9]), the conclusions given in Theorems 2.2, 2.3 and 2.4 can be obtained. Then the proof of the local superconvergence is thereby complete.

§4. Full Discretization of the Collocation Equations

The Convergence results established in the previous section were derived under the assumption that the integrals occurring in the collocation equation (1.7) can be evaluated analytically. Since this is rarely possible in concrete applications, there arises a question of how to approximate these integrals. The discretized collocation equation (1.7) will then yield an approximation $\bar{u} \in S_{m+r-1}^{(r-1)}(Z_N)$ which, in general, is different from u.

If m-point quadrature formulas with abscissas based on the collocation parameters are employed to carry out this additional discretization step, then we have (for ease of notation, we still use the notation u instead of $\bar{u}$ in this section)

$$\begin{aligned} u_n^{(r)} &= f(t_{nj}, u_n(t_{nj}), \cdots, u_n^{(r-1)}(t_{nj})) \\ &\quad + h_n \sum_{k=1}^{m} w_{jk} k(t_{nj}, t_n + c_j c_k h_n, u_n(t_n + c_j c_k h_n), \cdots, u_n^{(r-1)}(t_n + c_j c_k h_n)) \\ &\quad + \sum_{i=0}^{n-1} h_i \sum_{k=1}^{m} w_k k(t_{nj}, t_{nk}, u_i(t_{ik}), \cdots, u_i^{(r-1)}(t_{ik})),\ j = 1, \cdots, m; n = 0, \cdots, N-1. \end{aligned} \tag{4.1a}$$

Here, we have $w_{jk} := c_j w_k (1 \le j, k \le m)$, where

$$w_k := \int_0^1 L_k(s) ds\,. \tag{4.1b}$$

The collocation approximation $u \in S_{m+r-1}^{(r-1)}(Z_N)$ may be determined by (4.1), (1.8) and (1.9).

It can be shown, along the lines of [5], that if the discretized version of the collocation equation is derived by means of interpolatory m-point quadrature formulas based on the collocation parameters, then the resulting collocation approximation and its derivatives exhibit again the local superconvergence behavior described in Theorems 2.2, 2.3 and 2.4.

Finally, we shall give an example for the case $r = 3$. For the cases $r = 1$ and $r = 2$, one can see [5] and [2].

For $r = 3$, the fully discretized version of the collocation equations (4.1a) may be rewritten as

$$\begin{aligned} Y_{nj} &= f(t_{nj}, A_{nj}^1, A_{nj}^2, A_{nj}^3) \\ &\quad + h_n c_j \sum_{k=1}^{m} w_k k(t_{nj}, t_n + c_j c_k h_n, B_{njk}^1, B_{njk}^2, B_{njk}^3) \\ &\quad + \sum_{i=0}^{n-1} h_i \sum_{k=1}^{m} w_k k(t_{nj}, t_{nk}, C_{ki}^1, C_{ki}^2, C_{ki}^3), \end{aligned} \tag{4.2}$$

where

$$A_{nj}^1 := x_n + h_n c_j y_n + (h_n c_j)^2 z_n/2 + h_h^3 (\sum_{k=1}^{m} a_k^3(c_j) Y_{nk})/2; \tag{4.3a}$$

$$x_n := u_n(t_n), y_n := u_n'(t_n), z_n := u_n''(t_n); \tag{4.3b}$$

integro-differential equations

$$\begin{aligned} y^{(r)} &= f(t, y(t), \cdots, y^{(r-1)}(t)) \\ &+ \int_0^t (t-s)^{-\alpha} f(t, s, y(s), \cdots, y^{(r-1)}(s)) ds, \quad t \in I, 0 < \alpha < 1, \end{aligned} \tag{5.1}$$

with $y^{(j)}(0) = y_{j0}(j = 1, \cdots, r-1), r \geq 1$, the given initial values. We can follow the work of Brunner (1985 a) and (1985 b) (compare also [9]) and the method used in this paper, to give a detailed numerical analysis. This will be considered in a subsequent paper.

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$$a_k^q(s) := \int_0^s (s-z)^{q-1} L_k(z)dz; \quad (\text{cf.}(1.8)) \tag{4.3c}$$

$$A_{nj}^2 := y_n + h_n c_j z_n + h_n^2 (\sum_{k=1}^m a_k^2(c_j) Y_{nk}); \tag{4.3d}$$

$$A_{nj}^3 := z_n + h_n (\sum_{k=1}^m a_k^1(c_j) Y_{nk}); \tag{4.3e}$$

$$B_{njk}^1 := x_n + h_n c_j c_k y_n + (h_n c_j c_k)^2 z_n/2 + h_n^3 (\sum_{s=1}^m a_s^3(c_j c_k) Y_{ns})/2; \tag{4.3f}$$

$$B_{njk}^2 := y_n + h_n c_j c_k z_n + h_n^2 (\sum_{s=1}^m a_s^2(c_j c_k) Y_{ns}); \tag{4.3g}$$

$$B_{njk}^3 := z_n + h_n (\sum_{s=1}^m a_s^1(c_j c_k) Y_{ns}); \tag{4.3h}$$

$$C_{ki}^1 := x_i + h_i c_k y_i + (h_i c_k)^2 z_i/2 + h_i^3 (\sum_{s=1}^m a_s^3(c_k) Y_{is})/2; \tag{4.3i}$$

$$C_{ki}^2 := y_i + h_i c_k z_i + h_i^2 (\sum_{s=1}^m a_s^2(c_k) Y_{is}); \tag{4.3j}$$

$$C_{ki}^3 := z_i + h_i (\sum_{s=1}^m a_s^1(c_k) Y_{is}). \tag{4.3k}$$

From (4.2) and (4.3), the values $Y_{nj}(j = 1, \cdots, m)$ can be obtained. Once $Y_{nj} := u^{(3)}(t_{nj})(j = 1, \cdots, m)$ have been computed, the values of the collocation approximation u and its derivatives u' and u'' at the next mesh point $t = t_{n+1}$ may be found by means of the formulas (1.9).

Again, as we mentioned before, the resulting collocation approximation and its derivatives obtained by (4.2), (4.3), (1.9) exhibit the local superconvergence behavior described in Theorems 2.2, 2.3 and 2.4.

§5. Conclusion and Remarks

In this paper we considered the numerical solution of high-order Volterra integro-differential equations by means of collocation techniques in polynomial spline spaces $S_{m+r-1}^{(r-1)}(Z_N)$. The attainable order of local superconvergence of the numerical methods used was analyzed in detail. It was found that, for (first-order) ordinary differential equations, Volterra or Fredholm integral equations, or integro-differential equations of Volterra type, the analysis of superconvergence of collocation methods could be carried out (see Brunner (1981) and Brunner & Houwen (1986)). We believe that most results for first order equations may be generalized to high-order equations.

The technique of employing vector notations used in the proof of Theorem 2.1 seems to be applicable to some high-order equations of other forms. For example, for the Abel type