Journal of Computational Mathematics, Vol.8, No.2, 1990, 178–182.

A DIRECT METHOD FOR THE LINEAR COMPLEMENTARITY PROBLEM *1)

Zhou Shu-zi
(*Dept. of Math, Hunan University, Changsha, Hunan, China*)

One of the effective approaches for finding the numerical solutions of some free boundary problems is to reduce the problems into corresponding variational inequalities and then to discretize them by finite difference methods or finite element methods (see, for instance, [1–5]). Following this way we often obtain the so-called linear complementarity problem: find $w \in R^n$ such that

$$Aw \geq p, \quad w \geq q, \quad (Aw - p)^T(w - q) = 0, \tag{1}$$

where A is an $n \times n$ real matrix, and $p, q \in R^n$. Let $u = w - q$. Then (1) is reduced to the following form: find $u \in R^n$ such that

$$Au \geq b, u \geq 0, u^T(Au - b) = 0. \tag{2}$$

Most of the conventional algorithms for solving Problem (2) are iterative methods, and none of the exisiting direct methods for (2) are polynomial time algorithms[3]. We propose in this paper a new direct method for (2) which is a polynomial time method provided matrix A is a Stieltjes matrix.

Denote by v_i the i-th component of vector v, and by a_{ij} the element of matrix A. The subscript set $N = \{1, \cdots, n\}$. Suppose $B \subset N$. Denote by $v(B)$ and $A(B)$ respectively the subvector of v and the principal submatrix of A corresponding to the subscript set B. Let u be the solution of Problem (2). Denote by P and Z the subscript sets corresponding to the positive components and zero components of u respectively, i.e.

$$P = \{i \in N : u_i > 0\}, Z = \{i \in N : u_i = 0\}.$$

It is easy to see that

$$A(P)u(P) = b(P). \tag{3}$$

If $u(P)$ is known, then so is u. But there is a difficulty for finding $u(P)$—the subscript set P is unknown. This difficulty is similar to that appearing in solving free boundary problems. In the latter case the free boundary is unknown.

Suppose A is a stieltjes matrix (or simply S-matrix). It means that: (i) A is symmetric and positive definite; and (ii) $a_{ij} \leq 0$ for $i \neq j$. It is well known that (2) has a unique solution in this case. We now establish a lemma.

Lemma 1. *If A is a S-matrix and u is the solution of* (2), *then*

$$Z^* \supset P \supset P_0, \tag{4}$$

* Received September 29, 1987.
1) The Project Supported by National Natural Science Foundation of China.

where Z^* is the subscript set corresponding to the zero components of $Au - b$, and P_0 the subscript set corresponding to the positive components of b.

Proof. It follows from (2) that if $u_i > 0$, then $(Au - b)_i = 0$. This means $Z^* \supset P$. If $b_i > 0$, then it follows from $(Au - b)_i \geq 0, u \geq 0$ and $a_{ij} \leq 0$ for $i \neq j$ that

$$u_i \geq a_{ii}^{-1}(-\sum_{j\neq i} a_{ij}u_j + b_i) \geq a_{ii}^{-1}b_i > 0.$$

It means $P_0 \subset P$. The lemma has been proved.

The basic idea for constructing our new direct method is as follows: starting from the given subscript set P_0, expand it step by step and obtain finally the subscript set P in finite steps. Now we give the inductive definition of the algorithm.

Algorithm DM. (a) Suppose a subscript set P_k is known. Solve the following system by Gauss elimination:

$$A(P_k)u^{(k)}(P_k) = b(P_k). \tag{5}$$

Let

$$u_i^{(k)} = \begin{cases} u_i^{(k)}(P_k), & i \in P_k, \\ 0, & i \in N\backslash P_k. \end{cases} \tag{6}$$

(b) Calculate for $i \in N\backslash P_k$

$$C_i^{(k)} = (Au^{(k)} - b)_i. \tag{7}$$

(c) Let $M_k = \{i \in N\backslash P_k : C_i^{(k)} < 0\}$. If $M_k = \emptyset$, then stop computing and output $u^{(k)}$. If $M_k \neq \emptyset$, then let $P_{k+1} = P_k \cup M_k$ and repeat (a)–(c) after replacing k by $k+1$.

The following two theorems indicate that solution u of (2) is obtained in finite steps of Algorithm DM.

Theorem 1. *Suppose A is a S-matrix and M_k is defined in Algorithm DM. If $M_k = \emptyset$, then $u^{(k)} = u$.*

Theorem 2. *Suppose A is a S-matrix and m_0, m are respectively the cardinals of P_0, P. Then there exists a nonnegative integer $k_0 \leq m - m_0$ such that $M_{k_0} = \emptyset$.*

These theorems also indicate that we may find solution u of (2) by solving systems (5) for $k = 0, 1, \cdots, k_0$. Because $k_0 \leq n$, Algorithm DM is a polynomial time algorithm. In order to prove these theorems we need the following lemmas.

Lemma 2. *Suppose A is a S-matrix and $u^{(0)}(P_0)$ is defined by (5) for $k = 0$. Then $u^{(0)}(P_0) > 0$.*

Proof. Since A is a S-matrix, so is its principal submatrix $A(P_0)$. Suppose $A(P_0)$ is reducible. Because $A(P_0)$ is symmetric, we may reduce the system

$$A(P_0)u^{(0)}(P_0) = b(P_0)$$

to several irreducible subsystems

$$A(Q_l)u^{(0)}(Q_l) = b(Q_l), \quad l = 1, \cdots, r,$$

where Q_l's are subscript sets, $Q_i \cap Q_j = \emptyset$ for $i \neq j$ and $Q_1 \cup Q_2 \cup \cdots \cup Q_r = P_0$. Since $A(Q_l)$ is still a principal submatrix of A, it is an irreducible S-matrix. So we have $A^{-1}(Q_l) > 0$ ([8, p.61]) and

$$u^{(0)}(Q_l) = A^{-1}(Q_l)b(Q_l) > 0.$$

The proof is complete.

Lemma 3. *Suppose A is a S-matrix and $u^{(k)}$ is defined by* Algorithm DM. *Then we have*

$$u^{(k)} \geq u^{(k-1)} \geq 0.$$

Proof. According to Lemma 2, we have $u^{(0)}(P_0) > 0$. By (6),

$$u_i^{(0)} = \begin{cases} u_i^{(0)}(P_0), & i \in P_0, \\ 0, & i \in N \backslash P_0. \end{cases}$$

So $u^{(0)} \geq 0$. Letting $k = 0$ in (5), we obtain

$$\sum_{j \in P_0} a_{ij} u_j^{(0)} = b_i, \qquad i \in P_0.$$

It may be rewritten as

$$\sum_{j \in P_1} a_{ij} u_j^{(0)} = b_i, \qquad i \in P_0 \tag{8}$$

because $P_1 = P_0 \cup M_0 \supset P_0$ and $u_j^{(0)} = 0$ for $j \in P_1 \backslash P_0$. It follows from the definition of M_0 that we have for $i \in P_1 \backslash P_0 = M_0$

$$\sum_{j=1}^{n} a_{ij} u_j^{(0)} = c_i^{(0)} + b_i < b_i.$$

Then

$$\sum_{j \in P_1} a_{ij} u_j^{(0)} < b_i, \quad i \in P_1 \backslash P_0. \tag{9}$$

Combining (8) and (9) we obtain

$$A(P_1)u^{(0)}(P_1) \leq b(P_1).$$

On the other hand, we know that

$$A(P_1)u^{(1)}(P_1) = b(P_1).$$

So it is clear that

$$A(P_1)[u^{(1)}(P_1) - u^{(0)}(P_1)] \geq 0.$$

Since $A(P_1)$ is a S-matrix, it must be a monotone matrix[8]. Then we have $u^{(1)}(P_1) \geq u^{(0)}(P_1)$ and

$$u^{(1)} \geq u^{(0)} \geq 0.$$

The proof may be completed by induction.

Lemma 4. *Suppose the conditions of* Lemma 3 *hold. Then* $P_k \subset P$ *and* $u^{(k)} \leq u$.

Proof. We have $P_0 \subset P$ by Lemma 1. Noting that $u_j \geq 0$ and $a_{ij} \leq 0$ for $j \neq i$, we obtain for $i \in P_0$ that

$$(A(P_0)u(P_0))_i = \sum_{j\in P_0} a_{ij}u_j = \sum_{j\in P} a_{ij}u_j - \sum_{j\in P\backslash P_0} a_{ij}u_j \geq \sum_{j\in P} a_{ij}u_j = b_i.$$

It means

$$A(P_0)u(P_0) \geq b(P_0).$$

It follows from this fact and (5) that

$$A(P_0)[u(P_0) - u^{(0)}(P_0)] \geq 0.$$

Since $A(P_0)$ is a monotone matrix, we obtain

$$u(P_0) \geq u^{(0)}(P_0).$$

On the other hand, $u_i \geq 0 = u_i^{(0)}$ for $i \in N\backslash P_0$. So $u \geq u^{(0)}$.

Now we show that $P_1 \subset P$. Suppose it is not true. Then there exists a subscript i such that $i \in P_1$ and $i \bar{\in} P$. Since $P_1 = P_0 \cup M_0$ and $P_0 \subset P$, we have $i \in M_0$. It follows from $i \bar{\in} P$ that $u_i = 0$. By the definition of M_0 we know $u_i^{(0)} = 0$. We have proved $u \geq u^{(0)}$. So

$$a_{ij}u_j^{(0)} \geq a_{ij}u_j \qquad \text{for} \quad j \neq i.$$

Then we have

$$C_i^{(0)} = \sum_{j=1}^{n} a_{ij}u_j^{(0)} - b_i = \sum_{j\neq i} a_{ij}u_j^{(0)} - b_i \geq \sum_{j\neq i} a_{ij}u_j - b_i = (Au - b)_i \geq 0.$$

It contradicts $i \in M_0$. This contradiction proves $P_1 \subset P$.

We conclude by similar argument and induction that $P_k \subset P$ and $u^{(k)} \leq u$.

Proof of Theorem 1. Since $M_k = \emptyset$, we have

$$C_i^{(k)} = (Au^{(k)} - b)_i \geq 0, \qquad i \in N\backslash P_k.$$

It follows from (5) and (6) that

$$(Au^{(k)} - b)_i = 0, \qquad i \in P_k.$$

So we obtain $Au^{(k)} \geq b$. On the other hand, we know that $u^{(k)} \geq 0$ by Lemma 3. Then $u^{(k)}$ satisfies the first and second inequalities of (2). It follows from (5) and (6) that $(Au^{(k)} - b)_i = 0$ for $i \in P_k$ and $u_i^{(k)} = 0$ for $i \in N\backslash P_k$. So $u^{(k)}$ satisfies the third equation of (2). According to the uniqueness of the solution of (2) we conclude that $u^{(k)} = u$.

Proof of Theorem 2. If it is not the case, then we have

$$M_k \neq \emptyset, \qquad k = 0, 1, \cdots, m - m_0.$$

Denote by m_k the cardinal of P_k. Since the intersection of P_k and M_k is empty, and since

$$P_{k+1} = P_k \cup M_k,$$

we obtain

$$m_{k+1} \geq m_k + 1, \qquad k = 0, 1, \cdots, m - m_0.$$

We derive from it that

$$m_{m-m_0+1} \geq m_{m-m_0} + 1 \geq \cdots \geq m_0 + m - m_0 + 1 = m + 1. \tag{10}$$

On the other hand, by Lemma 4 we have

$$P_{m-m_0+1} \subset P.$$

So

$$m_{m-m_0+1} \leq m.$$

It contradicts (10) and the theorem has been proved.

Remark 1. We may construct a similar algorithm for Problem (1) and it is not necessary to reduce this problem into Problem (2).

Remark 2. Numerical experiments indicate that we may replace P_0 by any given subset of P in Algorithm DM. But it is an open problem to prove the reasonableness of this replacement.

References

[1] Feng Kang, Unilateral variational principle, *Computer Applications and Applied Math.*, 12, 1975, 1-8.

[2] R. Glowinski et al., Numerical Analysis of Variational Inequalities, North-Holland, 1981.

[3] R. W. Cottle et al. (eds.), Variational Inequalities and Complementarity Problems, Wiley, 1980.

[4] R. W. Cottle et al., On the solution of large, structured linear complementarity problems, the partitioned case, *Appl. Math. Optim.*, **4** (1978), 347-363.

[5] J. X. Wang et al., Multigrid method for elasticity problems, *JCM*, 4 (1986), 154-163.

[6] C. W. Cryer, The efficient solution of linear complementarity problems for tridiagonal Minkowski matrices, *ACM Trans. Math. Software*, **9** (1983), 199-214.

[7] S. Z. Zhou, On the convergence of an algorithm of Uzawa's type for Saddle-point problems, *Kexue Tongbao*, **30** (1985), 1531-1534.

[8] D. M. Young, Iterative Solution of Large Linear Systems, AP, 1971.