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CONVERGENCE THEORY FOR AOR METHOD*

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Abstract

In this paper we give some sufficient conditions for the convergence of the AOR method, introduced by Hadjidimos [5], which include the ones from [1], [2], [5], [6], [7], [9], [10], [11] and [12] and which show that the necessary condition given in [8] for the convergence of the AOR method is not valid. We give general conditions for the class of H-matrices, but they are not always easy to check in practice. Consequently, we give some more practical conditions concerning some subclasses of H-matrices.

§1. Introduction

Among the various iterative methods which are used for the numerical solution of the linear system

$$Ax = b,$$

where $A \in C^{n,n}$ is a nonsingular matrix with nonzero diagonal entries, and $x, b \in C^n$ with x unknown and b known, the completely consistent linear stationary iterative schemes of first degree play a very important role. Such an iterative method, called the accelerated overrelaxation (AOR) method, was introduced by Hadjidimos in [5]. Since the introduction of the AOR method, many properties as well as unmerical results concerning this method have been given. There are many papers dealing with the linear systems with a matrix which is strictly diagonally dominant (SDD), irreducible diagonally dominant (IDD), or generalized diagonally dominant (GDD) is an M– or H– matrix (cf. [1], [5], [6], [9], [10], [11], [12], [17], [18]). in [2] and [7] some new classes of linear systems have been considered. The purpose of this paper is: i) to present some further basic results concerning the convergence of the AOR method when the matrix A is an H-matrix (all of the mentioned classes are H-matrices), and ii) to give more practical sufficient conditions for the convergence of the AOR method when the matrix A belongs to some special subclasses of H-matrices.

Let $A = D - T - S$ be the decomposition of the matrix A into its diagonal, strictly lower and strictly upper triangular parts, respectively and let $\omega, \sigma \in R, \omega \neq 0$. The associated AOR method can be written as

$$x^{k+1} = M_{\sigma,\omega} x^k + d, \quad k = 0, 1, \cdots, x^0 \in C^n,$$

where $M_{\sigma,\omega} = (D - \sigma T)^{-1}((1-\omega)D + (\omega - \sigma)T + \omega S), \quad d = \omega(D - \sigma T)^{-1} b.$

Some special cases of this method are

$$\text{AOR} \begin{array}{l} \xrightarrow{\omega=\sigma} \text{SOR} \xrightarrow{\omega=1} \text{Gauss-Seidel} \\ \xrightarrow[\sigma=0]{} \text{JOR} \xrightarrow[\omega=1]{} \text{Jacobi} \end{array}$$

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The AOR method has some connection with the extrapolation principle, since it is an extrapolation of either the Jacobi method (case $\sigma = 0$) or the SOR method (case $\sigma \neq 0$, where the extrapolation parameter is ω/σ). This fact and many numerical examples (cf. [1], [5]) show the superiority of the AOR method.

§2. Preliminaries

We shall use the following notations:

$$N = \{1, 2, \cdots, n\}, \quad N(i) = N \setminus \{i\}, \quad i \in N.$$

For any matrix $A = [a_{ij}] \in C^{n,n}$ (= set of all complex $n \times n$ matrices) and $i \in N, \alpha \in [0, 1]$, we define

$$P_i(A) = \sum_{j \in N(i)} |a_{ij}|, \quad Q_i(A) = \sum_{j \in N(i)} |a_{ji}|,$$

$$P_{i,\alpha}(A) = \alpha P_i(A) + (1 - \alpha) Q_i(A), \quad Q_i^*(A) = \max_{j \in N(i)} |a_{ji}|,$$

$$Q_i^{(r)}(A) = \max_{t_r \in \theta_r} \sum_{j \in t_r} |a_{ji}|,$$

where $r \in N$ and θ_r is the set of all choices $t_r = \{i_1, \cdots, i_r\}$ of different indices from N.

Definition 2.1. *A real square matrix whose off-diagonal elements are all non-positive is called L-matrix.*

Definition 2.2. *A regular L-matrix A for which $A^{-1} \geq 0$ is called M-matrix.*

In [3] we have proved the following two theorems.

Theorem 2.1. *Let A be an L-matrix, whose diagonal elements are all positive such that at least one of the following conditions is satisfied:*

(i) $a_{ii} > P_i(A), i \in N$(SDD).

(ii) $a_{ii} > P_{i,\alpha}(A), i \in N$, for some $\alpha \in [0, 1]$.

(iii) $a_{ii} > P_i^{\alpha}(A) Q_i^{1-\alpha}(A), i \in N$, for some $\alpha \in [0, 1]$.

(iv) $a_{ii} a_{jj} > P_i(A) p_j(A), i \in N, j \in N(i)$.

(v) $a_{ii} a_{jj} > P_i^{\alpha}(A) Q_i^{1-\alpha} p_j^{\alpha}(A) Q_j^{1-\alpha}(A), i \in N, j \in N(i)$, for some $\alpha \in [0, 1]$.

(vi) For each $i \in N$ it holds that $a_{ii} > P_i(A)$ or

$$a_{ii} + \sum_{j \in J} a_{jj} > Q_i(A) + \sum_{j \in J} Q_j(A), \text{ where } J := \{i \in N : a_{ii} \leq Q_i(A)\}.$$

(vii) $a_{ii} > \min(P_i(A), Q_i^*(A)), i \in N$ and $a_{ii} + a_{jj} > P_i(A), i \in N, j \in N(i)$.

(viii) $a_{ii} > Q_i^{(p)}(B), i \in N$ and $\sum_{j \in t_p} a_{ii} > \sum_{j \in t_p} P_i(A), t_p \in \theta_p$, for some $p \in N$.

(ix) There exists $i \in N$ such that

$$a_{ii}(a_{jj} - P_j(A) + |a_{ji}|) > P_i(A)|a_{ji}|, j \in N(i).$$

Then A in an M-matrix.

Note that SDD matrices satisfy all of the conditions (i)–(ix).

For any matrix $A = [a_{ij}] \in C^{n,n}$, we define $M(A) = [m_{ij}] \in R^{n,n}$ as follows

$$m_{ii} = |a_{ii}|, i \in N, m_{ij} = -|a_{ij}|, i \in N, j \in N(i).$$

Definition 2.3. *A matrix A is called H-matrix iff $M(A)$ is an M-matrix.*

Definition 2.4. *A matrix A is called a generalized diagonally dominant* (GDD) *matrix iff there exists a regular diagonal matrix W, so that AW is* SDD.

Theorem 2.2. *Let A be a matrix whose elements satisfy at least one of the conditions* (i)–(ix) *in* Theorem 2.1, *where all diagonal elements of A are replaced by their modules. Then A is an H-matrix.*

Remark. Any irreducible diagonally dominant matrix is an H-matrix too (see [16]).

Theorem 2.3. *A matrix A is* GDD *if ang only if it is an H-matrix.*

Proof. Let A be GDD. Then there exists a regular diagonal matrix W such that AW is SDD. Then AW is an H-matrix, i.e. $M(AW) = M(A)M(W)$ is an M-matrix. Since $M(W)$ is regular and $M(W) > 0$, it follows that

$$(M(A))^{-1} = M(W)(M(AW))^{-1} \geq 0.$$

Conversely, if A is an H-matrix, i.e. if $M(A)$ is an M-matrix, then there exists a vector $z \in R^{n,n}, z > 0$, such that $M(A)z > 0$. It means that

$$|a_{ii}|z_i > \sum_{j\in N(i)} |a_{ij}|z_j \text{ for each } i \in N$$

and we can choose the matrix $W =$ diag $(z_1, \cdots, z_n)$.

§3. The Convergence of AOR Method

We shall begin our convergence analysis with the case that the matrix A is SDD.

In [2] we proved the following upper bound for the spectral radius of the matrix of the AOR method, $M_{\sigma,\omega}$:

(1) $p(M_{\sigma,\omega}) \leq \max_{1\leq i\leq n} ((|1-\omega|+|\omega-\sigma|P_i(L)+|\omega|P_i(U))/(1-|\sigma|P_i(L))$, if $1-|\sigma|P_i(L) > 0$.

Here $L = D^{-1}S, U = D^{-1}T$.

Theorem 3.1. *Let A be a strictly diagonally dominant matrix. Then the* AOR *method converges for:*

(i) $0 \leq \sigma < 2/(1 + P(M_{0,1}(M(A)))) =: s,$
$0 < \omega < \max\{2\sigma/(1 + P(M_{\sigma,\sigma})), 2/(1 + \max_i P_i(L + U)) =: t\}$, or

(ii) $\max_i(-\omega(1 - P_i(L + U)) + 2\max(0, \omega - 1))/2P_i(L) < \sigma < 0, 0 < \omega < t$, or

(iii) $t \leq \sigma < \min_i(\omega(1 + P_i(L) - P_i(U)) + 2\min(0, 1 - \omega))/2P_i(L), 0 < \omega < t.$

Proof. It is easy to verify that for each σ, which satisfies one of the conditions (i)–(iii), we have

$$1 - |\sigma|P_i(L) > 0, \quad i \in N.$$

(i) Since A is a SDD matrix, $M(A)$ is an M-matrix, and from [16] it follows that for $0 < \sigma < s, P(M_{\sigma,\sigma}) < 1$ holds. It is known that for $\sigma \neq 0, M_{\sigma,\omega} = (1-\omega/\sigma)E + \omega/\sigma M_{\sigma,\sigma}$. If $0 < \omega/\sigma < 2/(1+P(M_{\sigma,\sigma}))$, by using the extrapolation theorem[6], we conclude $P(M_{\sigma,\omega}) < 1$.

It remains to analyse the case $2\sigma/(1 + P(M_{\sigma,\sigma})) \leq \omega < t, o \leq \sigma < s$. Since $\sigma < 2\sigma/(1 + P(M_{\sigma,\sigma}))$, it follows that $0 \leq \sigma < \omega < t$. Because

$$0 \leq \sigma < \omega < 2/(1 + P_i(L + U)) \Rightarrow |1 - \omega| + (\omega - \sigma)P_i(L) + \omega P_i(U) < 1 - \sigma P_i(L),$$

from (1) we obtain $P(M_{\sigma,\omega}) < 1$. (ii) and (iii) can be proved similarly, by using the inequality (1).

As the following example shows, our area of convergence for the parameters σ and ω is weaker than the one from [12], and of course, it is weaker than the others from the cited literature, which are related to the class of SDD matrices. The same example shows that Theorem 2 from [8] is not valid.

Example 1. Let

$$A = \begin{bmatrix} 4 & 3 \\ 1 & 3 \end{bmatrix}.$$

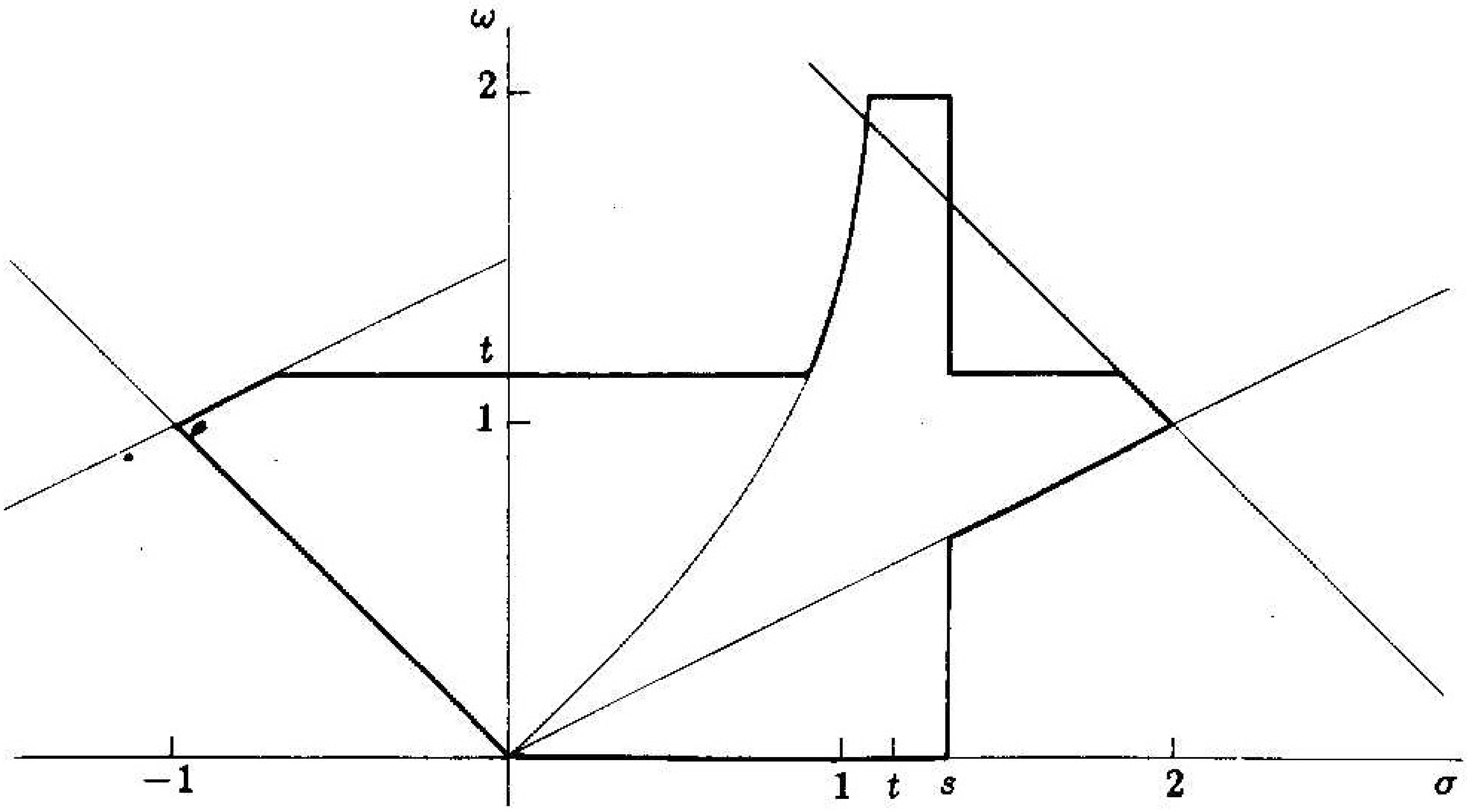

Fig. 1

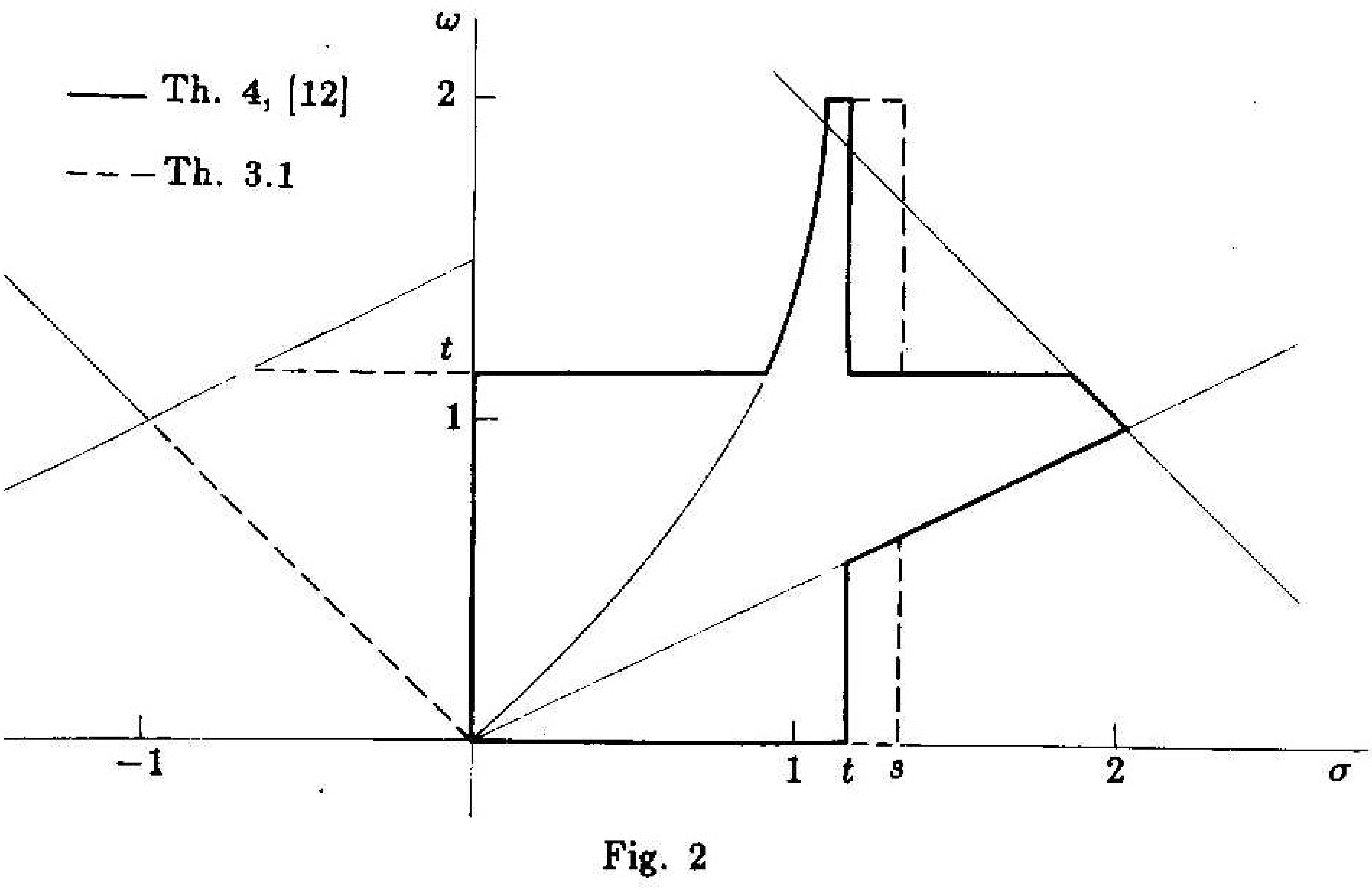

Fig. 2

From Theorem 3.1 we obtain the following area of convergence:

(i) $0 \le \sigma < \frac{4}{3}, 0 < \omega < \max\{8/7, 2\sigma/(1 + P(M_{\sigma,\sigma}))\}$ or

(ii) $0 < \omega \le 1, -\omega < \sigma < 0$ or $1 < \omega < 8/7, 2\omega - 3 < \sigma < 0$, or

(iii) $0 < \omega \le 1, 8/7 \le \sigma < 2\omega$ or $1 < \omega < 8/7, 8/7 \le \sigma < 3 - \omega$.

Figure 1 is a geometrical interpretation of the above area of convergence. This area is larger than the one from Theorem 4 in [12] (Fig. 2).

Now, we can use the result of Theorem 3.1 to improve the area of convergence for the parameters σ and ω in case that A is an H-matrix, i.e. GDD.

Since $P(M_{0,1}(M(A))) = P(M_{0,1}(M(AW)))$ and $P(M_{\sigma,\omega}(A)) = P(M_{\sigma,\omega}(AW))$ for regular matrix W, we obtain the following theorem.

Theorem 3.2. *If A is an H-matrix (i.e. GDD) and the parameters σ and ω are chosen as in Theorem 3.1, where the matrices L and U are replaced by LW and UW respectively, then $P(M_{\sigma,\omega}(A)) < 1$.*

Corollary 3.2.1. *If A is an IDD matrix or an M-matrix or a matrix, whose elements satisfy at least one of the conditions (i)–(ix) in Theorem 2.2, and if σ and ω are as in Theorem 3.2, then the AOR method converges.*

Let A be the matrix from Example 1. The area of convergence for the matrix $M(A)$, obtained now by Corollary 3.2.1 (see Fig. 1), is still larger than the one from Theorem 8 in [12] (which relates only to the class of M-matrices). Namely, in our case it is possible to choose the parameter σ negative.

Note that if we do not know the matrix W, we can choose the parameters as follows:

$$0 \le \sigma < s, \quad 0 < \omega < \max\{1, 2\sigma/(1 + P(M_{\sigma,\sigma})\}.$$

Of course, instead of each spectral redius we can use a norm of the corresponding matrix. The convergence will be still present.

Evidently, the case $0 \le \sigma < 1, 0 < \omega < 1$ is always included. The statement which is related to the convergence of the AOR method in this case, for the class of IDD matrices, is formulated in [5], but our opinion is that the proof is not complete. The author does not consider the case $\lambda = -1, \omega = 2\sigma$.

From the other side, sometimes the coefficient matrix A possesses some extra basic property, like one of (i)–(ix) in Theorem 2.2. Then we can say something else about the convergence of the AOR method. By the following theorem we shall illustrate how to obtain the convergence intervals for σ and ω without computation of the matrix $M_{\sigma,\sigma}$ (which may be weaker than the ones from the above theorems, as Example 2 shows).

Theorem 3.3. *Let A be a matrix whose elements satisfy the following condition*

$$1 > P_{i,\alpha}(D^{-1}A), \quad i \in N.$$

Then $P(M_{\sigma,\omega}) < 1$ for :

(i) $0 \le \sigma < \max\{s, \min_i 2/(1 + P_{i,\alpha}(L + U)) =: t'\}$,

$0 < \omega < \max\{t', 2\sigma/(1 + P(M_{\sigma,\sigma}))\}$ or

(ii) (iii) as in Theorem 3.1, where each P_i is replaced by $P_{i,\alpha}$.

Proof. In [2] we have obtained the upper bound for $P(M_{\sigma,\omega})$, which is the same as (1), where each P_i is replaced by $P_{i,\alpha}$. By that inequality we can prove the following implication

$$0 \leq \sigma < t' \Rightarrow P(M_{\sigma,\sigma}) < 1,$$

and just as in the proof of Theorem 4, we complete the proof here.

In a similar way we can construct intervals of convergence for σ and ω in the cases (ii), (iv), (vi) and (viii) from Theorem 2.2.

Example 2. For the same Matrix as in Example 1, by Theorem 3.3 for $\alpha = 0.5$, we obtain $P_{i,\alpha}(L+U) = 13/24, t' = 48/37 > t = 8/7$, and the following area of convergence:

(i) $0 \leq \sigma \leq 4/3, 0 < \omega < \max\{48/37, 2\sigma/(1+P(M_{\sigma,\sigma}))\}$, or

(ii) $0 < \omega \leq 1, -11\omega/8 < \sigma < 0$ or $1 < \omega < 48/37, 37\omega/8 - 6 < \sigma < 0$, or

(iii) $0 < \omega \leq 1, 48/37 \leq \sigma < 19\omega/8$ or $1 < \omega < 48/37, 48/37 \leq \sigma < 6 - 29\omega/8$.

In the following Fig. 3 we can see that this area is weaker than the one from Fig. 1.

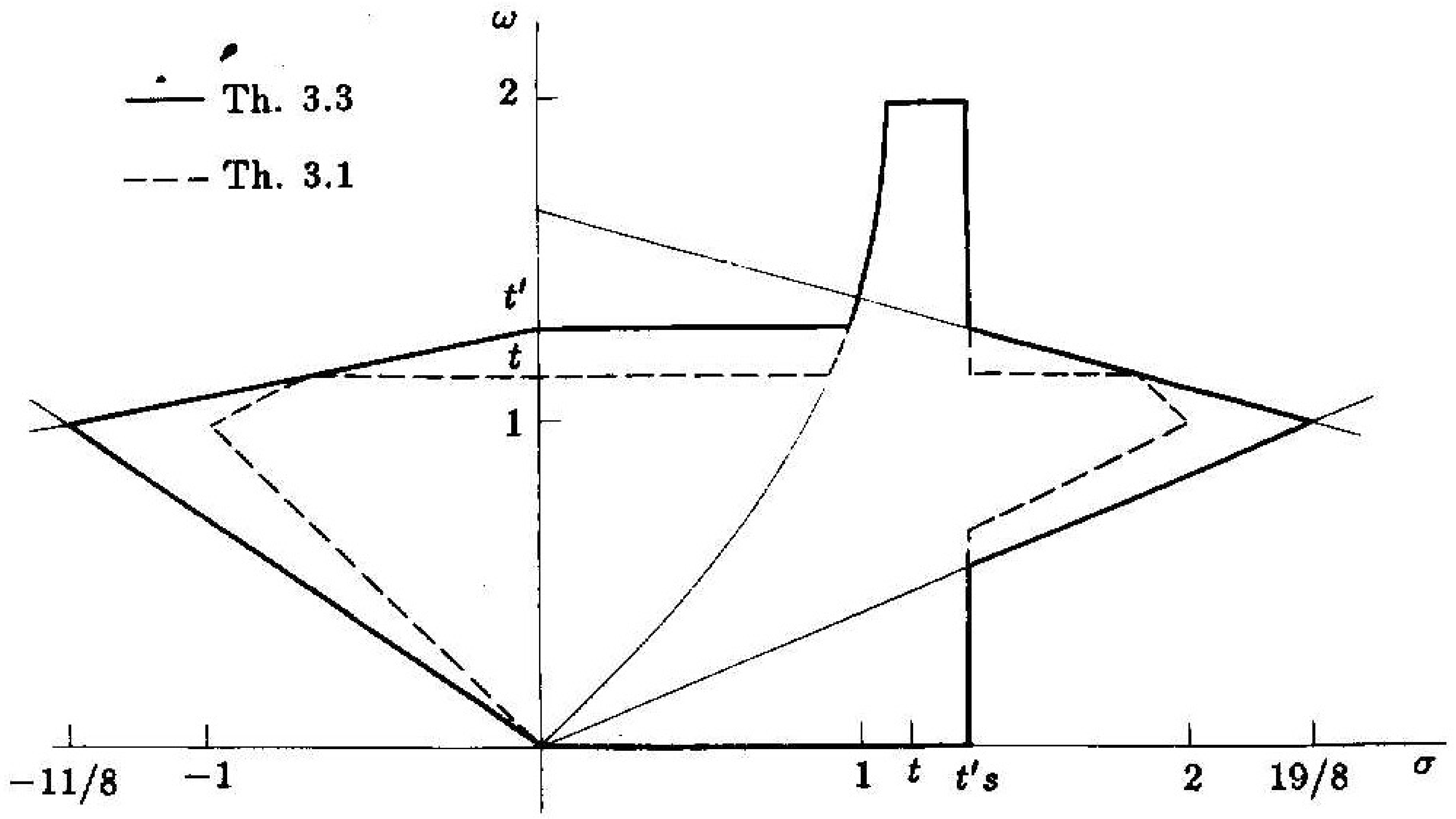

Fig. 3

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