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A SPLITTING ITERATION METHOD FOR A SIMPLE CORANK-2 BIFURCATION PROBLEM*1)

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Abstract

A splitting iteration method is introduced to approximate a simple corank-2 bifurcation point of a nonlinear equation with small extended systems. This iteration method converges linearly with an adjustable speed and needs little extra computational work.

§1. Introduction

Let E be a Hilbert space, and $G : E \times I\!R$ a nonlinear C^3-mapping. We consider the nonlinear equation

$$G(u, \lambda) = 0 \tag{1.1}$$

and its corank-2 bifurcation problems. We assume that there is a point (u_0, λ_0) in $E \times I\!R$ satisfying

(H1) $$G_0 := G(u_0, \lambda_0) = 0$$

and

(H2) D_uG_0 is a Fredholm operator with index 0 and zero is one of its eigenvalues with algebraic multiplicity 2; furthermore,

$$\text{a)}\quad \dim(\text{Null}(D_uG_0)) = 2, \qquad \text{b)}\quad D_\lambda G_0 \in \text{Range}(D_uG_0). \tag{1.2}$$

The main aim of this paper is to introduce an efficient method for accurate approximation of the simple corank-2 bifurcation point (u_0, λ_0) of (1.1) and the null vectors of $D_uG_0, D_uG_0^*$ which are used in path following of (1.1) around (u_0, λ_0) (cf. [2, 7, 13, 15, 16]).

For highly singular problems of (1.1), Allgower and Böhmer[1], Beyn[2] and Mezel[10] have discussed some general principles on the extended systems; particularly, also see [18], [20] and [3] for simple corank-2 bifurcation problems. All these extended systems are at least three times larger than the original equation (1.1) and the equations in these

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systems are intrinsically dependent on each other. Consequently, the linearized equations in Newton-like iterations have to be solved directly or successively with the help of some intermediate unknowns, which leads to large computational efforts, especially for PDE problems.

On the other hand, by unfolding Rabier and Reddien[14] transformed the highly singular equation into a generalized turning point problem and then made up minimally extended systems with some implicitly defined scalar equations. But, the convergence of Newton's method and its implementation were not discussed.

We will set up various small extended systems for (1.1) and its linearized problem at the bifurcation point (u_0, λ_0), and introduce a splitting iteration method to approximate simultaneously the point (u_0, λ_0), the null vectors of $D_uG_0, D_uG_0^*$ in a coupled way. This iteration method converges with an adjustable speed and its computational cost at each iteration step remains at the same level as that for the regular solutions of (1.1).

In Section 2 we state the definition of a corank-2 bifurcation point type-I of (1.1) and show its base independent property. Section 3 discusses the splitting iteration method and its convergence. Finally, we present in Section 4 two simple numerical example showing the behaviour of the method.

§2. Corank-2 Type-I Bifurcation Points

We introduce in this section a corank-2 bifurcation point type-I of (1.1) and its base independent property. In the following, we assume that the conditions (H1) and (H2) are satisfied and the mapping G is C^3-continuous. We see easily from statement (1.2b) that

$$\dim(N(DG_0)) = 3.$$

On the other hand, if

$$D_\lambda G_0 \notin R(D_uG_0),$$

equation (1.1) can be transformed into a simple bifurcation problem under symmetries and other parametrizations[19].

Under the conditions (H1)–(H2), the Fredholm operator theory shows that there are elements $\phi_i, \phi_i^* \in E, i = 1, 2$, such that

$$\begin{cases} V_1 := N(D_uG_0) = \text{Span}\,[\phi_1, \phi_2], \langle \phi_1, \phi_j \rangle = \delta_{ij}, \\ \tilde{V}_1 := N(D_uG_0^*) = \text{Span}\,[\phi_1^*, \phi_2^*], \langle \phi_i^*, \phi_j \rangle = \delta_{ij}, i, j = 1, 2, \end{cases} \tag{2.1}$$

and

$$\begin{cases} V_2 := R(D_uG_0) = \{u \in E, \langle \phi_i^*, u \rangle = 0, \quad i = 1, 2\} \\ \tilde{V}_2 := R(D_uG_0) = \{u \in E, \langle \phi_i, u \rangle = 0, \quad i = 1, 2\}; \end{cases} \tag{2.2}$$

furthermore,

$$E = V_1 \oplus V_2 = \tilde{V}_1 \oplus \tilde{V}_2, \tag{2.3}$$

Remark 2.1. Lemma 2.1 indicates that the corank-2 bifurcation point type-I of (1.1) is independent of the basis of the space E.

Without loss of generality, we choose f in E, such that[1,2,5,10,14]

$$|\langle f, \Phi_1\rangle| + |\langle f, \Phi_2\rangle| > 0, \quad \text{i.e.,} \quad f \notin R(D_u G_0^*). \tag{2.5}$$

Lemma 2.2. *If the element f is chosen as above, there is a basis $\{\Phi_1, \tilde{\Phi}_2\}$ of V_1 and basis $\{\Phi_1^*, \tilde{\Phi}_2^*\}$ of $\tilde{V}_1$, such that*

$$\langle f, \tilde{\Phi}_1\rangle = 0, \quad \langle \tilde{\Phi}_i, \tilde{\Phi}_j\rangle = \langle \tilde{\Phi}_i^*, \tilde{\Phi}_j\rangle = \delta_{ij}, \quad i,j = 1,2. \tag{2.6}$$

Proof. Let $\{\Phi_1, \Phi_2\}, \{\Phi_1^*, \Phi_2^*\}$ be the basis of the spaces $V_1, \tilde{V}_1$ in (2.1) respectively. If

$$\langle f, \Phi_1\rangle = 0,$$

we get (2.6) immediately. If

$$\langle f, \Phi_1\rangle \neq 0 \quad \text{and} \quad \langle f, \Phi_2\rangle = 0,$$

setting

$$\tilde{\Phi}_1 := \Phi_2, \quad \tilde{\Phi}_2 := \Phi_1 \quad \text{and} \quad \tilde{\Phi}_1^* := \Phi_2^*, \quad \tilde{\Phi}_2^* := \Phi_1^*,$$

we obtain (2.6). Otherwise, it holds that

$$\langle f, \Phi_1\rangle \cdot \langle f, \Phi_2\rangle \neq 0.$$

Let

$$\xi := \Phi_2 - \langle f, \Phi_2\rangle / \langle f, \Phi_1\rangle \cdot \Phi_1 \neq 0, \quad \tilde{\Phi}_1 := \xi / \sqrt{\langle \xi, \xi\rangle}$$

and

$$\eta := \Phi_2 - \langle \Phi_2, \tilde{\Phi}_1\rangle \tilde{\Phi}_1, \quad \tilde{\Phi}_2 := \eta / \sqrt{\langle \eta, \eta\rangle},$$

where $\eta \neq 0$ follows from the fact $\langle \eta, \Phi_1\rangle = \langle f, \Phi_2\rangle / [\langle f, \Phi_1\rangle \cdot \langle \xi, \xi\rangle] \neq 0$. It is easy to verify

$$\langle f, \tilde{\Phi}_1\rangle = 0, \quad \langle \tilde{\Phi}_i, \tilde{\Phi}_j\rangle = \delta_{ij}, \quad i,j = 1,2.$$

On the other hand, there are $a_{ij} \in I\!R, i,j = 1,2$, such that

$$\tilde{\Phi}_1 = a_{11}\Phi_1 + a_{12}\Phi_2, \quad \tilde{\Phi}_2 = a_{21}\Phi_1 + a_{22}\Phi_2$$

and the matrix

$$A := \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \neq 0$$

is nonsingular. Denote

$$B := \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

as the inverse of A. We choose

$$\tilde{\Phi}_i^* = b_{1i}\Phi_1^* + b_{2i}\Phi_2^* \in \tilde{V}_1, \quad i = 1,2$$

which consist of a basis of $\tilde{V}_1$. From (2.1) it follows that

$$\langle \tilde{\Phi}_i^*, \tilde{\Phi}_j\rangle = \delta_{ij}, \quad i,j = 1,2.$$

This completes the proof.

We will denote the bases of $V_1, , \tilde{V}_1$ in Lemma 2.2 still as $\{\Phi_1, \Phi_2\}$, $\{\Phi_1^*, \Phi_2^*\}$. Due to the base independence property, the condition (H3) will be considered under these bases. Furthermore, the parameter ε will be adjusted constantly in the subset of $I\!R^4$ in which (H3) holds.

§3. A Splitting Iteration Method for Corank-2 Bifurcation Points Type-I

In this section, we discuss a splitting iteration method for the approximation of the corank-2 bifurcation point (u_0, λ_0) and the basis of the null spaces $N(D_uG_0)$ and $N(D_uG_0^*)$ in Lemma 2.2.

Denote $E_1 := E_2 \times (E_3)^4, E_2 := E \times I\!R^4$ and $E_3 := E \times I\!R^2$ equipped with product norms respectively. We express all elements x in E_1 with five blocks as $x := (x_0, x_1, x_2, x_3, x_4)$ including unfolding constants c_i in $I\!R$:

$$\begin{cases} x_0 := (u, \lambda, c_1, c_2, c_3) \in E_2, \\ x_i := (u_j, c_{2i+2}, c_{2i+3}) \in E_3, \quad i = 1(1)4. \end{cases} \tag{3.1}$$

Then we define five mappings $H_0 : E_1 \to E_2$ and $H_i : E_1 \to E_3$, $i = 1(1)4$, and consider five small extended systems for (1.1) and its linearizations. First of all,

$$H_0(x) := \begin{pmatrix} G(u, \lambda) + c_1 u_1/m + c_2 u_2/m^2 \\ \langle u_3, D_u G u_1 \rangle /m^4 + \varepsilon_1 c_3 \\ \langle u_4, D_u G u_1 \rangle /m^4 + \varepsilon_2 c_3 \\ \langle u_3, D_u G u_2 \rangle /m^5 + \varepsilon_3 c_3 \\ \langle u_4, D_u G u_2 \rangle /m^5 + \varepsilon_4 c_3 \end{pmatrix} = 0, \tag{3.2}$$

where $m > 0$ is a normalizing parameter and $\varepsilon := (\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)^T \in I\!R^4$ is a vector of control parameters. We regard (3.2) as a system for unknown $x_0 = (u, \lambda, c_1, c_2, c_3)$ in E_2, and

$$H_1(x) := \begin{pmatrix} D_u G(u, \lambda) u_1 + c_4 u_1/m + c_5 u_2/m^2 \\ [\langle u_1, u_1 \rangle - m^2]/2m \\ \langle f, u_1 \rangle \end{pmatrix} = 0 \tag{3.3}$$

as a system for unknown $x_1 = (u_1, c_4, c_5)$ in E_3. At the same time, system

$$H_2(x) := \begin{pmatrix} D_u G(u, \lambda) u_2 + c_6 u_1/m + c_7 u_2/m^2 \\ \langle u_1, u_2 \rangle /m \\ [\langle u_2, u_2 \rangle - m^4]/2m^2 \end{pmatrix} = 0 \tag{3.4}$$

is considered for unknown $x_2 = (u_2, c_6, c_7)$ and

$$H_3(x) := \begin{pmatrix} D_uG(u,\lambda)^* u_3 + c_8 u_1/m + c_9 u_2/m^2 \\ [\langle u_1, u_3\rangle - m^4]/m \\ \langle u_2, u_3\rangle/m^2 \end{pmatrix} = 0 \tag{3.5}$$

for unknown $x_3 = (u_3, c_8, c_9)$ in E_3. Finally,

$$H_4(x) := \begin{pmatrix} D_uG(u,\lambda)^* u_4 + c_{10} u_1/m + c_{11} u_2/m^2 \\ \langle u_1, u_4\rangle/m \\ [\langle u_2, u_4\rangle - m^5]/m^2 \end{pmatrix} = 0 \tag{3.6}$$

is a system for unknown $x_4 = (u_4, c_{10}, c_{11})$ in E_3.

Since mapping G is C^3-continuous, mappings H_i, $i = 0(1)4$, are obviously well defined and C^2-continuous in E_1. In addition, observing the extended system (3.2), we may find that the unknown c_3 is superfluous. In fact, we can eliminate it among the four scalar equations and add the resulting three scalar equations to (1.1) to make up a minimally extended system for (1.1)[11,12,5,14]. In particular, if the condition (H3) holds for ε_i=const. e_i and $i \in \{1,2,3,4\}$, the unknown c_3 can be solved explicitly from the scalar equations in (3.2).

Nevertheless, with this additional unfolding parameter c_3, we have the base independence property indicated in Lemma 2.1. The regularity and the solution of (3.2) are then independent of the basis of E and the unfolding forms of the accompanying equations in (3.3)–(3.6). This allows us to choose the starting value for the splitting iteration method more freely than in [11], [12] where the convergence of the Newton-like method is achieved merely in some sectors in the neighborhood of the exact solution of the extended systems. For this reason, we prefer using the extended system (3.2) which includes many possible minimally extended systems for (1.1) by changing the control parameter ε in $\mathbb{R}^4$.

Denote

$$\begin{cases} x^* := (x_0^*, x_1^*, x_2^*, x_3^*, x_4^*) \ \text{ with } \ x_0^* := (u_0, \lambda_0, 0) \in E_2, \\ x_i^* := (m^i \Phi_i, 0) \in E_3, \quad x_{i+2}^* := (m^3 \Phi_i^*, 0) \in E_3, \quad i = 1, 2. \end{cases} \tag{3.7}$$

We obtain from statements (2.1)–(2.4), (2.6) and definitions (3.2)–(3.7)

$$H_i(x^*) = 0, \quad i = 0(1)4. \tag{3.8}$$

Remark 3.2. Equality (3.8) is independent of the choice of the control parameter ε in $\mathbb{R}^4$.

Let

$$\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \cdots, \tilde{x}_4) := (u, \lambda, 0, u_1, 0, u_2, 0, u_3, 0, u_4, 0) \in E_1 \tag{3.9}$$

be a restriction of x in E_1 which annihilates the unfolding parameters c_i in x. We calculate directly in (3.2)–(3.6) and get

$$A_0(x,\varepsilon) := D_{x_0} H_0(x) = m^{-5}$$

$$\times \begin{pmatrix} m^5 D_uG(u,\lambda) & m^5 D_\lambda G(u,\lambda) & m^4 u_1 & m^3 u_2 & 0 \\ m\langle u_3, D_{uu}Gu_1\rangle & m\langle u_3, D_{u\lambda}Gu_1\rangle & 0 & 0 & \varepsilon_1 m^5 \\ m\langle u_4, D_{uu}Gu_1\rangle & m\langle u_4, D_{u\lambda}Gu_1\rangle & 0 & 0 & \varepsilon_2 m^5 \\ \langle u_3, D_{uu}Gu_2\rangle & \langle u_3, D_{u\lambda}Gu_2\rangle & 0 & 0 & \varepsilon_3 m^5 \\ \langle u_4, D_{uu}Gu_2\rangle & \langle u_4, D_{u\lambda}Gu_2\rangle & 0 & 0 & \varepsilon_4 m^5 \end{pmatrix}, \tag{3.10}$$

$$A_1(x) := D_{x_1}H_1(\tilde{x}) = \begin{pmatrix} D_uG(u,\lambda) & u_1/m & u_2/m^2 \\ \langle u_1, \cdot\rangle/m & 0 & 0 \\ \langle f, \cdot\rangle & 0 & 0 \end{pmatrix}, \tag{3.11}$$

$$A_2(x) := D_{x_2}H_2(\tilde{x}) = \begin{pmatrix} D_uG(u,\lambda) & u_1/m & u_2/m^2 \\ \langle u_1, \cdot\rangle/m & 0 & 0 \\ \langle u_2, \cdot\rangle/m^2 & 0 & 0 \end{pmatrix}, \tag{3.12}$$

and $D_{x_i}H_i(\tilde{x}) = A_2(x)^*, i = 3, 4$, the adjoint operator of $A_2(x)$.

Theorem 3.1. *Under Conditions* (H1)–(H3), *the operators* $A_0(x^*, \varepsilon)$ *and* $A_i(x^*)$, $i = 1, 2$, *are nonsingular on* E_2 *and* E_3 *respectively. Furthermore, they are independent of the parameter* $m > 0$.

Proof. From (3.7) and (3.10)–(3.12), we see immediately that $A_0(x^*, \varepsilon)$ and $A_i(x^*), i = 1, 2$, are independent of the parameter $m > 0$.

Let $y_0 = (v, \theta, d_1, d_2, d_3)$ be arbitrary in E_2. We consider the equation

$$D_{x_0}H_0(x^*)x_0 = y_0$$

for $x_0 = (u, \lambda, c_1, c_2, c_3) \in E_2$. More precisely,

$$\begin{aligned} &\text{a)}\ D_uG_0u + D_\lambda G_0\lambda + c_1\Phi_1 + c_2\Phi_2 = v, \\ &\text{b)}\ \langle\Phi_1^*, D_{uu}G_0\Phi_1 u\rangle + \langle\Phi_1^*, D_u\lambda G_0\Phi_1\rangle\lambda + \varepsilon_1 c_3 = \theta, \\ &\text{c)}\ \langle\Phi_2^*, D_{uu}G_0\Phi_1 u\rangle + \langle\Phi_2^*, D_u\lambda G_0\Phi_1\rangle\lambda + \varepsilon_2 c_3 = d_1, \\ &\text{d)}\ \langle\Phi_1^*, D_{uu}G_0\Phi_2 u\rangle + \langle\Phi_1^*, D_u\lambda G_0\Phi_2\rangle\lambda + \varepsilon_3 c_3 = d_2, \\ &\text{e)}\ \langle\Phi_2^*, D_{uu}G_0\Phi_2 u\rangle + \langle\Phi_2^*, D_u\lambda G_0\Phi_2\rangle\lambda + \varepsilon_4 c_3 = d_3. \end{aligned} \tag{3.13}$$

Taking a dual product of $\Phi_i^*, i = 1, 2$, with equation (3.13a), we derive from the statements (2.1)–(2.4)

$$c_i = \langle\Phi_i^*, v\rangle, \quad i = 1, 2 \quad \text{and} \quad u = \tilde{u} + \xi_1\Phi_1 + \xi_2\Phi_2 + \lambda v_0$$

where $\tilde{u}$ is uniquely determined in V_3, ξ_1, ξ_2, λ are arbitrary in $I\!R$ and

$$V_2 = V_3 \oplus \text{Span}\,[v_0].$$

Substituting them into (3.13b)–(3.13e), we obtain a system for $\xi_1, \xi_2, \lambda, c_3$ in $I\!R^4$:

$$M_0(\varepsilon) \cdot (\xi_1, \xi_2, \lambda, c_3)^T = (\theta, d_1, d_2, d_3)^T - \text{vec}\,(\tilde{u}),$$

where $\text{vec}(\tilde{u}) := (\langle\Phi_1^*, D_{uu}G_0\Phi_i\tilde{u}\rangle, \langle\Phi_2^*, D_{uu}G_0\Phi_i\tilde{u}\rangle, i = 1, 2)^T \in I\!R^4$. Under the condition (H3), we can solve this system uniquely. Therefore, equation (3.13) is uniquely solvable for arbitrary y_0 in E_2. In other words, $A_0(x^*, \varepsilon)$ is nonsingular on E_2.

Similiarly, it can be proven that the operators $A_i(x^*), i = 1, 2$, are regular on E_3. Moreover,

$$A_1(x^*)^{-1} = \begin{pmatrix} \tilde{P}_2[D_u G_0|V_2]^{-1}P_1 & \Phi_1 & \Phi_2/\langle f, \Phi_2\rangle \\ \langle \Phi_1^*, \cdot\rangle & 0 & 0 \\ \langle \Phi_2^*, \cdot\rangle & 0 & 0 \end{pmatrix}, \tag{3.14}$$

and

$$A_2(x^*)^{-1} = \begin{pmatrix} \tilde{P}_2[D_u G_0|V_2]^{-1}P_1 & \Phi_1 & \Phi_2 \\ \langle \Phi_1^*, \cdot\rangle & 0 & 0 \\ \langle \Phi_2^*, \cdot\rangle & 0 & 0 \end{pmatrix}, \tag{3.15}$$

where

$$P_1 := I - \langle \Phi_1^*, \cdot\rangle\Phi_1 - \langle \Phi_2^*, \cdot\rangle\Phi_2,$$

$$P_2 := I - \langle \Phi_1, \cdot\rangle\Phi_1 - \langle \Phi_2, \cdot\rangle\Phi_2$$

and

$$\tilde{P}_2 := I - \langle \Phi_1, \cdot\rangle\Phi_1 - (\langle f, \cdot\rangle/\langle f, \Phi_2\rangle)\Phi_2$$

are different projections from E onto V_2 and I is the identity operator in E.

Since mappings $H_i, i = 0(1)4$, are C^2-continuous, there is a constant $\delta_0 > 0$, such that operators $A_0(x, \varepsilon)$ and $A_i(x), i = 1, 2$, remain regular for all x in the neighborhood $B(x^*, \delta_0)$ of x^*,

$$B(x^*, \delta_0) := \{x \in E_1, \|x_i - x_i^*\| \le \delta_0,\ i = 0(1)4\}. \tag{3.16}$$

We set up a fixed point iteration to approximate x^* in E_1. Let x^0 be arbitrary in $B(x^*, \delta_0)$. For $k = 0, 1, \cdots$, we do

$$x^{k+1} := F(x^k) = (f_0(x^k), f_1(x^k), f_2(x^k), f_3(x^k), f_4(x^k)), \tag{3.17}$$

where

$$f_i(x) := x_i - D_{x_i}H_i(\tilde{x})^{-1} \cdot H_i(x), \quad i = 0(1)4, \quad \forall\ x \in E_1. \tag{3.18}$$

According to the continuity of mappings $H_i, i = 0(1)4$, and the definition (3.18), mapping F is well defined and C^1-continuous (at least) in $B(x^*, \delta_0)$. Moreover, equality (3.8) implies

$$x^* = F(x^*),$$

i.e., the element x^* is a fixed point of F in E_1. Denote

$$\Delta x^k = (\Delta x_0^k, \Delta x_1^k, \Delta x_2^k, \Delta x_3^k, \Delta x_4^k,) := x^{k+1} - x^k.$$

We calculate the components of $F(x^k)$ in the iteration (3.17) independently and call it a splitting iteration method.

Algorithm 3.1. Choose x^0 in $B(x^*, \delta_0)$ arbitrarily. For $k = 0, 1, \cdots$:

Step 1. *Solve the equations*

$$D_{x_i}H_i(\tilde{x}^k) \cdot \Delta x_i^k = -H_i(x^k), \quad i = 0(1)4. \tag{3.19}$$

Step 2. Let $x^{k+1} = x^k + \Delta x^k$ and go back to step 1 until the given criterion is safisfied.

As we have seen, five linear equations have to be solved at every k-th iteration in Algorithm 3.1, but there are only three different coefficient operators $A_0(x,\varepsilon), A_i(x), i = 1,2$, used there. Basides, the structure of Algorithm 3.1 offers a possibility to solve the five equations simultaneously and to realize it in a parallel computation proccess.

Remark 3.3. If the operator D_uG_0 is self-adjoint, so is $A_2(x^*)$. In this case, the element u_i in (3.3)–(3.4) will be directly used as u_{i+2} in (3.2), $i = 1,2$, and the last two components in $F(x)$ will be omitted.

After directly calculating the partial derivatives of F at its fixed point x^* in (3.18), we state here simply the non-vanishing terms in matrix form:

$$D_{x_i} f_0(x^*) = 0, \quad i = 0(1)4,$$

$$D_{(u,\lambda)} f_1(x^*) = A_1(x^*)^{-1} \cdot \begin{pmatrix} mDD_uG_0\Phi_1 \\ 0 \\ 0 \end{pmatrix},$$

$$D_{((u,\lambda),u_1)} f_2(x^*) = A_2(x^*)^{-1} \cdot \begin{pmatrix} m^2DD_uG_0\Phi_2 & 0 \\ 0 & m\Phi_2 \\ 0 & 0 \end{pmatrix},$$

and

$$D_{((u,\lambda),u_1,u_2)} f_{i+2}(x^*) = [A_2(x^*)^*]^{-1} \cdot \begin{pmatrix} m^3[D_uDG_0]^*\Phi_i^* & 0 & 0 \\ 0 & m^2\Phi_i & 0 \\ 0 & 0 & m\Phi_i \end{pmatrix}, \quad i = 1,2.$$

The product norms in $E_i, i = 1,2,3$, yield

$$\|D(f_0(x^*)\| = 0,$$

$$\|D(f_1(x^*)\| \le m\|A_1(x^*)^{-1}\| \cdot \|DD_uG_0\Phi_1\|$$

and

$$\|Df_2(x^*)\| \le m(m\|A_2(x^*)^{-1}\| \cdot \|DD_uG_0\Phi_2\| + 1),$$

$$\|Df_{i+2}(x^*)\| \le m\{m^2\|A_2(x^*)^{-1}\| \cdot \|(D_uDG_0)^*\Phi_i^*\| + (m+1)\|\Phi_i^*\|\}, \quad i = 1,2.$$

Lemma 3.1. *Choosing* $m \in (0,1]$, *such that*

$$m^{-1} \le 20 \cdot \max\{\|A_1(x^*)^{-1}\| \cdot \|D_uDG_0\Phi_1\|, [\|A_2(x^*)^{-1}\| \cdot \|D_uDG_0\Phi_2\| + 1], [\|A_2(x^*)^{-1}\| \cdot \|(D_uDG)^*\Phi_i^*\| + 2\|\Phi_i^*\|], \quad i = 1,2\} \tag{3.20}$$

we have

$$\|Df_i(x^*)\| \le 1/20, \quad i = 0(1)4. \tag{3.21}$$

Proof. It follows directly from the estimates above and the definitions of the product norms in $E_i, i = 1,2,3$.

Theorem 3.2. *Let mapping G be C^3-continuous and conditions* (H1)–(H3) *be satisfied. Furthermore, if the parameter $m > 0$ is chosen small enough and inequality* (3.20) *is satisfied, there is a constant δ_1 in $(0, \delta_0]$, such that for all x in $B(x^*, \delta_1)$, we have*

$$\|Df_i(x)\| \le 1/10, \quad i = 0(1)4. \tag{3.22}$$

Moreover, for every initial value x^0 in $B(x^*, \delta_1)$, the iteration (3.17) *is well defined for $k = 0, 1, \cdots$ and*

$$\|x_i^k - x_i^*\| \le \left(\frac{1}{2}\right)^k \cdot \delta_1, \quad i = 0(1)4. \tag{3.23}$$

Proof. Under the C^3-Continuity of G and conditions (H1)–(H3), we know from the definitions of H_i and f_i that mapping f_i is C^1-continuous in $B(x^*, \delta_0)$. Hence, Theorem 3.1 and Lemma 3.1 show that there is a constant δ_1 in $(0, \delta_0]$ such that (3.22) is satisfied.

On the other hand, the estimate (3.23) holds obviously for $k = 0$ and arbitrary x^0 in $B(x^*, \delta_1)$. We assume that it is also true for arbitrary k in $|\ |$. Then we get from Taylor's formula[6]

$$\|x_i^{k+1} - x_i^*\| = \|f_i(x^k) - f(x^*)\| \le \int_0^1 \|Df_i[x^* + t(x^k - x^*)\ \|dt \cdot \|x^k - x^*\|$$

$$\le (1/10) \cdot \|x^k - x^*\| \le (1/2)^{k+1} \cdot \delta_1, \quad i = 0(1)4.$$

This completes the proof.

Remark 3.4. The normalizing parameter $m > 0$ can be used to adjusce the convergence rate of (3.17).

In order to make full use of intermediate results in Algorithm 3.1, we rewrite it in the Gauss-Seidel form.

Algorithm 3.2. Let $x^0 \in B(x^*, \delta_1)$ be arbitrary. For $k = 0, 1, \cdots$:

Step 1. Compute x_0^{k+1}

$$x_0^{k+1} = f_0(x^k).$$

Step 2. Let $y_1^k := (x_0^{k+1}, x_1^k, \cdots x_4^k)$ and calculate

$$x_1^{k+1} = f_1(y_1^k).$$

Step 3. Define $y_2^k := (x_0^{k+1}, x_1^{k+1}, x_2^k, \cdots, x_4^k)$; then do

$$x_2^{k+1} = f_2(y_2^k).$$

Step 4. Denote $y_3^k := (x_0^{k+1}, \cdots, x_2^{k+1}, x_3^k, x_4^k)$ and compute

$$x_3^{k+1} = f_3(y_3^k).$$

Step 5. Let $y_4^k := (x_0^{k+1}, \cdots, x_3^{k+1}, x_4^k)$ and evaluate

$$x_4^{k+1} = f_4(y_4^k).$$

Step 6. Take $x^{k+1} := (x_0^{k+1}, x_1^{k+1}, \cdots, x_4^{k+1})$ and go back to Step 1 until the required accuracy is attained.

Note that in Algorithms 3.1 and 3.2 the second derivatives of G are involved only in the computations of x_0^{k+1}. As a matter of fact, these are some directional derivatives of G which can be approximated by difference quotients without altering the convergence of the algorithms; see for example, [5], [11] and [12].

If $E = I\!R^n$, we define a function $g : I\!R^n \times I\!R \times I\!R^n \to I\!R^n$,

$$g(u,\lambda,v) := \langle v, G(u,\lambda)\rangle.$$

Let $p^k > 0$ be a parameter sequence tending to 0 for $k \to \infty$. We approximate the terms involving the second derivatives of G in Algorithms 3.1 and 3.2 as follows:

$$\langle v, D_u G w\rangle = D_u g(u,\lambda,v)w \approx 1/p^k \cdot [g(u+p^k w,\lambda,v) - g(u,\lambda,v)] \tag{3.24}$$

and

$$\langle v, D_{uu} G_w\rangle = D_{uu} g(u,\lambda,v)w \approx (1/p^k)^2 \cdot ([g(u+p^k(w+e_i),\lambda,v) + g(u+p^k w,\lambda,v) - g(u+p^k e_i,\lambda,v) - g(u,\lambda,v)], \quad i = 1,\cdots,n, \tag{3.25}$$

where $e_i, i = 1,\cdots,n$, are unit coordinate vectors in $I\!R^n$.

In addition, the operators $A_0(x,\varepsilon)$ and $A_i(x), i = 1,2$, possess a large common part $(D_u G, u_1, u_2)$ which can be utilized in solving the five linear equations in the above algorithms.

In fact, $A_0(x,\varepsilon)$ and $A_i(x), i = 1,2$, are $(n+4)\times(n+4)$ and $(n+2)\times(n+2)$ matrices respectively. At every k-th step in (3.17), we do firstly a LR triangularization of the $n \times n$ matrix $D_u G^k$ using an improved column pivotation which permutes the present column with the last one if the pivoted element is smaller than the given tolerance. With $(n^3 - n)/3$ operations, we obtain

$$D_u G^k = P \cdot L \cdot R \cdot Q, \tag{3.26}$$

where P, Q are permutation matrices. If x^k is near x^*, the last two lines of the upper triangular matrix R become small; see also [4].

Substituting the triangularition (3.26) into $A_0(x^k,\varepsilon)$ and $A_i(x^k)$, we rewrite them in the form (3.26) and eliminate the first $n-2$ elements in the last four and two rows of the resulting R_i with $2n(n+1)$ and $n(n+1)$ operations respectively. Their trangularizations are achieved by further decompositions of 6×6 and 4×4 matrices, which are done directly and the computational cost is neglectable for large n.

Thereafter, the resolution of (3.19) is accomplished by 3 backward substitutions for $i = 0,1,2$, and 2 forward substitutions for $i = 3,4$ with $5n(n+1)/2$ operations. Hence, the main computational cost of the k-th step in the splitting iteraiton is $n^3/3 + 13n^2/2 + 37n/6$ operations which is at the same level $n^3/3$ as that for the regular solutions of (1.1).

§4. Numerical Examples

Usually, there are two ways to set up a splitting iteration proccess in the discrete spaces. One is to discretize equation (1.1) first and then make up formally the extended system in the discrete form; the other is to make up mappings $H_i, i = 0(1)4$, and then discretize them. Since all the systems are regular, there is no difference in the consistence and stability of the resulting approximate problems[1].

Example 4.1[18]. Consider a system in $I\!R^3$

$$G(u,\lambda,\omega) := \begin{pmatrix} 2x_1 - \lambda e^{x_1} + \omega(x_2 + x_3) \\ 2x_2 - \lambda e^{x_2} + \omega(x_1 + x_3) \\ 2x_3 - \lambda e^{x_3} + \omega(x_1 + x_2) \end{pmatrix} = 0, \tag{4.1}$$

where $u = (x_1, x_2, x_3)^T \in I\!R^3$ and $\lambda, \omega \in I\!R$.

This is a steady state system for the temperature of exothermic reaction of three cells with symmetric contact; ω is the contact coefficient.

Making up the extended systems (3.2)–(3.6) for (4.1), we carry out the iteration (3.17) and obtain a corank-2 biffurcation point of (4.1) for $\omega = 0.5$ in accordance with [18]:

$$u_0 = (2.5, 2.5, 2.5)^T \in I\!R^3, \quad \lambda_0 = 0.205212,$$

(see Table 1) where we have fixed the normalizing parameter $m = 1$ and the control parameter $\varepsilon^* = (1,2,4,2)^T \in I\!R^4$, and

$$f = (1,0,0)^T \in I\!R^3.$$

The same numerical results are achieved for ε around ε^*.

Example 4.2. Let us consider a two dimensional buckling state problem with initial deformation

$$\begin{cases} \Delta u + \lambda q(u) = 0, & \text{in } \Omega := [0,1] \times [0,1], \\ u = 0, & \text{on } \partial\Omega \end{cases} \tag{4.2}$$

where $q(u) := u + xyu^2$ which is evidently C^∞-continuous.

Table 1

k	u^k	Φ_1^k	Φ_1^k	λ^k
0	(1, 1, 1.)	(0, −2, 1.)	(1, 0, −1)	1
1	(2.4749, 2.4290, 2.4749)	(.0000. − .9563, 1.0872)	(.9073, −.3399, −.5926)	.552988
2	(2.5218, 2.4589, 2.5228)	(.0000, −.6769, .8286)	(.8314, −.4166, −.4201)	.236617
3	(2.5017, 2.4883, 2.4968)	(.0000, −.6999, .7224)	(.8349, −.3935, −.3868)	.205955
4	(2.4986, 2.4989, 2.4998)	(.0000, −.7077, .7066)	(.8174, −.4078, −.4078)	.205319
5	(2.4999, 2.5000, 2.4999)	(.0000, −.7071, −.7071)	(.8164, −.4082, −.4082)	.205212
6	(2.5000, 2.5000, 2.5000)	(.0000, −.7071, −.7071)	(.8164, −.4082, −.4082)	.205212

Let $E := H_0^1(\Omega)$ and $\langle \cdot\,,\,\cdot \rangle$ be the dual product on $E^* \times E$. We define a mapping $T : E^* \to E$ implicitly by

$$a(Tg, v) = \langle g, v \rangle, \quad \forall v \in E,$$

where $a(\cdot\,,\,\cdot)$ is a bilinear form on $E \times E$,

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v dx dy, \quad \forall u, v \in E.$$

Obviously, the operator T is well defined, linear and self-adjoint in E. Furthermore, it is compact from $L^2(\Omega)$ into E. On the other hand, the inequality

$$\int_\Omega q(u)^2 dx dy \le \int_\Omega (u^2 + 2|u|^3 + u^4) dx dy, \quad \forall u \in E$$

and

$$E \subset L^4(\Omega) \subset L^3(\Omega) \subset L^2(\Omega)$$

ensure that q maps E into $L^2(\Omega)$. We write (4.2) in a weak form

$$\begin{cases} \text{Find}\,(u,\lambda) \in E \times I\!R, \text{ such that} \\ G(u,\lambda) := u + \lambda T q(u) = 0. \end{cases} \tag{4.3}$$

Since $\lambda \in R$ and T is compact, $D_u G(u,\lambda)$ is a Fredholm operator on E with index 0 (cf. [17]):

$$D_u G(u,\lambda) = I + \lambda T D_u q(u).$$

Observing the spectrum of the operator T (see for example [9]), we see that

$$(u_0, \lambda_0) := (0, 5\pi^2)$$

in one of the corank-2 bifurcation points of (4.3) (resp. (4.1)). Moreover, we know analytically

$$N(D_u G_0) = N(D_u G_0^*) = \text{Span}\,[\varphi_1, \varphi_2],$$

and

$$D_\lambda G_0 = 0$$

where

$$\varphi_1 = \varphi_1^* = 2\sin \pi x, \sin 2\pi y, \quad \varphi_2 = \varphi_2^* = 2\sin 2\pi x, \sin \pi y.$$

Conditions (H1)–(H2) follow from the Riesz-Schauder theory[17]. At the same time, statements (2.1)–(2.4) imply

$$\nu_0 = 0.$$

Elementary calculation yields

$$q_{i_j} = D_{uu} G_0 \varphi_i \varphi_j = \lambda_0 T D_{uu} q_0 \varphi_i \varphi_j, \quad D_{u\lambda} G_0 = T,$$

$$q_{i0} = D_u D G_0(\nu_0, 1)\varphi^i = T\varphi_i, \quad q_{00} = D^2 G_0(\nu_0, 1) = 0.$$

According to the Ritz representation theorem, we have

$$a_{11,1} = \langle \varphi_1, q_1 \rangle = a(\lambda_0 T(2xy\varphi_1\varphi_1), \varphi_1) = \lambda_0 \iint_a 2xy\varphi_1^3 dxdy = 160/9\pi,$$

$$a_{11,2} = \langle \varphi_2, q_{11} \rangle = \lambda_0 \iint_a 2xy\varphi_1^2\varphi_2 dxdy = -\frac{85}{6}\pi,$$

$$a_{12,1} = -\frac{85}{6}\pi, \quad a_{12,2} = -8, \quad a_{22,1} = -8, \quad a_{22,2} = -\frac{200}{9},$$

$$a_{10,1} = a(\varphi_1, T\varphi_1) = 1, \quad a_{10,2} = 0, \quad a_{20,1} = 0, \quad a_{20,2} = 1.$$

Therefore

$$M_0(\varepsilon) = \begin{bmatrix} \frac{160}{9\pi} & -\frac{85}{6}\pi & 1 & \varepsilon_1 \\ -\frac{85}{6}\pi & -8 & 0 & \varepsilon_2 \\ -\frac{85}{6}\pi & -8 & 0 & \varepsilon_3 \\ -8 & -\frac{200}{9} & 1 & \varepsilon_4 \end{bmatrix}.$$

Elementary calculation shows that, if $\varepsilon_2 \neq \varepsilon_3$, then $\det(M_0(\varepsilon)) \neq 0$. This provides condition (H3). Therefore, (u_0, λ_0) is a corank-2 bifurcation point type–I of (4.3).

Using finite element methods, we discretize Ω with Courant's triangular elements and set up formally extended systems (3.2)–(3.6) for the discrete problems of (4.3). Then, we carry out Algorithm 3.2 and obtain convergent approximations for the discret values of $(u_0, \lambda_0), \varphi_1, \varphi_2$; see for example Table 2. Due to the discretizations, the bifurcation point of (4.3) moves correspondingly for the discrete problems, e.g., for $h = 1/7$, it holds that

$$(u_0^k, \lambda_0^k) = (0, 46.60305),$$

etc. where we have used the discrete L^2-norm. The starting value for Algorithm 3.2 is the discrete values of the functions

$$(u^0, \lambda^0, \Phi_1^0, \Phi_2^0) = (.5x, 44.\Phi_1 + 2x, \Phi_2 - 2y)$$

and the function f is chosen to be

$$f = x.$$

Table 2 ($h = 1/7$)

k	$\|u_h^k - 0\|$	$\|\lambda_{0h} - \lambda_h^k\|$	$\|H_{0h}^k\|$	$\|H_{1h}^k\|$	$\|H_{2h}^k\|$
0	.2384344	2.603050	4.122181	.3894015	.8078239
1	.1762093	.8623657	1.202055	.3303478	.4886288
2	.1732754	3.115203	.5123317	.0282068	.0602275
3	.0405147	.2347106	.0811900	.0195719	.0218195
4	.0011108	.0009378	.0034578	.0009297	.0014470
5	.0000051	.0000152	.0000180	.0000012	.0000013
6	.0000001	.0000004	.0000007	.0000006	.0000008

All the computations were done with single precision on a Vax/vms 11/750.

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