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CHEBYSHEV APPROXIMATION OF THE ANALYTICAL SOLUTION OF DIRICHLET PROBLEM*

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Abstract

In this paper linear programming method for minimax approximation is used to obtain an approximation to the analytical solution of a Dirichlet problem using the logarithmic potential function as an approximating function. This approach has the advantage of producing a better approximation than that using other solution of the potential equation as an approximating or basis function for a problem in $n = 2$ dimensions.

§1. Introduction

It has been stated that a problem involving the position of a point in space may be regarded as two dimensional whenever it may be made to depend on two real coordinates. An infinite straight wire of constant density λ produces such a field. The attraction is perpendicular to it and it is in accordance with the law of the inverse first power of the distance. Its magnitude in attraction units is $2\lambda/r$ where r is the distance of the attracted unit particle from the wire. The potential of such a particle is $2\lambda\log(1/r)$ where the constant which may be added to the potential has been determined so that the potential vanishes at a unit distance from the particle.

It is known that continuous distributions of matter attracting according to the law of inverse first power are interpretable as distributions of matter attracting according to Newton's law on infinite cylinders or throughout volumes bounded by infinite cylinders whose densities are the same at all points of the generators of the cylinders or of lines parallel to them. Since the total mass of such a cylinder does not vanish, its potential cannot vanish at infinity. It can only become infinite. In order to make the zero of the potential to be defined, it is made to vanish at a unit distance from the attracted particle in the case of a particle, and, in the case of a continuous distribution, by integrating the potential of a unit particle multiplied by the density over the curve or area occupied by matter. The potential is then defined by the integrals

$$u = \int_C \lambda\log(1/r)ds, \quad u = \int_A\int \sigma\log(1/r)ds \tag{1.1}$$

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for distributions on curves and over areas respectively.

These potentials which are regarded as plane material curves or plane laminas whose elements attract according to the law of the inverse first power are distinguished from the potentials of curves and laminas whose elements attract according to Newton's law by calling them the logarithmic potentials.

Logarithmic potentials are the limiting forms of Newtonian potentials[6].

A Unit Source and Principal Solution

Assuming that the discontinuity of u at Q consists of a unit source, that is, the yield q of a source Q is defined as the outward gradient of its field u, and denoting the distance from Q by ρ, we have[11]

$$q = \int_K \frac{\partial u}{\partial \rho} ds \tag{1.2}$$

where K is a circle of arbitrarily small radius about Q. Assuming that in the immediate neighbourhood of the source, u depends only on ρ, then we get by transformation

$$q = \int_{-\pi}^{\pi} \frac{du}{d\rho} \rho d\psi = 2\pi\rho \frac{du}{d\rho} \tag{1.3}$$

A unit source is therefore given by

$$1 = 2\pi\rho \frac{du}{d\rho} \quad \text{or} \quad \frac{du}{d\rho} = \frac{1}{2\pi\rho} \tag{1.4}$$

and

$$u = \frac{1}{2\pi} \log \rho + \text{constant} \quad \text{for} \quad \rho \to 0. \tag{1.5}$$

For arbitrary ρ, we have

$$u = U \log \rho + V, \quad \rho = \{(x-\xi)^2 + (y-\eta)^2\}^{1/2} \tag{1.6}$$

where U and V are analytic funcitons of (x, y) and (ξ, η) such that U becomes $1/2\pi$ when $\rho \to 0$.

A solution like (1.6) is known as the principal solution of the differential equation $M(u) = 0$ where

$$M(u) = \frac{\partial^2 Au}{\partial x^2} + 2\frac{\partial^2 Bu}{\partial x \partial y} + \frac{\partial^2 Cu}{\partial y^2} - \frac{\partial Du}{\partial x} - \frac{\partial Eu}{\partial y} + Fu. \tag{1.7}$$

In the case of the potential equaiton $\Delta u = 0$, the principal solution corresponds to the logarithmic potential

$$u = \frac{1}{2\pi} \log \rho \ \text{ for all } \rho. \tag{1.8}$$

For the three space-dimensional case, the analogue of (1.4) is

$$\frac{\partial u}{\partial \rho} = \frac{1}{4\pi\rho^2}, \quad u = \frac{1}{4\pi\rho} + \text{constant} \tag{1.9}$$

where ρ is as defined previously and $4\pi\rho^2$ is the surface area of the sphere of radius ρ enclosing Q.

The principal solution of the potential equaiotn $\Delta u = 0$, is therefore,

$$u = \begin{cases} \log(\rho) & \text{for } n = 2, \\ \rho^{2-n} & \text{for } n > 2 \end{cases} \tag{1.10}$$

up to a multiplicative constant, for an n space-dimensional problem.

It can be deduced from the foregoing that a unit source at the point Q_i gives the potential

$$u_i = \frac{1}{2\pi} \log \rho_i + \text{constant} \tag{1.11}$$

as the solution of the potential equation $\Delta u_i = 0$ for $n = 2$ and that, given a set of unit sources at the points $Q_i (i = 1, 2, 3, \cdots, M)$,

$$u = a_0 + \sum_{i=1}^{M} a_i u_i \tag{1.12}$$

is the solution of the potential equation

$$\Delta u = 0 \tag{1.13}$$

where $a_i (i = 0, 1, 2, \cdots, M)$ are constants to be determined such that the boundary conditions

$$u_B = f, \tag{1.14}$$

where f is a function of the space variables, are satisfied as closely as possible by (1.12). Such a closeness can be achieved by minimizing the maximum differences between (1.12) and (1.14). This can be done by the linear programming technique. This approximation will then be in the L_∞-norm.

The solution (1.10) has been referred to by many authors [4, 7] as the fundamental solution of the Laplacian equation (1.13) and Rudolf Mathon [7], using approximation in the L_2-norm, has used it as a basis function for an approximate analytical solution of the problem (1.13), (1.14) in two space variables.

The use of approximation in the L_∞-norm has, to the knowledge of this author, not been made in the solution of the Dirichlet's problem (1.13), (1.14).

In the next section, the L_∞-norm approximation technique is described. In section 4, the technique for choosing the positions $Q_i (i = 1, 2, 3, \cdots, M)$ of the unit sources to produce the potential function (1.12) which is the best approximation to (1.14) is presented. In section 5, results obtained for solved experimental problems are tabulated with results obtained in the L_2-norm for comparison.

§2. The Aproximation as a Linear Programming Problem

The following notations will be sued:

$$\begin{aligned} a &= \{a_i \mid i = 0, 1, 2, \cdots, M\}, \\ x &= \{x^i \mid i = 1, 2\}, \\ X &= \{x_k \mid k = 1, 2, \cdots, N\} \end{aligned} \tag{2.1}$$

where $x \in R^n, a \in R$ and a^* is the calculated value of a. Let

$$f_k = f(x_k), \quad u_{ik} = u_i(x_k). \tag{2.2}$$

Let D denote the domain of solution and δD, its boundaries. If

$$\Psi(a, x) = a_0 + \sum_{i=1}^{M} a_i u_i(x), \tag{2.3}$$

then the approximation problem becomes

$$\max_X |\Psi(a^*, X) - f(X)| \le \min_a \left\{ \max_X |\Psi(a, X) - f(X)| \right\}. \tag{2.4}$$

Using the notation

$$\max_X |\Psi(a, X) - f(X)| = w, \tag{2.5}$$

the equivalent linear programming problem is

$$\text{minimize } w, \tag{2.6}$$

subject to the constraints

$$\begin{aligned} \Psi(a^*, X) + w &\ge f(X), \\ \Psi(a^*, X) - w &\le f(X). \end{aligned} \tag{2.7}$$

Usually, for a problem of the type we are trying to solve here, $N > M$. It is then easier to solve the dual problem

$$\text{maximize } (f(X))^T \cdot (s - t) \tag{2.8}$$

subject to the constraints

$$\begin{aligned} &(s + t)^T \cdot ((s + t)/|s + t|) \le 1, \\ &(u_i(X))^T \cdot (s - t) \le 0, \quad i = 0, 1, 2, \cdots, M, \\ &u_0(X) = 1 \end{aligned} \tag{2.9}$$

where

$$\begin{cases} s = \{s_i |\ i = 1, 2, \cdots, N\}, \quad t = \{t_i |\ i = 1, 2, \cdots, N\}, \quad s_i \in R \text{ and } t_i \in R, \\ s_i > 0 \text{ and } t_i \equiv 0 \text{ when } f_i > 0, \quad s_i \equiv 0 \text{ and } t_i > 0 \text{ when } f_i < 0, \end{cases} \tag{2.10}$$

T denotes transposition and $\cdot$ denotes the dot product. When slack variables are used to reduce (2.9) to a system of linear algebraic equations, there are more unknowns than equations and the Simplex Method can then be used to solve the resulting system of equations. The initial tableau for the simplex algorithm can be condensed to the form

Table 2.1. Initial Tableau of the Simplex Method

$c \rightarrow$	0	0	0	$\cdots$	0	f_1	f_2	f_3	$\cdots$	f_N
Variables in Basis	w	a_0	a_1	$\cdots$	a_M	s_1	s_2	s_3	$\cdots$	s_N
w	1	0	0	$\cdots$	0	1	1	1	$\cdots$	1
a_0	0	1	0	$\cdots$	0	1	1	1	$\cdots$	1
a_1	0	0	1	$\cdots$	0	u_{11}	u_{12}	u_{13}	$\cdots$	u_{1N}
$\vdots$										
$\vdots$										
a_M	0	0	0	$\cdots$	1	u_{M1}	u_{M2}	u_{M3}	$\cdots$	u_{MN}
$z - c \rightarrow$	0	0	0	$\cdots$	0	$-f_1$	$-f_2$	$-f_3$	$\cdots$	$-f_N$

From the tableau it can be seen that the column under s_i or t_i has $M+2$ rows. It is unnecessary to store the columns under $(t_1, t_2, t_3, \cdots, t_N)$ since each column is easily obtainable using relation

$$s_i + t_i = 2w \tag{2.11}$$

where each of s_i, t_i and w now represents the $(M+2)$ rows under it in the tableau at any stage of the solution process.

§3. Location of the Unit Sources

Let the ith unit source be located at the point P_i where $P_i \notin DU\delta D$ and let its angular distance from the base line, $\theta = 0$ be θ_i. Let P_B be an arbitrary point on the boundary δD whose angular distance from the horizontal is θ_B. Denote the distance between the point $P_B \in \delta D$ and $P_i \notin DU\delta D$ by r_i. Let $0 \in DU\delta D, |0P_B| = d$, and $|0P_i| = R_i$ such that

$$r_i = (R_i^2 + d^2 - 2R_i d\cos(\theta_B - \theta_i))^{1/2}, \quad 0 \le \theta_B, \theta_i \le 2\pi. \tag{3.1}$$

By Fourier series expansion of the general power of the distance between two points (see [1]), we have

$$\log r_i = \log R_i - \sum_{k=1}^{\infty} \frac{1}{k}(d/R_i)^k \cos k(\theta_B - \theta_i), \tag{3.2}$$

From (3.2), we must have

$$r_i = R_i \exp\Big\{ -\sum_{k=1}^{\infty} \frac{1}{k}(d/R_i)^k \cos k\chi_i \Big\} \tag{3.3}$$

where

$$\chi_i = \theta_i - \theta_B, \quad -\pi \le \chi_i \le \pi. \tag{3.4}$$

The nearer R_i is to d in value, the smaller is the value of the negative exponential funciton and the farther, the bigger. Therefore, even when $\theta_j = \theta_i$, if $R_j > R_i$, then we must have $r_j > r_i$ for a given i and j.

As both R_i and θ_i vary within the limits $0 < R_i < \infty$ and $0 < \theta_i < 2\pi$ respectively, it is required that r_i should decrease towards d for the best approximation. In order to facilitate this decrease the method of steepest descent is proposed for the search for the optimal values of the set of R_i and θ_i. This method is described subsequently.

After each iteration of the search method, the set of computed values of R_i and θ_i are then used to compute a better set of values of the coefficients a_i of the approximation (1.12).

This solution process is continued until the error w is minimized.

When $n = 3, \chi_i$ in (3.3) is given by the equation

$$\cos\chi_i = \cos\theta_i\cos\theta_B + \sin\theta_i\sin\theta_B\cos(\psi_i - \psi_B) \tag{3.5}$$

where $0 < \psi_i, \psi_B < 2\pi$. The range of values of R_i and θ_i are as defined previously.

Choice of Origin of Approximation

The origin of approximation from which the $R_i(i = 1, 2, \cdots, M)$ are to be measured is fixed in the domain of solution. The positions of the unit sources are variable while the positions of the sinks (the interpolation points on the boundaries of the domain of solution) are fixed on δD. Any point in the domain of solution D is a variable point also. Hence, the origin of approximation should be at an interpolation point (a fixed point on the boundaries). This is consistent with the definition of r_i as a function of the coordinates of the source (the approximation point) and the sink (the interpolation point). See equation (1.6).

Since each R_i is measured from the origin of approximation, it can change only relative to the origin. The change is independent of the interpolation points.

Therefore, in calculating the changes the error function defined by

$$e = u - f \tag{3.6}$$

where u and f are as defined in (1.12) and (1.14), $d = 0$ since, at the origin of approximation, $d \equiv 0$ for all values of θ_i. The change in R_i is also independent of its angular position in the domain of solution. The angular position of the approximation point can, therefore, be arbitrarily fixed in the domain of solution. This is the mode of approximation followed in this work.

The Search Method

The error funciton is given by (3.6). Consider a column vector function $F(Q)$.

Let the directional derivative of the function $F(Q)$ be denoted by $DF(Q^{(0)}, v)$ where v is a directional vector. Then

$$DF(Q^{(0)}, v) = \lim_{h\to 0}\frac{F(Q^{(0)} + hv) - F(Q^{(0)})}{h} = v^T\nabla F(Q^{(0)}). \tag{3.7}$$

Consider all the directional vectors $v \in R^n$ such that, for a given $Q^{(0)} \in R^n$, we have

$$v^T \nabla F(Q^{(0)}) < 0. \tag{3.8}$$

Then for sufficiently small positive h, this implies that

$$F(Q^{(0)} + hv) > F(Q^{(0)}). \tag{3.9}$$

That is, if the minimum of $F(Q)$ on R^n is required and for $Q^{(0)} \in R^n$, the gradient of $F(Q)$ does not vanish, then a sufficiently small move in a direction v that satisfies (3.8) will result in a function decrease. Among all directions v having some bounded length, say $\|v\| < 1$, that particular direction that ptoduces the steepest descent in the value of $F(Q)$ for $Q^{(0)}$ for which $\nabla F(Q^{(0)}) \neq 0$ is the optimal solution of the nonlinear programming problem

$$\min_v v^T \nabla F(Q^{(0)}) = \sum_{j=1}^{M} \frac{\partial F(Q^{(0)})}{\partial Q_j} v_j \tag{3.10}$$

subject to

$$\|v\| = \{\Sigma(v_j)^2\}^{1/2} < 1 \tag{3.11}$$

is (see [2, p.291])

$$v^* = -\nabla F(Q^{(0)}) / \|\nabla F(Q^{(0)})\|. \tag{3.12}$$

The steepest descent in the function value is thus in the direction of the negative gradient. The method of steepest descent can therefore be described as follows: Given $Q^{(0)} \in R^n$, compute, for $k = 0, 1, \cdots$, the sequence of values

$$Q^{(k+1)} = Q^{(k)} - h_k \nabla F(Q^{(k)}) \tag{3.13}$$

where $h_k \geq 0$ satisfies the condition

$$F(Q^{(k)} - h_k^* \nabla F(Q^{(k)})) = \min_{h \geq 0} F(Q^{(k)} - h_k \nabla F(Q^{(k)})). \tag{3.14}$$

Consider the function

$$F^k(h) = F(Q^{(k)} - h \nabla F(Q^{(k)})). \tag{3.15}$$

Then we must have

$$F^k(h_k) = F(Q^{(k)} - h_k \nabla F(Q^{(k)})) = F(Q^{(k+1)}) \tag{3.16}$$

and we can write $F^{(k+1)} = F(Q^{(k+1)})$. Let $g^k(h)$ and $H^k(h)$ denote the gradient vector and the Hessian matrix of $F^k(h)$ respectively. Then

$$g^k(h) = -g_F^k(h) \nabla F(Q^{(k)}) \tag{3.17}$$

and

$$H^k(h) = \nabla F(Q^{(k)})^T H_F^k(h) \nabla F(Q^{(k)}) \tag{3.18}$$

where T denotes transposition. The value of h which minimizes the value of the funciton (3.15) is the solution of the equation

$$g^k(h) = 0. \tag{3.19}$$

This can be obtained by the Newton's Iteration .

$$h = h_k - (H^k(h_k))^{-1}(g^k(h_k))^T \tag{3.20}$$

starting with $h = h_k$. The values $Q^{(k)}$ of the vector Q will become unchanging when $h = 0$ is reached. Then, (3.20) will become

$$h_k = (H^k(h_k))^{-1}(g^k(h_k))^T. \tag{3.21}$$

Using (3.21), the relation (3.13) becomes

$$Q^{(k+1)} = Q^{(k)} + (\nabla F(Q^{(k)})^T H_F(h_k)\nabla F(Q^{(k)}))^{-1}$$
$$\times(\nabla F(Q^{(k)})^T g_F(h_k)\nabla F(Q^{(k)}))^T. \tag{3.22}$$

This is the recurrence relation for computing the vector $Q \notin DU\delta D$ of the locations of the unit sources to produce the values of the potential which are specified on the boundaries δD of the solution domain of the problem. Here

$$e = F(Q) \tag{3.23}$$

where e is as defined in (3.6).

In order to facilitate the comparison of our results with others, the experimental problem is also solved in the L_2 norm using the search method proposed here and an indirect method for computing the best values of the vector a.

In the Euclidean (L_2) approximation, the function $F(Q)$ is denoted by $F(Q)_E$ where

$$F(Q)_E = \sum_{x\in\delta D} F(Q)\cdot F(Q) \tag{3.24}$$

where ($\cdot$) denotes a dot product. Then,

$$g^k_E(h) = -2\sum_{x\in\Delta D} F^k(h)\cdot g^k_F(h)\nabla F(Q^{(k)}). \tag{3.25}$$

Since $F^k(h) \neq 0$, (3.25) vanishes identically if

$$g^k(h) = -g^k_F(h)\nabla F(Q^{(k)}) \tag{3.26}$$

vanishes identically. Then, this will be the equation (3.17) and the same amount of computation is equired in both methods for obtaining the final value of the vector of locations $Q \notin DU\delta D$ of the unit sources required to produce the best approximation to the solution of the problem.

§4. Experimental Problems

Problem 1. The stream function ψ for steady irrotational two-dimensional flow, parallel to the xoy plane, of incompressible, inviscid fluid satisfies Laplace's equation at all points inside the field of flow. Calculate a solution for flow through the channel shown below given that ABCD is the streamline $\psi = 0$, EFG is the streamline $\psi = 1$, and ψ varies linearly across AE and GD. $AB = AE = EF = FG = GD = DC = 1$.

An approximation to the continuous solution of this problem was computed in the domain shown in Fig. 4.1 in the L_p norm for $p = 2$ and $p = \infty$ using the logarithmic potential function solution of the Laplacian equation as approximating function and the results obtained are shown in Tables 4.1 to 4.3 for comparison.

Computations were done on IBM 370/145 mainframe digital computer.

Table 4.4 shows that the values computed by the use of this method is dependent on the number (N) of boundary points and the number (m) of approximation points.

Computations were done on the Apple II+ microcomputer. The results obtained for Problem 2 below show that

(i) this method produces the best results for solution domains where none of the interior angles is an acute angle,

(ii) the origin of approximation should be situated on the boundary at a corner of the domain which is on a line of symmetry of the domain since the assumption is that the sources are equidistant from the chosen centre and outside the domain,

(iii) where the domain of solution has acute-angled corners, other types of approximating (basis) functions (e.g. the harmonic functions) would produce better results.

Problem 2. Boundary values: $f(s,t) = \sinh(-\pi(t+1)/2)\sin(\pi(s+1)/2)/\sinh(-\pi)$. Boundary points: 42, spaced 0.2 on AB&BC with additional points $(0.05,-1)$, $(0.1,-1)$, $(0.9,-1)$, $(1,-0.9)$, and $(1,0.9)$. On AD and DC, uniform spacing with respect to s-axis at 0.1 and 0.2 respectively, together with the points $(0.05,-0.775)$, $(0.75,-0.625)$, $(0.8,-0.52)$, and $(0.99,-0.92)$.

True solution: $u = f$.

Domain of solution: See Figure 4.2.

Computations were done on IBM 370/145 mainframe digital Computer.

Table 4.5 shows the computed values for a centre of approximation at $(0.8,-0.6)$, a corner boundary point of the domain of solution. The starting line for all angular measurements is BD. Four of the sources have fixed directions BD, DB for $m \geq 2$ and DA, DC for $m \geq 4$.

Table 4.6 shows the results obtained with the approximation centre at the point $(29/34,-3/17) \approx (0.85,-0.18)$ and an initial radius of 2 units. The starting line for all angular measurements is BD and the sources are in the directions BD and DB for $m \geq 2$, DA and DC for $m \geq 4$.

Computed values of the LP maximum error using Harmonic Functions as approximating functions {3,5} are shown in Table 4.7 for comparison with values in Tables

4.5 and 4.6.

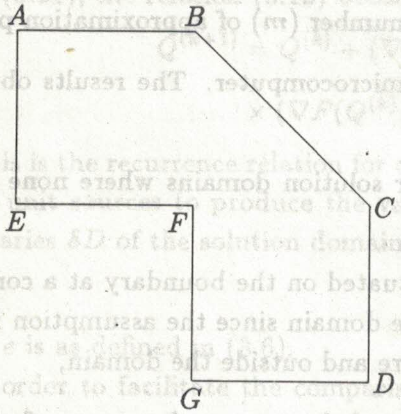

Fig. 4.1. Domain of solution for Problem 1.

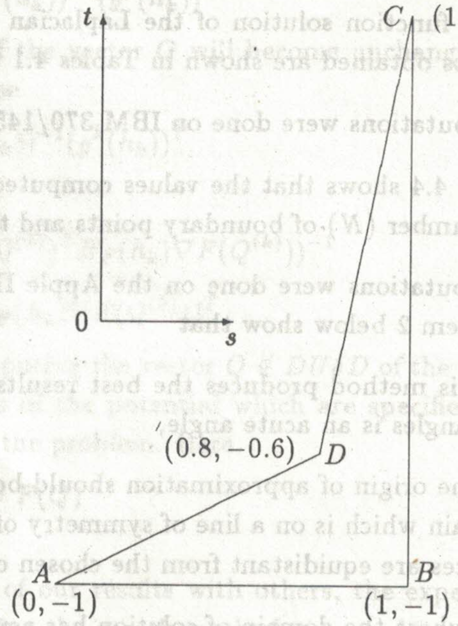

Fig. 4.2. Domain of solution for Problem 2.

Table 4.1.

Number of Unit Sources	L_∞ maximum error at nodal points	L_2 maximum error at nodal points	L_∞ maximum error at non-nodal points	L_2 max. error at non-nodal points
2	0.2359	0.6010	0.2357	0.5884
4	0.1939	0.4494	0.1936	0.4205
6	0.1787	0.4335	0.1787	0.3938
8	0.1779	0.4103	0.1772	0.3686

Table 4.2

No. of Unit Sources	Components of the distance vector (R) from the origin F							
	1	2	3	4	5	6	7	8
2	1.9364	1.4633						
4	1.6678	1.8164	0.9826	1.8163				
6	1.6318	1.5480	1.8400	0.9030	1.8400	1.5480		
8	1.7188	1.5640	1.6257	1.9192	0.9498	1.9192	1.6257	1.5640

Table 4.3

No. of Unit Sources	Norm	Components of coefficient of approximation vector a 0	1	2	3	4	5	6	7	8
2	L_2	0.2539	−0.3209	0.1759						
	L_∞	1.4408	0.1514	1.5146						
4	L_2	0.1717	−0.3048	−0.1800	0.4632	−0.1800				
	L_∞	0.8763	0.0495	0.0541	1.1687	0.0541				
6	L_2	0.1718	−0.2293	−0.2687	0.0048	0.5248	0.0048	−0.2687		
	L_∞	2.0358	0.5543	0.5366	0.5174	1.5387	0.5174	0.5367		
8	L_2	0.1345	−0.2100	−0.2329	−0.2354	0.0936	0.5140	0.0937	−0.2354	−0.2329
	L_∞	1.5998	0.1699	0.4042	0.3023	0.0849	1.5631	0.0849	0.3023	0.4042

Table 4.4

$m \backslash N \rightarrow$	28	42	56	70	112
2	0.2335	0.2334	0.2333	0.2333	0.9948
4	0.2334	0.2344	0.2336	0.2335	1.8164
6	0.2197	0.2220	0.2210	0.2218	1.9421
8	0.2115	0.2096	0.2115	0.2109	1.5002
10	0.1962	0.1973	0.1978	0.1982	0.6465

Table 4.5

No. of Sources	Values of computed LP max. error at the specified radius (R) from the centre of approximation 1.7	4.9	5.1
2	0.0198	0.0227	0.0227
4	0.0146	0.0085	0.0085
6	0.0136	0.0081	0.0080
8	0.0051	0.0020	0.0020
10	0.0057	0.0012	0.0012
12	0.0028	0.0002	0.0002
14	0.0025	0.0002	0.0002
16	0.0004	0.4E−04	0.1E−04

Table 4.6

No. of Sources	Computed Lp. max. error	Distance from the centre of approximation of the unit sources							
		1	2	3	4	5	6	7	8
2	0.0124	4.1658	2.2757						
4	0.0111	4.1703	7.2402	4.2829	4.8337				
6	0.0106	2.5914	2.6975	1.7778	2.5507	3.4240	2.4982		
8	0.0101	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000	2.0000

Table 4.7

No. of approximating functions	computed Lp max. error
2	0.0230
4	0.0088
6	0.0064
8	0.0012
10	2.4E−04
12	3.9E−05
14	8.2E−06
16	7.9E−07
18	7.9E−08

§5. Conclusion

The results of the numerical example sloved in this paper and tabulated in Tables 4.1 to 4.3 have shown that the use of the logarithmic potential as a basis function for an approximate analytical solution of Dirichlet's problem produces better results in the Chebyshev norm than in the Euclidean norm for the same number of unit sources and about the same amount of computational work. Best results are obtained when the approximation centre is chosen on the boundary on a line of symmetry of the domain of solution.

Although the analyses and the results of the computations showed that the results become unreliable in the Chebyshev approximation for a fairly large number (much greater than shown on the Tables) of approximation points, the results in the Euclidean approximation shows a more normal form at that stage. It is apparent from the results (Table 4.1) that the tatio of the number of unit sources must be greater than $4(L_2)$: $1(L_\infty)$ for the errors in the approximations to be equal in magnitude.

This method produces better results when used on regular domains (domains without acute interior angles).

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