Journal of Computational Mathematics, Vol.14, No.1, 1996, 40–53.

TRACE AVERAGING DOMAIN DECOMPOSITION METHOD WITH NONCONFORMING FINITE ELEMENTS*

J. Gu X. Hu
(*Department of Applied Mathematics, Tsinghua University, Beijing, China*)

Abstract

We consider, in this paper, the trace averaging domain decomposition method for the second order self-adjoint elliptic problems discretized by a class of nonconforming finite elements, which is only continuous at the nodes of the quasi-uniform mesh. We show its geometric convergence and present the dependence of the convergence factor on the relaxation factor, the subdomain diameter H and the mesh parameter h. In essence, this method is equivalent to the simple iterative method for the preconditioned capacitance equation. The preconditioner implied in this iteration is easily invertible and can be applied to preconditioning the capacitance matrix with the condition number no more than $O\big((1+\ln \frac{H}{h}) max(1+H^{-2}, 1+\ln \frac{H}{h})\big)$.

1. Introduction

Domain decomposition refers to numerical methods for obtaining solutions of scientific and engineering problems by combining solutions to problems posed on physical subdomains, or, more generally, by combining solutions to appropriately constructed subproblems. It has been a subject of intense interest recently because of its suitability for implementation on high performance computer architectures. Some papers are listed in the references herein, which indicate that much progress has been made in the study of nonoverlap domain decomposition methods , also known as the substructuring methods. It is rather complicated in the case of multi-subdomains with the internal cross points. A cross point is defined to be the common boundary point of more than two subdomains. With the techniques of the separation of the internal cross points from other mesh nodes, Bramble et al.[2,3,4,5], Widlund[18], constructed different preconditioners for the algebraic system of equations which arise from the following self-adjoint elliptic problems via conforming finite element methods:

$$u \in H_0^1(\Omega): \ a(u,v) = (f,v), \quad \forall\, v \in H_0^1(\Omega), \tag{1.1}$$

* Received June 4, 1994.

where

$$a(u,v)=\int_\Omega\left[\sum_{i,j=1}^{2}a_{ij}(x)\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+a_0(x)uv\right],\quad (f,v)=\int_\Omega fv, \tag{1.2}$$

$\Omega\subseteq\Re^2$ is a bounded polygonal open domain, $f\in H^{-1}(\Omega)$, $a_0(x)\geq 0$, $a_{ij}(x)$, $i,j=1,2$, $a_0(x)$ are piecewise smooth and bounded functions in Ω, (a_{ij}) is a symmetric, uniformly positive definite matrix in Ω. All their preconditioners can be inversed easily in parallel and precondition the stiff matrix with the condition number no more than $O((1+\ln\frac{H}{h})^2)$, where H,h are the subdomain diameter and the fine mesh parameter, respectively. Bourgat et al.[1] introduced an iterative substructuring method with the trace averaging operator to deal with the internal cross points, and illustrated its efficiency in the conforming discrete case with plenty of numerical experiments. Later, Chu[9] gave the theoretical proof of its convergence.

The present paper is concerned with the construction of efficient iterative schemes for solving (1.1) discretized by a class of nonconforming finite elements, which is only continuous at the mesh nodes. Let $\Omega_h=\{e\}$ be a quasi-uniform mesh of Ω, where h is the mesh parameter and e, a triangle or a quadrilateral, represents typical element in Ω_h. Let the nonconforming finite element space

$$\begin{aligned}S^h(\Omega)=&\{v_h:\ \ v_h=\theta_h+w_h,\ \theta_h\in T^h(\Omega),\ w_h(x)=0,\ \forall\text{ node }x\in\overline{\Omega},\\&\quad w_h|_e\text{ is a finite order polynomial, }\forall\ e\in\Omega_h\},\end{aligned}$$

where

$$T^h(\Omega)=\{\theta_h\in C(\Omega):\ \theta_h|_e\text{ is linear (bilinear) if } e\text{ is a triangle (quadrilateral), }\forall\, e\in\Omega_h\}.$$

Here, a node $x\in\overline{\Omega}$ is defined to be the vertex of some $e\in\Omega_h$. In practice, there are many nonconforming finite elements which are only continuous at the mesh nodes, e.g. Wilson elements[19], triangle membrane elements[8], etc. Denote

$$\begin{aligned}&S_0^h(\Omega)=\{v_h\in S^h(\Omega):\ v_h(x)=0,\ \forall\text{ node }x\in\partial\Omega\},\\&A(u,v)=\sum_{e\subset\Omega}\int_e\left[\sum_{i,j=1}^{2}a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j}+a_0uv\right].\end{aligned}$$

Then, the nonconforming finite element discrete problem for (1.1) is

$$u_h\in S_0^h(\Omega):\ A(u_h,v_h)=(f,v_h),\ \forall\ v_h\in S_0^h(\Omega). \tag{1.3}$$

In the two-subdomain nonoverlap cases, Gu[12] proposed and analysed a series of algorithms for solving (1.3) via the extension theorem for nonconforming elements[13]. In the multi–subdomain nonoverlap cases, many preconditioners for (1.3) have been constructed successfully, based on the conforming interpolation operator and the essential estimates[12,9]. All are as efficient as their counterparts in the conforming discrete cases. Furthermore, we note that a hierarchical basis multilevel method with

Crouzeix–Raviart elements[11] introduced by Oswald[17] is better than the ones for conforming elements[6,20,21]. So, we think that some further results may be obtained in the nonconforming discrete cases.

The goal of this paper is to extend the trace averaging domain decomposition algorithm[1] to the solving of (1.3). Let $\{\Omega_i\}_{i=1}^N$ be a nonoverlap subdomain division of Ω. In each iteration of our algorithm, the Dirichlet subproblems set on subdomains Ω_i are solved simultaneously, which is then followed by the parallel solving of the Neumann subproblems posed on Ω_i, where the discrete trace averaging operators r_0^i play a key role in the interchange of information between subdomains. r_0^i is a matrix (cf. Sect. 2), which is only related to the nodes on $\partial\Omega_i \backslash \partial\Omega$ and to the numbers of subdomains by which a node is shared. Therefore, the existence of the internal cross points doesn't affect the efficiency of our algorithm. It is proved to be geometrically convergent and is essentially equivalent to the simple iterative method applied to the preconditioned capacitance equation. Furthermore, the preconditioner implied in the iteration can be inversed easily in parallel and the condition number of the preconditioned capacitance matrix is bounded by $O((1+\ln\frac{H}{h})max(1+H^{-2}, 1+\ln\frac{H}{h}))$.

The remainder of the paper is organized as follows. The domain decomposition algorithm for (1.3) is described in Sect. 2. Its convergence analysis is established in Sect. 3, while its matrix analysis is presented in Sect. 4.

We remark that some domain decomposition algorithms have been developed for the solving of (1.1) which is discretized by another class of nonconforming elements with their continuity only at the edge midpoints of the elements of the mesh, e.g. Crouzeix–Raviart elements, which can be referred to [12,14,15].

2. Domain Decomposition Method

In what follows, we assume that there exists an unique solution u_h of (1.3) which converges to the solution u of (1.1) (e.g. [8,19]) and there exists a positive constant α s.t. for $a_0(x)$ of (1.2)

$$a_0(x) \geq \alpha, \quad \forall\, x \in \Omega.$$

Let $\{\Omega_i\}_{i=1}^N$ be a subdivision of Ω, which satisfies:

A$_1$. Ω_i is an open triangle or an open quadrilateral, $\bigcup\limits_{i=1}^{N} \overline{\Omega}_i = \overline{\Omega}$;

A$_2$. $\{\Omega_i\}_{i=1}^N \triangleq \Omega_H$, which is supposed to be a quasi-uniform mesh of Ω with H as its mesh parameter. Generally, $H \gg h$, therefore, Ω_H, Ω_h are called the coarse mesh and the fine mesh, respectively ;

A$_3$. Ω_H is compatible with the fine mesh Ω_h, i.e. $e \cap \Omega_i$ is either empty or e for all $e \in \Omega_h$ and $\Omega_i \in \Omega_H$.

Denote $\Gamma = \bigcup\limits_{i=1}^{N} \partial\Omega_i \backslash \partial\Omega$. Let $\{\nu_j\}$ be the set of vertices of $\{\Omega_i\}_{i=1}^N$, i.e. the set of coarse mesh nodes. The open edge in Γ with endpoints ν_i, ν_j is denoted to be

Γ_{ij}. Let $\{\chi_k\}_{k=1}^m$ be the set of the nodes on Γ (ordered in some way). Denote $I_i = \{k : \chi_k \in \partial\Omega_i \backslash \partial\Omega, \ k = 1, 2, \cdots, m\}$. Let m_i be the number of elements of I_i. For $i = 1, 2, \cdots, N$, the matrix $r_0^i \in \Re^{m\times m}$ satisfies

$$(r_0^i)_{lj} = \begin{cases} \frac{1}{k}, & l = j \in I_i, \ \chi_j \text{ is a common boundary point of } \\ & \qquad\qquad k \text{ subdomains} \\ & \\ 0, & l \notin I_i \ \text{ or } j \notin I_i \end{cases}$$

where $(r_0^i)_{lj}$ represents the (l, j)th element of r_0^i, $l, j = 1, 2, \cdots, m$. Obviously, we have

$$\sum_{i=1}^{N} r_0^i = I, \tag{2.1}$$

r_0^i is called the discrete trace averaging operator. The construction of r_0^i is an originality of the present approach. We define the discrete trace operator $r_0^h : \ S_0^h(\Omega) \longrightarrow \Re^m$ as follows:

$$\forall \ v \in S_0^h(\Omega), \qquad r_0^h v \in \Re^m, \qquad (r_0^h v)(j) = v(\chi_j), \qquad j = 1, 2, \cdots, m.$$

And, we introduce the following notations:

$$\begin{aligned} S^h(\Omega_i) &= \{v \in S_0^h(\Omega) : \ \ v(x) = 0, \ \forall \text{ interpolation point } x \in \Omega \backslash \overline{\Omega}_i\}, \\ S_0^h(\Omega_i) &= \{v \in S_0^h(\Omega) : \ \ v(x) = 0, \ \forall \text{ interpolation point } x \in \Omega \backslash \Omega_i\}, \\ A_i(u, v) &= \sum_{e\subset\Omega_i} \int_e \left[\sum_{k,j=1}^{2} a_{kj} \frac{\partial u}{\partial x_k} \frac{\partial v}{\partial x_j} + a_0 uv \right], \\ (f, v)_i &= \int_{\Omega_i} fv, \qquad i = 1, 2, \cdots, N. \end{aligned}$$

Here, an interpolation point x is related to the definition of $S^h(\Omega)$.

Now we are in a position to describe the trace averaging domain decomposition method for the solving of the nonconforming discrete problem (1.3).

Algorithm 2.1.

Step 1 Choose arbitrary $\lambda^0 = (\lambda_1^0, \lambda_2^0, \cdots, \lambda_m^0)^T \in \Re^m$. Set n:=0 .

Step 2 For $i = 1, 2, \cdots, N$, solve in parallel

$$\begin{cases} u_i^n \in S^h(\Omega_i) \\ \\ A_i(u_i^n, \theta) = (f, \theta)_i, & \forall \ \theta \in S_0^h(\Omega_i) \\ \\ u_i^n(\chi_j) = \lambda_j^n, & \forall \text{ node } \chi_j \in \partial\Omega_i \backslash \partial\Omega \\ \\ u_i^n(x) = 0, & \forall \text{ node } x \in \partial\Omega_i \cap \partial\Omega \end{cases}$$

Step 3 For $i = 1, 2, \cdots, N$, solve in parallel

$$\begin{cases} \psi_i^n \in S^h(\Omega_i) \\ A_i(\psi_i^n, \theta) = \sum\limits_{j=1}^{N} \Big[A_j(u_j^n, Tr_j^{-1}(r_0^i r_0^h \theta)) - (f, Tr_j^{-1}(r_0^i r_0^h \theta))_j \Big], \quad \forall\, \theta \in S^h(\Omega_i) \end{cases}$$

Step 4 Choose relaxation factor $0 < \rho < 1$, compute

$$\lambda^{n+1} = \lambda^n - \rho \sum_{j=1}^{N} r_0^j r_0^h \psi_j^n .$$

Set $n := n + 1$. Then, go back to Step 2 until some reasonable stopping criterion is satisfied.

Remark 2.1. In Algorithm 2.1, for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_m)^T$, $Tr_j^{-1}(\lambda)$ means any element of the set

$$T_j(\lambda) = \{w \in S^h(\Omega_j) : \quad w(\chi_k) = \lambda_k, \quad \forall \text{ node } \chi_k \in \partial\Omega_j \backslash \partial\Omega\}.$$

Its arbitrariness is implied by Step 2 and the fact that $w_1 - w_2 \in S_0^h(\Omega_j)$, $\forall\, w_1, w_2 \in T_j(\lambda)$.

The detailed analysis of Algorithm 2.1 will be given in Section 3 and Section 4. From now on, c and C (with or without subscript) will denote generic positive constants which are independent of h, H, Ω_i and the functions appearing with them.

3. Convergence Analysis

For $i = 1, 2, \cdots, N$, denote: $|v|_{1,\Omega_i,h} = (\sum\limits_{e \subset \Omega_i} |v|^2_{H^1(e)})^{\frac{1}{2}}$. Define the interplation operator $I_h : S^h(\Omega) \longrightarrow C(\overline{\Omega})$ as follows:

$$\forall\, v \in S^h(\Omega), \quad I_h v \in T^h(\Omega), \quad (I_h v)(x) = v(x), \qquad \forall \text{ node } x \in \overline{\Omega}.$$

Lemma 3.1. *If Ω_h is quasi-uniform, then for $i = 1, 2, \cdots, N$*

$$||v - I_h v||_{L^\infty(\Omega_i)} \le c|v|_{1,\Omega_i,h}, \quad \forall\, v \in S^h(\Omega_i), \tag{3.1}$$

$$||v - I_h v||_{L^2(\Omega_i)} \le c||v||_{L^2(\Omega_i)}, \quad \forall\, v \in S^h(\Omega_i), \tag{3.2}$$

$$|v - I_h v|_{1,\Omega_i,h} \le c|v|_{1,\Omega_i,h}, \quad \forall\, v \in S^h(\Omega_i). \tag{3.3}$$

Proof. It follows from the interpolation theorem and an "inverse property"[10] implied by the quasi-uniformness of the mesh Ω_h that

$$\begin{aligned} ||v - I_h v||_{L^\infty(\Omega_i)} &\le \max_{e \subset \Omega_i} ||v - I_h v||_{L^\infty(e)} = ||v - I_h v||_{L^\infty(e_0)} \\ &\le ch|v|_{H^2(e_0)} \le c|v|_{H^1(e_0)} \le c|v|_{1,\Omega_i,h}, \end{aligned}$$

where e_0 is some element in Ω_i. So (3.1) holds.

(3.2) (3.3) can be established in the same manner.

Lemma 3.2. *If Ω_h is quasi-uniform, then for $i = 1, 2, \cdots, N$*

$$||v||^2_{L^\infty(\Omega_i)} \le c\Big[H^{-2}||v||^2_{L^2(\Omega_i)} + (1+\ln\frac{H}{h})|v|^2_{1,\Omega_i,h}\Big], \quad \forall\ v \in S^h(\Omega_i). \tag{3.4}$$

Proof. Lemma 3.3 in [2] gives

$$||I_h v||^2_{L^\infty(\Omega_i)} \le c\Big[H^{-2}||I_h v||^2_{L^2(\Omega_i)} + \ln\frac{H}{h}|I_h v|^2_{H^1(\Omega_i)}\Big].$$

It follows from (3.2) and (3.3) that

$$||I_h v||_{L^2(\Omega_i)} \le ||v||_{L^2(\Omega_i)} + ||v - I_h v||_{L^2(\Omega_i)} \le c||v||_{L^2(\Omega_i)},$$
$$|I_h v|_{H^1(\Omega_i)} \le |v|_{1,\Omega_i,h} + |v - I_h v|_{1,\Omega_i,h} \le c|v|_{1,\Omega_i,h}.$$

With the triangle inequality $||v||_{L^\infty(\Omega_i)} \le ||v - I_h v||_{L^\infty(\Omega_i)} + ||I_h v||_{L^\infty(\Omega_i)}$, (3.1) and the above three inequalities, we eventually obtain (3.4).

Lemma 3.3.[12] *Suppose $\hat\Gamma_h$ is a quasi-uniform mesh of the interval $\hat\Gamma = [0, H]$. Let $w(x)$ be a piecewise linear continuous function defined on $\hat\Gamma_h$ with $w(0) = 0$. Then,*

$$\int_{\hat\Gamma} \frac{(w(x))^2}{x} dx \le c(1+\ln\frac{H}{h})||w||^2_{L^\infty(\hat\Gamma)}. \tag{3.5}$$

Lemma 3.4. *Let ν be any vertice of the subdomain $\Omega_i \in \Omega_H$. If $w \in S^h(\Omega_i)$ satisfies*

$$\begin{cases} A_i(w,\theta) = 0, & \forall\ \theta \in S^h_0(\Omega_i) \\ w(x) = 0, & \forall \text{ node } x \in \partial\Omega_i,\ x \ne \nu \end{cases}$$

then

$$A_i(w,w) \le c|w(\nu)|^2.$$

Proof. Let v be a piecewise linear continuous function defined on $\partial\Omega_i$ s.t.

$$v(x) = w(x), \quad \forall \text{ node } x \in \partial\Omega_i.$$

It follows from [12], or the proof of Theorem 3 in [13] that

$$A_i(w,w) \le c||v||^2_{\frac{1}{2},\partial\Omega_i}. \tag{3.6}$$

Now, let's compute $||v||^2_{\frac{1}{2},\partial\Omega_i}$. Without loss of generality, we assume that Ω_i is a triangle with $\Gamma_i,\ i = 1, 2, 3$ as its three edges, and ν is the common endpoint of Γ_1, Γ_3. It follows from the definition of the norm of the Sobolev space $H^{\frac{1}{2}}(\partial\Omega_i)$[16] that

$$||v||^2_{\frac{1}{2},\partial\Omega_i} = \int_{\partial\Omega_i}\int_{\partial\Omega_i} \frac{|v(x)-v(y)|^2}{|x-y|^2} ds(x)ds(y) + \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i} v^2 ds = S_1 + S_2, \tag{3.7}$$

$$S_2 = \frac{1}{|\partial\Omega_i|}\int_{\partial\Omega_i} v^2 ds \le \frac{1}{CH}\int_0^h (1-\frac{1}{h}\xi)^2 d\xi \le c, \tag{3.8}$$

$$S_1 = \sum_{k,j=1}^{3}\int_{\Gamma_k}\int_{\Gamma_j}\frac{|v(x)-v(y)|^2}{|x-y|^2}ds(x)ds(y). \tag{3.9}$$

Here, $|\partial\Omega_i|$ is the perimeter of Ω_i. By quasi-uniformness of Ω_H and the fact that $h \ll H$, simple calculation gives

$$\begin{aligned}
&\int_{\Gamma_1}\int_{\Gamma_2} = \int_{\Gamma_2}\int_{\Gamma_1} \le c|v(\nu)|^2, \qquad &&\int_{\Gamma_1}\int_{\Gamma_1} < 2|v(\nu)|^2,\\
&\int_{\Gamma_1}\int_{\Gamma_3} = \int_{\Gamma_3}\int_{\Gamma_1} < 2|v(\nu)|^2, \qquad &&\int_{\Gamma_2}\int_{\Gamma_2} = 0,\\
&\int_{\Gamma_2}\int_{\Gamma_3} = \int_{\Gamma_3}\int_{\Gamma_2} \le c|v(\nu)|^2, \qquad &&\int_{\Gamma_3}\int_{\Gamma_3} < 2|v(\nu)|^2.
\end{aligned}$$

Combining the above inequalities and (3.6)—(3.9) completes the proof of Lemma 3.4 .

Let u_h be the solution of (1.3), $\{u_i^n\}_{i=1}^N$ the sequence resulted from Algorithm 2.1. For $i = 1, 2, \cdots, N$, let $u_i \in S^h(\Omega_i)$, such that

$$u_i(x) = \begin{cases} u_h(x), & \forall \text{ interpolation point } x \in \overline{\Omega}_i \\ \\ 0, & \forall \text{ interpolation point } x \in \overline{\Omega}\backslash\overline{\Omega}_i \end{cases}$$

Then, $\varepsilon_i^n = u_i^n - u_i$ satisfies

$$\text{(I)} \quad \varepsilon_i^n \in S^h(\Omega_i),\ \varepsilon_i^n(x) = \varepsilon_j^n(x),\ \ \forall \text{ node } x \in \partial\Omega_i \cap \partial\Omega_j$$

$$A_i(\varepsilon_i^n, \theta) = 0, \ \ \forall\ \theta \in S_0^h(\Omega_i)$$

$$\text{(II)} \quad A_i(\psi_i^n, \theta) = \sum_{j=1}^N A_j(\varepsilon_j^n, Tr_j^{-1}(r_0^i r_0^h \theta)), \ \ \forall\ \theta \in S^h(\Omega_i)$$

Lemma 3.5. $\sum_{i=1}^N A_i(\varepsilon_i^n, \varepsilon_i^n) \le \sum_{i=1}^N A_i(\psi_i^n, \psi_i^n).$

Proof. By (I) and (2.1), we obtain that $Tr_j^{-1}(\sum_{i=1}^N r_0^i r_0^h \varepsilon_i^n) - \varepsilon_j^n \in S_0^h(\Omega_j)$. Furthermore, with (I), the substitution of θ in (II) with ε_i^n and the Schwarz inequality, it is easy to see that

$$\begin{aligned}
\sum_{j=1}^N A_j(\varepsilon_j^n, \varepsilon_j^n) &= \sum_{j=1}^N A_j(\varepsilon_j^n, Tr_j^{-1}(\sum_{i=1}^N r_0^i r_0^h \varepsilon_i^n))\\
&= \sum_{i=1}^N\sum_{j=1}^N A_j(\varepsilon_j^n, Tr_j^{-1}(r_0^i r_0^h \varepsilon_i^n)) = \sum_{i=1}^N A_i(\psi_i^n, \varepsilon_i^n)\\
&\le \Big[\sum_{i=1}^N A_i(\psi_i^n, \psi_i^n)\Big]^{\frac{1}{2}}\Big[\sum_{i=1}^N A_i(\varepsilon_i^n, \varepsilon_i^n)\Big]^{\frac{1}{2}},
\end{aligned}$$

which implies that Lemma 3.5 holds.

Lemma 3.6. *Denote* $\varphi_i^n = \frac{1}{\rho}(\varepsilon_i^n - \varepsilon_i^{n+1})$. *we have*

$$\sum_{i=1}^N A_i(\varepsilon_i^n, \varphi_i^n) = \sum_{i=1}^N A_i(\psi_i^n, \psi_i^n).$$

Proof. The substitution of θ in (I) with $\varphi_i^n - Tr_i^{-1}(\sum_{j=1}^N r_0^j r_0^h \psi_j^n) \in S_0^h(\Omega_i)$ gives

$$\sum_{i=1}^N A_i(\varepsilon_i^n, \varphi_i^n) = \sum_{i=1}^N A_i(\varepsilon_i^n, \sum_{j=1}^N Tr_i^{-1}(r_0^j r_0^h \psi_j^n)).$$

It follows from (II) with $\theta = \psi_i^n$ that

$$\sum_{i=1}^N A_i(\psi_i^n, \psi_i^n) = \sum_{i=1}^N \sum_{j=1}^N A_j(\varepsilon_j^n, Tr_j^{-1}(r_0^i r_0^h \psi_i^n)) = \sum_{j=1}^N A_j(\varepsilon_j^n, \sum_{i=1}^N Tr_j^{-1}(r_0^i r_0^h \psi_i^n)).$$

With the above two equalities, we come to the conclusion.

Lemma 3.7. *Let* Γ_{jk} *be an open edge of* Ω_i *with endpoints* ν_j, ν_k. *If* $w \in S^h(\Omega_i)$ *satisfies*

$$\begin{cases} A_i(w,\theta) = 0, & \forall\ \theta \in S_0^h(\Omega_i) \\ w(x) = \frac{1}{2}\psi_i^n(x), & \forall \textit{ node } x \in \Gamma_{jk} \\ w(x) = 0, & \forall \textit{ node } x \in \partial\Omega_i \backslash \Gamma_{jk} \end{cases}$$

then

$$A_i(w,w) \le \tau A_i(\psi_i^n, \psi_i^n), \quad \textit{where } \tau \le c(1 + \ln\frac{H}{h})\max(1 + H^{-2}, 1 + \ln\frac{H}{h}).$$

Proof. Let $\tilde{w},\ \tilde{w}_\perp \in S^h(\Omega_i)$ satisfy respectively

$$\begin{cases} A_i(\tilde{w},\theta) = 0, & \forall\ \theta \in S_0^h(\Omega_i) \\ \tilde{w}(x) = \frac{1}{2}\psi_i^n(x), & \forall \text{ node } x \in \partial\Omega_i, \text{where } x \text{ is not any vertex of } \Omega_i \\ \tilde{w}(x) = 0, & \forall \text{ vertex } x \text{ of } \Omega_i \end{cases}$$

$$\begin{cases} A_i(\tilde{w}_\perp,\theta) = 0, & \forall\ \theta \in S_0^h(\Omega_i) \\ \tilde{w}_\perp(x) = 0, & \forall \text{ node } x \in \partial\Omega_i, \text{where } x \text{ is not any vertex of } \Omega_i \\ \tilde{w}_\perp(x) = \frac{1}{2}\psi_i^n(x), & \forall \text{ vertex } x \text{ of } \Omega_i \end{cases}$$

By (II), we have

$$\tilde{w} + \tilde{w}_\perp = \frac{1}{2}\psi_i^n \qquad \text{in } \Omega_i. \tag{3.10}$$

Denote $v = I_h\tilde{w}|_{\partial\Omega_i}$, $v_{jk} = \begin{cases} v, & \text{on } \Gamma_{jk} \\ 0, & \text{on } \partial\Omega_i \backslash \Gamma_{jk} \end{cases}$. It follows from [12], or the proof of Theorem 3 in [13] that

$$A_i(w,w) \leq c||v_{jk}||^2_{H^{\frac{1}{2}}_{00}(\Gamma_{jk})} \leq c\Big\{||v||^2_{\frac{1}{2},\partial\Omega_i} + \int_{\Gamma_{jk}} (\frac{(v_{jk}(x))^2}{|x-\nu_j|} + \frac{(v_{jk}(x))^2}{|x-\nu_k|})ds(x)\Big\}. \tag{3.11}$$

From (3.2), (3.3), we know that

$$||I_h\tilde{w}||^2_{H^1(\Omega_i)} = ||I_h\tilde{w}||^2_{L^2(\Omega_i)} + |I_h\tilde{w}|^2_{1,\Omega_i,h} \leq c\{||\tilde{w}||^2_{L^2(\Omega_i)} + |\tilde{w}|^2_{1,\Omega_i,h}\},$$

by which the trace theorem and the definition of $A(\cdot,\cdot)$ give

$$||v||^2_{\frac{1}{2},\partial\Omega_i} = ||I_h\tilde{w}||^2_{\frac{1}{2},\partial\Omega_i} \leq cA_i(\tilde{w},\tilde{w}) \leq c\{A_i(\tilde{w}+\tilde{w}_\perp, \tilde{w}+\tilde{w}_\perp) + A_i(\tilde{w}_\perp,\tilde{w}_\perp)\}.$$

Using (3.10), Lemma 3.4 and the triangle inequality, we obtain

$$\begin{aligned} ||v||^2_{\frac{1}{2},\partial\Omega_i} &\leq c\{A_i(\psi_i^n,\psi_i^n) + \max_{\text{vertex } x \text{ of } \Omega_i} (\psi_i^n(x))^2\} \\ &\leq c\{A_i(\psi_i^n,\psi_i^n) + ||\psi_i^n||^2_{L^\infty(\Omega_i)}\}. \end{aligned} \tag{3.12}$$

It follows from Lemma 3.3 that

$$\begin{aligned} &\int_{\Gamma_{jk}} \Big(\frac{(v_{jk}(x))^2}{|x-\nu_j|} + \frac{(v_{jk}(x))^2}{|x-\nu_k|}\Big)ds(x) \\ &\leq c(1+\ln\frac{H}{h})||v_{jk}||^2_{L^\infty(\Gamma_{jk})} = c(1+\ln\frac{H}{h}) \max_{\text{node } x\in\Gamma_{jk}} (v_{jk}(x))^2 \\ &\leq c(1+\ln\frac{H}{h})||\psi_i^n||^2_{L^\infty(\Gamma_{jk})} \leq c(1+\ln\frac{H}{h})||\psi_i^n||^2_{L^\infty(\Omega_i)}. \end{aligned} \tag{3.13}$$

Now, Lemma 3.2 and the substitution of (3.12), (3.13) into (3.11) imply Lemma 3.7 holds.

Lemma 3.8. *Let* Γ_{ij} *be an open edge with endpoints* ν_i, ν_j. *Suppose* $\overline{\Gamma}_{ij} = \partial\Omega_k \cap \partial\Omega_l$ *and* $w \in S^h(\Omega_k)$ *satisfies*

$$\begin{cases} A_k(w,\theta) = 0, & \forall\ \theta \in S_0^h(\Omega_k) \\ w(x) = \frac{1}{2}\psi_l^n(x), & \forall \textit{ node } x \in \Gamma_{ij} \\ w(x) = 0, & \forall \textit{ node } x \in \partial\Omega_k\backslash\Gamma_{ij} \end{cases}$$

then $A_k(w,w) \leq \tau A_l(\psi_l^n,\psi_l^n)$, *where* τ *is the same as that in Lemma 3.7*.

Proof. Let $\tilde{w} \in S^h(\Omega_l)$ satisfy

$$\begin{cases} A_l(\tilde{w},\theta) = 0, & \forall\, \theta \in S_0^h(\Omega_l) \\ \tilde{w}(x) = \frac{1}{2}\psi_l^n(x), & \forall \text{ node } x \in \Gamma_{ij} \\ \tilde{w}(x) = 0, & \forall \text{ node } x \in \partial\Omega_l \backslash \Gamma_{ij} \end{cases}$$

It follows from Theorem 3 in [13] and Lemma 3.7 that

$$A_k(w,w) \le cA_l(\tilde{w},\tilde{w}) \le \tau A_l(\psi_l^n,\psi_l^n).$$

Therefore, the lemma holds.

With the above lemmas, we state and prove the main result of this paper as follows.

Theorem 3.9. *If* $0<\rho<\frac{2}{\tau}$, *then the error sequence* $\{\varepsilon_i^n\}_{i=1}^N$ *of Algorithm 2.1 satisfies*

$$\sum_{i=1}^N A_i(\varepsilon_i^{n+1},\varepsilon_i^{n+1}) \le \kappa(\rho)\sum_{i=1}^N A_i(\varepsilon_i^n,\varepsilon_i^n), \tag{3.14}$$

where $\kappa(\rho) = 1-2\rho+\tau\rho^2$, *and* τ *is the same as that in Lemma 3.7. Furthermore, we have*

$$0<\kappa(\rho)<1, \qquad \text{if } \; 0<\rho<\frac{2}{\tau},$$

and $$\kappa(\rho_{opt}) = 1-\tfrac{1}{\tau} = \min_{0<\rho<\frac{2}{\tau}} \kappa(\rho)\ , \quad \textit{where} \quad \rho_{opt} = \tfrac{1}{\tau}\ .$$

Proof. For convenience, we introduce the following notations:

$$\xi_i = \{k:\ \Omega_k \in \Omega_H,\ meas(\Omega_k\cap\Omega_i)>0\},$$

$$\eta_j = \{k:\ \Omega_k \in \Omega_H,\ \nu_j \text{ is the vertex of } \Omega_k\}.$$

Let n_j be the number of the elements of η_j. Suppose $\nu_j \in \overline{\Omega}_i$, for $k\in\eta_j$, let $\hat{w}_i^k \in S^h(\Omega_i)$, s.t.

$$\begin{cases} A_i(\hat{w}_i^k,\ \theta) = 0, & \forall\, \theta \in S_0^h(\Omega_i) \\ \hat{w}_i^k(\nu_j) = \frac{1}{n_j}\psi_k^n(\nu_j) & \\ \hat{w}_i^k(x) = 0, & \forall \text{ node } x \in \partial\Omega_i,\ x \ne \nu_j \end{cases}$$

By Lemma 3.4 and Lemma 3.2, we have

$$\begin{aligned} A_i(\hat{w}_i^k,\ \hat{w}_i^k) &\le c|\psi_k^n(\nu_j)|^2 \\ &\le c\ max(H^{-2},\ 1+ln\frac{H}{h})A_k(\psi_k^n,\ \psi_k^n) \end{aligned}.$$

For $k \in \xi_i$, let $w_i^k \in S^h(\Omega_i)$, such that

$$\begin{cases} A_i(w_i^k,\ \theta) = 0, & \forall\, \theta \in S_0^h(\Omega_i) \\ w_i^k(x) = \frac{1}{2}\psi_k^n(x), & \forall \text{ node } x \in (\partial\Omega_i \cap \partial\Omega_k)\backslash\{\nu_j\} \\ w_i^k(x) = 0, & \forall \text{ node } x \in \overline{\partial\Omega_i(\backslash\partial\Omega_i \cap \partial\Omega_k)} \end{cases}$$

It follows from Lemma 3.7 and Lemma 3.8 that

$$A_i(w_i^k,\ w_i^k) \le \tau A_k(\psi_k^n,\ \psi_k^n).$$

Furthermore, it is easy to see that

$$\varphi_i^n = \frac{1}{\rho}(\varepsilon_i^n - \varepsilon_i^{n+1}) = \sum_{\nu_j \in \overline{\Omega}_i} \sum_{k \in \eta_j} \hat{w}_i^k + \sum_{k \in \xi_i} w_i^k. \tag{3.15}$$

So, with the triangle inequality, the above two inequalities, and the quasi-uniformness of Ω_H, we obtain

$$\begin{aligned} A_i(\varphi_i^n,\ \varphi_i^n) &\le c\Big\{\sum_{\nu_j \in \overline{\Omega}_i} \sum_{k \in \eta_j} A_i(\hat{w}_i^k,\ \hat{w}_i^k) + \sum_{k \in \xi_i} A_i(w_i^k,\ w_i^k)\Big\} \\ &\le \tau\Big\{\sum_{\nu_j \in \overline{\Omega}_i} \sum_{k \in \eta_j} A_k(\psi_k^n,\ \psi_k^n) + \sum_{k \in \xi_i} A_k(\psi_k^n,\ \psi_k^n)\Big\}, \end{aligned}$$

$$\sum_{i=1}^N A_i(\varphi_i^n, \varphi_i^n) \le \tau \sum_{i=1}^N A_i(\psi_i^n,\ \psi_i^n). \tag{3.16}$$

By Lemma 3.6, (3.16), Lemma 3.5, we see that for $0 < \rho < \frac{2}{\tau}$

$$\begin{aligned} \sum_{i=1}^N A_i(\varepsilon_i^{n+1}, \varepsilon_i^{n+1}) &= \sum_{i=1}^N A_i(\varepsilon_i^n, \varepsilon_i^n) - 2\rho \sum_{i=1}^N A_i(\varepsilon_i^n, \varphi_i^n) + \rho^2 \sum_{i=1}^N A_i(\varphi_i^n, \varphi_i^n) \\ &\le \sum_{i=1}^N A_i(\varepsilon_i^n, \varepsilon_i^n) - 2\rho \sum_{i=1}^N A_i(\psi_i^n, \psi_i^n) + \rho^2\tau \sum_{i=1}^N A_i(\psi_i^n, \psi_i^n) \\ &\le (1 - 2\rho + \tau\rho^2) \sum_{i=1}^N A_i(\varepsilon_i^n, \varepsilon_i^n)\ , \end{aligned}$$

which completes the proof of (3.14), and hence the theorem.

4. Matrix Analysis

Section 3 gives the convergence analysis of Algorithm 2.1 by the reduction of error energies. Now, we analyse Algorithm 2.1 from the algebraic viewpoint. (1.3) can be written as

$$KU = \begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{I\Gamma}^T & K_{\Gamma\Gamma} \end{bmatrix} \begin{bmatrix} U_I \\ U_\Gamma \end{bmatrix} = \begin{bmatrix} F_I \\ F_\Gamma \end{bmatrix}, \tag{4.1}$$

where

$$K=(A(\phi_i,\phi_j)),\quad F_I=((f,\phi_i^I)),\qquad F_\Gamma=((f,\phi_i^\Gamma)),$$
$$K_{II}=(A(\phi_i^I,\phi_j^I)),\quad K_{I\Gamma}=(A(\phi_i^I,\phi_j^\Gamma)),\quad K_{\Gamma\Gamma}=(A(\phi_i^\Gamma,\phi_j^\Gamma)),$$

$\{\phi_i\}$ represents the set of the basis functions of $S_0^h(\Omega)$, ϕ_i^I,ϕ_j^Γ are a basis function of $S_0^h(\Omega)$ in $\bigcup_{k=1}^N \Omega_k$ and that on Γ, respectively. By the construction of the nonconforming elements, U_Γ is the vector of the node values of u_h on Γ. With the block Gauss elimination, we obtain

$$SU_\Gamma=F_\Gamma-K_{I\Gamma}^TK_{II}^{-1}F_I=\tilde{F}_\Gamma, \tag{4.2}$$

where $S=K_{\Gamma\Gamma}-K_{I\Gamma}^TK_{II}^{-1}K_{I\Gamma}$ is the capacitance matrix of the stiff matrix K with regard to Γ. Of course, once the solution U_Γ of the capacitance equation (4.2) is known, the solution U of (4.1) can be obtained by solving in parallel the discrete problems on subdomains (cf. step 2 of Algorithm 2.1).

Let $\{\theta_j\}$ be the set of the basis function of $S^h(\Omega_i)$. θ_j^I, θ_k^Γ represent a basis function of $S^h(\Omega_i)$ in Ω_i and that on $\partial\Omega_i\backslash\partial\Omega$, respectively. Denote

$$K_{ii}=(A_i(\theta_j^I,\theta_k^I)),\quad K_{i\Gamma_i}=(A_i(\theta_j^I,\theta_k^\Gamma)),\quad K_{\Gamma_i\Gamma_i}=(A_i(\theta_j^\Gamma,\theta_k^\Gamma)),$$
$$S_i=K_{\Gamma_i\Gamma_i}-K_{i\Gamma_i}^TK_{ii}^{-1}K_{i\Gamma_i},\qquad R_i=((r_0^i)_{jk},k\in I_i)\in\Re^{m\times m_i},$$

where r_0^i,I_i,m_i are the same as those in Section 2. Correspondingly, S_i is called the capacitance matrix of K in Ω_i concerning $\partial\Omega_i\backslash\partial\Omega$. It is easy to see that Algorithm 2.1 is essentially equivalent to the following iterative method for (4.2):

$$\lambda^{n+1}=\lambda^n-\rho\sum_{i=1}^N R_iS_i^{-1}R_i^TS\lambda^n+\rho\sum_{i=1}^N R_iS_i^{-1}R_i^T\tilde{F}_\Gamma. \tag{4.3}$$

Obviously, (4.3) can be viewed as the simple iterative method for (4.2) preconditioned by

$$Q=(\rho\sum_{i=1}^N R_iS_i^{-1}R_i^T)^{-1}. \tag{4.4}$$

The simple iterative method converges much more slowly than the conjugate gradient method (CG) to solve the same symmetric positive definite algebraic equation, we consider applying CG to (4.2) with (4.4) as preconditioner. In fact, Q is an efficient preconditioner of the capacitance matrix S, because it can be inversed easily in parallel and the condition number of $Q^{-1}S$ is bounded by $O\Big((1+\ln\frac{H}{h})max(1+H^{-2},1+\ln\frac{H}{h})\Big)$, which is implied by the following theorem.

Theorem 4.1. *Let S,Q be as defined in (4.2) and (4.4), then for arbitrary $\lambda\in\Re^m\backslash\{0\}$,*

$$\rho\le\frac{(SQ^{-1}S\lambda,\lambda)}{(S\lambda,\lambda)}\le c\rho(1+\ln\frac{H}{h})max(1+H^{-2},1+\ln\frac{H}{h}). \tag{4.5}$$

Proof. Let ε_i, ψ_i be generated by Algorithm 2.1 with $f = 0$ and $\lambda^n = \lambda$. A more iteration gives $\bar{\varepsilon}_i, \bar{\psi}_i$. Denote $\varphi_i = \frac{1}{\rho}(\varepsilon_i - \bar{\varepsilon}_i)$. It is easy to see that

$$(S\lambda, \lambda) = \sum_{i=1}^{N} A_i(\varepsilon_i, \varepsilon_i), \quad (SQ^{-1}S\lambda, \lambda) = \rho \sum_{i=1}^{N} A_i(\varepsilon_i, \varphi_i).$$

So, by Lemma 3.6, Lemma 3.5, we get the left part of (4.5) .

On the other hand, it follows from Lemma 3.6 and (3.16) that

$$\begin{aligned}\sum_{i=1}^{N} A_i(\psi_i, \psi_i) &= \sum_{i=1}^{N} A_i(\varepsilon_i, \varphi_i) \\ &\le [\sum_{i=1}^{N} A_i(\varepsilon_i, \varepsilon_i)]^{\frac{1}{2}} [\sum_{i=1}^{N} A_i(\varphi_i, \varphi_i)]^{\frac{1}{2}} \\ &\le [\sum_{i=1}^{N} A_i(\varepsilon_i, \varepsilon_i)]^{\frac{1}{2}} [\tau \sum_{i=1}^{N} A_i(\psi_i, \psi_i)]^{\frac{1}{2}},\end{aligned}$$

which implies the right part of (4.5) holds, and hence the theorem.

Remark 4.1. When $N = 2$, Algorithm 2.1 degenerates to be two-subdomain domain decomposition method. By Theorem 3 in [13], we can prove that Algorithm 2.1 in this special case converges geometrically with its convergence factor independent of H, h and the optimal convergence factor exists.

Remark 4.2. It follows from the properties of Q defined by (4.4) that

$$\begin{bmatrix} K_{II} & K_{I\Gamma} \\ K_{I\Gamma}^T & Q + K_{I\Gamma}^T K_{II}^{-1} K_{I\Gamma} \end{bmatrix}$$

is an efficient preconditioner of the stiff matrix in (4.1).

References

[1] J.F. Bourgat, R. Glowinski, P.L. Tallec, and M. Vidrascu, Variational formulation and algorithm for trace operator in domain decomposition calculations. In: Chan, T.F., et al. eds. Domain Decomposition Methods. *SIAM*, Philadelphia, 1989, 3–16.

[2] J.H. Bramble, J.E. Pasciak and A.H. Schatz, The construction of preconditioners for elliptic problems by substructuring. I, *Math. Comp.*, 47(1986), 103–134

[3] J.H. Bramble, J.E. Pasciak and A.H. Schatz, The construction of preconditioners for elliptic problems by substructuring. II, *Math. Comp.*, 49(1987), 1–16

[4] J.H. Bramble, J.E. Pasciak and A.H. Schatz, The construction of preconditioners for elliptic problems by substructuring. III, *Math. Comp.*, 51(1988), 415–430

[5] J.H. Bramble, J.E. Pasciak and A.H. Schatz, The construction of preconditioners for elliptic problems by substructuring. IV, *Math. Comp.*, 53(1989), 1–24

[6] J.H. Bramble, J.E. Pasciak, and J. Xu, Parallel multilevel preconditioners, *Math. Comp.*, 55(1990), 1–22

[7] X. Cai, W.D. Gropp, and D.E. Keyes, Convergence rate estimate for a domain decomposition method, *Numer. Math.*, 61(1992), 153–169

[8] G.F. Carey, An analysis of finite element equations and mesh subdivision, *Comput. Methods Appl. Mech. Engrg.*, 9(1976), 165–179

[9] D. Chu, Some Problems in Domain Decomposition Methods for Elliptic Equations. Ph. D. thesis. Dept. of Applied Math., Tsinghua University, China, 1991.

[10] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, North-Holland, Amsterdam, 1978.

[11] M. Crouzeix, and P.A. Raviart, Conforming and nonconforming finite element methods for solving the stationary Stokes equations, *RAIRO Numer. Anal.*, 7-R3(1973), 33–76

[12] J. Gu, Domain Decomposition Methods with Nonconforming Finite Elements. Ph. D. thesis. Dept. of Applied Math., Tsinghua University, China, 1994.

[13] J. Gu, and X. Hu, On an essential estimate in the analysis of domain decomposition methods. *J. Comput. Math.*, 12(1994), 132–137

[14] J. Gu, and X. Hu, A simple domain decomposition algorithm with nonconforming elements. Submitted to The Proceedings of the 7th International Conference on Domain Decomposition, 1993.

[15] J. Gu, and X. Hu, An iterative substructuring method with nonconforming elements. Submitted to *Math. Comp.*, 1993.

[16] J.L. Lions, and E. Magenes, Non-homogeneous Boundary Value Problems and Applications. I, Grund. B. 181. Berlin, Springer, 1972.

[17] P. Oswald, On a hierarchical basis multilevel method with nonconforming P1 elements, *Numer. Math.*, 62(1992), 189–212

[18] O.B. Widlund, Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane, In Glowinski, R., et al., eds. First International Symposium on Domain Decomposition Methods for Partial Differential Equations, *SIAM*, Philadelphia, (1988), 113–128,

[19] E.L. Wilson and R.L. Taylor, Incompatible displacement models. In Proceedings of the Symposium on Numerical and Computer Methods in Structural Engineering (O.N.R., University of Illinois), 1971.

[20] J. Xu, Theory of Multilevel Method, Ph.D. thesis, Cornell University, 1989.

[21] H. Yserentant, On the multilevel splitting of finite element spaces, *Numer. Math.*, 49(1986), 379–412