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DOUBLE S-BREAKING CUBIC TURNING POINTS AND THEIR COMPUTATION *1)

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Abstract

In the paper we are concerned with double S-breaking cubic turning points of two-parameter nonlinear problems in the presence of $\mathcal{Z}_2$-symmetry. Three extended systems are proposed to determine double S-breaking cubic turning points. We show that there exist two kinds of singular point path passing through double S-breaking cubic turning point, One is the simple quadratic turning point path, the other is the pitchfork bifurcation point path.

1. Introduction

Many natural phenomena possess more or less exact symmetries, which are likely to be reflected in any sensible mathematical model. Idealizations such as periodic boundary conditions can produce additional symmetries. Phenomena whose models exhibit both symmetry and nonlinearity lead to problems which are challenging and rich in complexity. Problems with symmetries can show a rich bifurcation behaviour. The occurrence of multiple steady state bifurcation is mostly due to underlying symmetries. This gives rise to the difficulties to numerical computation. However, in the recent years, the tools provided by group theory and representation theory have proven to be highly effective in treating nonlinear problem involving symmetry. By these means, highly complicated situations may be decomposed into a number of simpler ones which are already understood or are at least easier to handle. In the presence of symmetries, the codimension of singularity reduces considerably and the symmetric systems have some special equivariance (see Golubitsky et al. [2], Werner and Spence[6]), which can simplify the bifurcation analysis near the multiple singular points and the numerical computation.

For the bifurcation analysis and numerical computation of double S-breaking quadratic turning points, See Werner[5], [6], also see recently Wu et al. [7] in which the authors presented a detail discussion in two-parameter dependent equations with Z_2-symmetry. However, the bifurcation analysis and numerical computation of double S-breaking cubic turning points seems to be rarely considered. The major aim of this

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paper is to present an approach for the computation of double S-breaking cubic turning points. The main idea is that, in a sense to be precise in Theorem 2.2 of Section 2, the double S-breaking cubic turning point is a simple quadratic turning point of the large extended system (1.13) provided certain conditions are satisfied. Hence, we can use the standard methods for simple turning points on this large extended system to give the required double S-breaking cubic turning points.

Consider the nonlinear problem

$$f(x, \lambda, \mu) = 0,\ f : X \times R^2 \to X\ , \tag{1.1}$$

where X is a Hilbert space and $f \in C^3$. We assume that f satisfies $\mathcal{Z}_2$- symmetry: there exist a linear operator $S \in L(X)$ such that $S \neq I,\ S^2 = I$ and

$$Sf(x, \lambda, \mu) = f(Sx, \lambda, \mu), \quad \forall (x, \lambda, \mu) \in X \times R^2\ . \tag{1.2}$$

Then X and its dual space X' are naturally splitted into

$$X = X_s \oplus X_a, \quad X' = X'_s \oplus X'_a \tag{1.3}$$

where

$$\begin{array}{llll} X_s & = \{x \in X \mid Sx = x\}, & X_a & = \{x \in X \mid Sx = -x\}, \\ X'_s & = \{\psi \in X' \mid \psi S = \psi\}, & X'_a & = \{\psi \in X' \mid \psi S = -\psi\}. \end{array}$$

It is easy to show that

$$\psi x = 0,\ \text{if } (\psi, x) \in X'_s \times X_a \text{ or } (\psi, x) \in X'_a \times X_s\ . \tag{1.4}$$

We specify λ as the bifurcation parameter, and μ the auxiliary parameter. (x, λ, μ) is called a singular point of (1.1) if $f(x, \lambda, \mu) = 0$ and $\dim N((f_x(x, \lambda, \mu)) \geq 1$. In this paper, we are concerned with double S-breaking cubic turning point.

Definition 1.1. *A point (x_0, λ_0, μ_0) is a double S-breaking turning point of (1.1) with respect to λ if*

$$f(x_0, \lambda_0, \mu_0) = 0,\ x_0 \in X_s, \tag{1.5a}$$

$$N(f_x^\circ) = span\{\phi_1, \phi_2\},\ \phi_1 \in X_s \backslash \{0\},\ \phi_2 \in X_a \backslash \{0\}, \tag{1.5b}$$

$$R(f_x^\circ) = \{x \in X \mid \psi_1 x = \psi_2 x = 0\},\ \psi_1 \in X'_s \backslash \{0\},\ \psi_2 \in X'_a \backslash \{0\}, \tag{1.5c}$$

$$\psi_1 f_\lambda^\circ \neq 0,\ \psi_i \phi_i \neq 0,\ i = 1, 2 \tag{1.5d}$$

where $N(f_x^\circ)$ is the null space of $f_x(x_0, \lambda_0, \mu_0)$. $R(f_x^\circ)$ is the range of $f_x(x_0, \lambda_0, \mu_0)$.

A double S-breaking turning point is called a double S-breaking quadratic turning point of (1.1) if

$$D_{111} \neq 0,\ D_{122} \neq 0,\ D_{212} \neq 0. \tag{1.6}$$

A double S-breaking turning point is called a double S-breaking cubic turning point of (1.1) if

$$D_{111} = 0,\ D_{122} = 0,\ D_{212} = 0, \tag{1.7}$$

where

$$D_{ijk} := \psi_i f^\circ_{xx}\phi_j\phi_k, \quad i,j,k=1,2.$$

Definition 1.2. *A point* (x_0,λ_0,μ_0) *is called a simple quadratic turning point of (1.1) with respect to* λ *if*

$$f(x_0,\lambda_0,\mu_0)=0, \tag{1.8a}$$
$$N(f^\circ_x)=span\{\phi\},\ \phi\in X\backslash\{0\}, \tag{1.8b}$$
$$R(f^\circ_x)=\{x\in X\mid \psi x=0\},\ \psi\in X'\backslash\{0\}, \tag{1.8c}$$
$$\psi f^\circ_\lambda\neq 0, \tag{1.8d}$$
$$\psi f^\circ_{xx}\phi\phi\neq 0, \tag{1.8e}$$
$$\psi\phi\neq 0. \tag{1.8f}$$

Definition 1.3. (x_0,λ_0,μ_0) *is called a pitchfork bifurcation point of (1.1) if*

$$f(x_0,\lambda_0,\mu_0)=0,\ x_0\in X_s, \tag{1.9a}$$
$$N(f^\circ_x)=span\{\phi\},\ \phi\in X_a\backslash\{0\}, \tag{1.9b}$$
$$R(f^\circ_x)=\{x\in X\mid \psi x=0\},\ \psi\in X'_a\backslash\{0\}, \tag{1.9c}$$
$$b_\lambda := \psi(f^\circ_{xx}\phi v_\lambda + f^\circ_{x\lambda}\phi)\neq 0, \tag{1.9d}$$
$$\psi\phi\neq 0 \tag{1.9e}$$

where v_λ *is defined by*

$$f^\circ_x v_\lambda + f^\circ_\lambda = 0, \quad v_\lambda\in X_s. \tag{1.10}$$

A pitchfork bifurcation point (x_0,λ_0,μ_0) *is called a quadratic pitchfork bifurcation point if*

$$b_z := \psi(f_{xxx}\phi^3 + 3f_{xx}v_z\phi)\neq 0 \tag{1.11}$$

where v_z *is defined by*

$$f^\circ_x v_z + f^\circ_{xx}\phi\phi = 0, \quad v_z\in X_s. \tag{1.12}$$

In [5], [6], the following two extended systems were used to compute double S-breaking quadratic turning point.

$$F_i(x,\phi,\lambda,\mu)=\begin{bmatrix} f(x,\lambda,\mu)\\ f_x\phi\\ l_i\phi-1\end{bmatrix}=0, \quad i=1,2 \tag{1.13}$$

where $l_1\in X'_s,\ l_2\in X'_a$ such that $l_1\phi_1-1=0$ and $l_2\phi_2-1=0$ respectively.

The following theorem ensures that the double S-breaking quadratic turning point can be detected by solving (1.13) via Newton's method.

Theorem 1.1.[6] *Let* (x_0,λ_0,μ_0) *be a double* S*-breaking quadratic turning point of* $f(x,\lambda,\mu)=0$ *with respect to* λ*. With* $F_i(x,\phi,\lambda,\mu)$ *given by* (1.13) *considered as a mapping on* $X_s\times X_\sigma\times R$ *for fixed* $\mu=\mu_0$*, then* $F_i(x,\phi,\lambda,\mu)=0$ *is regular at* $(x_0,\phi_i,\lambda_0,\mu_0)$*,* $i=1,2$*. Here and below* $\sigma=s$ *if* $i=1$ *and* $\sigma=a$ *if* $i=2$*.*

An outline of the paper is as follows. Section 2 contains a discussion about double S-breaking cubic turning point. We show that there exist two kinds of singular point path passing through the double S-breaking cubic turning point, one pitchfork bifurcation point path and one simple quadratic turning point path. In Section 3, we propose three extended systems, which could be used to calculate double S-breaking cubic turning point. Two numerical examples are given in Section 4.

2. Double S-Breaking Cubic Turning Points

Lemma 2.1. *Let* (x_0, λ_0, μ_0) *be a double* S*-breaking cubic turning point. With* $F_i(y,\mu)$ *mapping* $Y \times R \rightarrow Y = X_s \times X_\sigma \times R$*, we have*

(i). *For* $\sigma = s$, $N(F_y^\circ) = span\{\Phi_1^s\}$, $R(F_y^\circ) = \{y \in Y \mid \Psi_1^s y = 0\}$, *where*

$$F_y^\circ = F_{1y}(y_0, \mu_0),\ y_0 = (x_0, \phi_1, \lambda_0), \tag{2.1a}$$

$$\Phi_1^s = (\phi_1, z_1, 0)^T,\ \Psi_1^s = (\zeta_1, \psi_1, 0), \tag{2.1b}$$

$$f_x^\circ z_1 + f_{xx}^\circ \phi_1 \phi_1 = 0,\ z_1 \in X_s, l_1 z_1 = 0, \tag{2.1c}$$

$$\zeta_1 f_x^\circ + \psi_1 f_{xx}^\circ \phi_1 = 0,\ \zeta_1 f_\lambda^\circ + \psi_1 f_{\lambda x}^\circ \phi_1 = 0, \zeta_1 \in X_s' \tag{2.1d}$$

(ii). *For* $\sigma = a$, $N(F_y^\circ) = span\{\Phi_1^a\}$, $R(F_y^\circ) = \{y \in Y \mid \Psi_1^a y = 0\}$ *where*

$$F_y^\circ = F_{2y}(y_0, \mu_0),\ y_0 = (x_0, \phi_2, \lambda_0), \tag{2.2a}$$

$$\Phi_1^a = (\phi_1, z_0, 0)^T,\ \Psi_1^a = (\zeta_0, \psi_2, 0), \tag{2.2b}$$

$$f_x^\circ z_0 + f_{xx}^\circ \phi_1 \phi_2 = 0,\ z_0 \in X_a, l_2 z_0 = 0, \tag{2.2c}$$

$$\zeta_0 f_x^\circ + \psi_2 f_{xx}^\circ \phi_2 = 0, \zeta_0 f_\lambda^\circ + \psi_2 f_{\lambda x}^\circ \phi_2 = 0, \zeta_0 \in X_s'. \tag{2.2d}$$

We notice that ζ_1 *in (2.1d) and* ζ_0 *in (2.2d) are determined uniquely.*

Proof.

(i) For $\sigma = s$, consider

$$F_{1y}^\circ W = 0,\ w = (w_1, w_2, c_0)^T, \quad w_1 \in X_s, w_2 \in X_s, c_0 \in R. \tag{2.3}$$

Expanding (2.3) we have

$$f_x^\circ w_1 + c_0 f_\lambda^\circ = 0 \tag{2.4a}$$

$$f_{xx}^\circ \phi_1 w_1 + f_x^\circ w_2 + c_0 f_{\lambda x}^\circ \phi_1 = 0 \tag{2.4b}$$

$$l_1 w_2 = 0 \tag{2.4c}$$

Applying $< \psi_1, \cdot >$ to (2.4a), we have $c_0 \psi_1 f_\lambda^\circ = 0$. Thus $c_0 = 0$ since $\psi_1 f_\lambda^\circ \neq 0$. We may assume $w_1 = \alpha_1 \phi_1$. Substituting $w_1 = \alpha_1 \phi_1$ and $c_0 = 0$ into (2.4b) we derive

$$\alpha_1 f_{xx}^\circ \phi_1 \phi_1 + f_x^\circ w_2 = 0 \tag{2.5b}$$

Due to (2.5b) together with (2.4c), we may assume $w_2 = \alpha_1 z_1$, where z_1 is defined by

$$f_{xx}^\circ \phi_1 \phi_1 + f_x^\circ z_1 = 0, \quad z_1 \in X_s, l_1 z_1 = 0.$$

Therefore

$$N(F^\circ_{1y}) = \text{span}\{\Phi^s_1\}, \quad \Phi^s_1 = (\phi_1, z_1, 0)^T.$$

Consider

$$\xi \cdot F^\circ_{1y} = 0 \tag{2.6}$$

where $\xi = (\xi_1, \xi_2, c_1) \in X'_s \times X'_s \times R$. Expanding (2.6) we derive

$$\xi_1 f^\circ_x + \xi_2 f^\circ_{xx}\phi_1 = 0, \tag{2.7a}$$
$$\xi_2 f^\circ_x + c_1 l_1 = 0, \tag{2.7b}$$
$$\xi_1 f^\circ_\lambda + \xi_2 f^\circ_{\lambda x}\phi_1 = 0. \tag{2.7c}$$

From (2.7b), we obtain $c_1 l_1 \phi_1 = 0$, thus $c_1 = 0$ and we may assume $\xi_2 = \beta_1\psi_1$. Substituting ξ_2 into (2.7a), (2.7c), we then assume $\xi_1 = \beta_1\zeta_1$ where $\zeta_1 \in X'_s$ is defined by

$$\zeta_1 f^\circ_x + \psi_1 f^\circ_{xx}\phi_1 = 0, \ \zeta_1 f^\circ_\lambda + \psi_1 f^\circ_{\lambda x}\phi_1 = 0.$$

The proof of (i) is completed. Similarly we could prove (ii).

From Lemma 2.1 we know that $F_{iy}(y_0, \mu_0) = 0$, with $y \in Y = X_s \times X_\sigma \times R$, has one dimensional null space spanned by Φ^σ_1. To guarantee $(y_0, \mu_0) = (x_0, \phi_i, \lambda_0, \mu_0)$ is a simple quadratic turning point of $F_i(y, \mu) = 0$ with respect to μ, the following condition is assumed

$$F^\circ_\mu \notin R(F^\circ_{iy}) \tag{2.8}$$

which is equivalent to $\Psi^\sigma_1 F^\circ_\mu \neq 0$, i.e.

$$d^s_0 := \zeta_1 f^\circ_\mu + \psi_1 f^\circ_{\mu x}\phi_1 \neq 0, \quad \text{for } \sigma = s, \tag{2.8a}$$
$$d^a_0 := \zeta_0 f^\circ_\mu + \psi_2 f^\circ_{\mu x}\phi_2 \neq 0, \quad \text{for } \sigma = a, \tag{2.8b}$$

Now, we are in a position to state our main results.

Theorem 2.2. *Assume* (2.8). *Let* (x_0, λ_0, μ_0) *be a double S-breaking cubic turning point of* (1.1) *with respect to* λ. *Then*

(i) *For* $\sigma = s, (y_0, \mu_0) = (x_0, \phi_1, \lambda_0, \mu_0)$ *is a simple quadratic turning point of* $F_1(y, \mu) \mid_{X_s \times X_s \times R^2} = 0$ *with respect to* μ *if*

$$D^s_0 := \psi_1(f^\circ_{xxx}\phi^3_1 + 3f^\circ_{xx}\phi_1 z_1) \neq 0 \tag{2.9}$$

(ii) *For* $\sigma = a, (y_0, \mu_0) = (x_0, \phi_2, \lambda_0, \mu_0)$ *is a simple quadratic turning point of* $F_2(y, \mu) \mid_{X_s \times X_a \times R^2} = 0$ *with respect to* μ *if*

$$D^a_0 := \psi_2(f^\circ_{xxx}\phi^2_1\phi_2 + 2f^\circ_{xx}\phi_1 z_0 + f^\circ_{xx}\phi_2 z_1) \neq 0 \tag{2.10}$$

Proof.

(i) By a direct calculation, with $\Phi^s_1 = (\phi_1, z_1, 0)^T$, and $\Psi^s_1 = (\zeta_1, \psi_1, 0)$, we obtain

$$\Psi^s_1 F^\circ_{1yy}\Phi^s_1\Phi^s_1 = \zeta_1 f^\circ_{xx}\phi_1\phi_1 + \psi_1(f^\circ_{xxx}\phi^3_1 + 2f^\circ_{xx}\phi_1 z_1).$$

According to (2.1d) and (2.1c), we derive

$$\zeta_1 f_{xx}^\circ \phi_1\phi_1 = -\zeta_1 f_x^\circ z_1 = \psi_1 f_{xx}^\circ \phi_1 z_1$$

Due to (2.9),

$$\Psi_1^s F_{1yy}(y_0,\mu_0)\Phi_1^s\Phi_1^s = D_0^s \neq 0$$

and we complete the proof of (i).

(ii) Similarly,

$$\Psi_1^a F_{2yy}(y_0,\mu_0)\Phi_1^a\Phi_1^a = \zeta_0 f_{xx}^\circ \phi_1\phi_1 + \psi_2(f_{xxx}^\circ \phi_1^2\phi_2 + 2f_{xx}^\circ \phi_1 z_0)$$

According to (2.1c) and (2.2d), we have

$$\zeta_0 f_{xx}^\circ \phi_1\phi_1 = -\zeta_0 f_x^\circ z_1 = \psi_2 f_{xx}^\circ \phi_2 z_1$$

Due to (2.l0),

$$\Psi_1^a F_{2yy}(y_0,\mu_0)\Phi_1^a\Phi_1^a = \psi_2 f_{xxx}^\circ \phi_1^2\phi_2 + 2\psi_2 f_{xx}^\circ \phi_1 z_0 + \psi_2 f_{xx}^\circ \phi_2 z_1 \neq 0.$$

The proof of (ii) is completed.

The following corollary is the consequence of Theorem 2.2.

Corollary 2.3. *Assume* (2.8), (2.9) *and* (2.10). *Let* (x_0,λ_0,μ_0) *be a double* S-*breaking cubic turning point of* (1.1) *with respect to* λ. *Then there exist only solution branch* l_σ *of* $F_i(y,\mu)\mid_{X_s\times X_\sigma\times R^2}= 0$ *passing through* (y_0,μ_0). *Moreover the solution branch has tangent* $(\Phi_1^\sigma, 0)$ *at* (y_0,μ_0).

Theorem 2.2 is important practically as well as theoretically. It indicates clearly a procedure to be followed for the calculation of the double S-breaking cubic turning point (x_0,λ_0,μ_0). We apply the idea of the extended system once again. The twice extended system

$$F_i^2(y,\Phi_1^\sigma,\mu) \equiv \begin{pmatrix} F_i(y,\mu) \\ F_{iy}(y,\mu)\Phi_1^\sigma \\ L_i\Phi_1^\sigma - 1 \end{pmatrix} = 0$$

is regular at double S-breaking cubic turning point. Implicit function theorem insures that there are singular solution branches l_s and l_a of (1.1), which pass crossly through double S-breaking cubic turning point. Therefore, along l_s and l_a, 0 is always an eigenvalue of f_x.

We shall assume the following condition

$$\Delta := \det k \neq 0 \tag{2.11}$$

where

$$k := \begin{pmatrix} D_0^s & d_0^s \\ D_0^a & d_0^a \end{pmatrix}. \tag{2.12}$$

Theorem 2.4. *In addition to the conditions in Corollary* 2.3, *we assume that* (2.11) *holds. Then* l_s *and* l_a *in Corollary* 2.3 *correspond to simple quadratic turning* *point and pitchfork bifurcation point of* (1.1) *with respect to* λ *respectively, except for* (x_0, λ_0, μ_0).

Proof. We divide the proof into two steps.

(a). Let $(y(\varepsilon), \mu(\varepsilon)) = (x(\varepsilon), \phi(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon)) \in X_s \times X_\sigma \times R^2$ be the solution branch l_σ in Corollary 2.3. We first show that zero is a simple eigenvalue of $f_x(\varepsilon) := f_x(x(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon))$ for $\varepsilon \neq 0$.

Consider the following system

$$M(m, \varepsilon) = \begin{pmatrix} f_x(\varepsilon)\theta(\varepsilon) - \beta(\varepsilon)\theta(\varepsilon) \\ l_j\theta(\varepsilon) - 1 \end{pmatrix} = 0 \tag{2.13}$$

where

$j = 2$ if $\sigma = s$ and $j = 1$ if $\sigma = a$. $m = (\theta, \beta)$, $m_0 = (\phi_j, 0)$, $M : X_\delta \times R^2 \to X_\delta \times R$, $\delta = a$ if $\sigma = s$ and $\delta = s$ if $\sigma = a$.

It is easy to check that $M(m_0, 0) = 0$ and $M_m(m_0, 0) : X_\delta \times R \to X_\delta \times R$ is regular and hence there exists a unique solution path $m(\varepsilon) = (\theta(\varepsilon), \beta(\varepsilon))$ such that $m(0) = m_0$, $M(m(\varepsilon), \varepsilon) = 0$.

As for l_s it follows from Corollary 2.3 that $\dot{x}(0) = \phi_1, \dot{\lambda}(0) = 0, \dot{\mu}(0) = 0, \dot{\phi}(0) = z_1$. Differentiating $f_x(\varepsilon)\theta(\varepsilon) - \beta(\varepsilon)\theta(\varepsilon) = 0$ with respect to ε at $\varepsilon = 0$ and multiplying by $\psi_2(\dot{x}(0) = \dfrac{dx(\varepsilon)}{d\varepsilon} |_{\varepsilon=0}$ etc.) yields

$$\dot{\beta}(0) = 0, \quad f^\circ_{xx}\phi_1\phi_2 + f^\circ_x\dot{\theta}(0) = 0. \tag{2.14}$$

(2.14) implies that $\dot{\theta}(0) = z_0$.

Differentiating $f_x(\varepsilon)\theta(\varepsilon) - \beta(\varepsilon)\theta(\varepsilon) = 0$ with respect to ε at $\varepsilon = 0$ twice and multiplying by ψ_2 yields

$$\psi_2 f^\circ_{xxx}\phi_1^2\phi_2 + 2\psi_2 f^\circ_{xx}\phi_1 z_0 + \psi_2(f^\circ_{xx}\ddot{x}(0) + f^\circ_{x\lambda}\ddot{\lambda}(0) + f_{x\mu}\ddot{\mu}(0))\phi_2 - \ddot{\beta}(0)\psi_2\phi_2 = 0. \tag{2.15}$$

Differentiating $f(x(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon)) = 0$ twice with respect to ε at $\varepsilon = 0$ and multiplying by ζ_0 yields

$$\zeta_0 f^\circ_{xx}\phi_1\phi_1 + \zeta_0(f^\circ_x\ddot{x}(0) + f^\circ_\lambda\ddot{\lambda}(0) + f^\circ_\mu\ddot{\mu}(0)) = 0. \tag{2.16}$$

(2.15) together with (2.16) yields

$$\psi_2 f^\circ_{xxx}\phi_1^2\phi_2 + 2\psi_2 f^\circ_{xx}\phi_1 z_0 + \psi_2 f^\circ_{xx}\phi_2 z_1 + d_0^a\ddot{\mu}(0) - \ddot{\beta}(0)\psi_2\phi_2 = 0. \tag{2.17}$$

Generally,

$$\begin{aligned} \tfrac{d^2}{d\varepsilon^2} F_1(y(\varepsilon), \mu(\varepsilon)) |_{\varepsilon=0} = {} & F^\circ_{1yy}\dot{y}(0)^2 + 2F_{1y\mu}\dot{y}(0)\dot{\mu}(0) + F_{1\mu\mu}\dot{\mu}(0)^2 \\ & + F^\circ_{1y}\ddot{y}(0) + F^\circ_{1\mu}\ddot{\mu}(0) = 0. \end{aligned} \tag{2.18}$$

Since $\dot{y}(0) = (\dot{x}(0), \dot{\phi}(0), \ \dot{\lambda}(0)) = (\phi_1, z_1, 0), \dot{\mu}(0) = 0$, we obtain

$$f^\circ_{xx}\phi_1\phi_1 + f^\circ_x\ddot{x}(0) + f^\circ_\lambda\ddot{\lambda}(0) + f^\circ_\mu\ddot{\mu}(0) = 0, \tag{2.19a}$$

$$f^\circ_{xxx}\phi_1^3 + 2f^\circ_{xx}\phi_1 z_1 + f^\circ_{xx}\phi_1\ddot{x}(0) + f^\circ_x\ddot{\phi}(0) + f^\circ_{\lambda x}\phi_1\ddot{\lambda}(0) + f^\circ_{\mu x}\phi_1\ddot{\mu}(0) = 0. \tag{2.19b}$$

Multiplying (2.19a) by ζ_1 and (2.19b) by ψ_1 leads to

$$\psi_1 f_{xxx}^\circ \phi_1^3 + 2\psi_1 f_{xx}^\circ \phi_1 z_1 + \zeta_1 f_{xx}^\circ \phi_1 \phi_1 + d_0^s \ddot{\mu}(0) = 0.$$

Therefore,

$$\psi_1 (f_{xxx}^\circ \phi_1^3 + 3 f_{xx}^\circ \phi_1 z_1) + d_0^s \ddot{\mu}(0) = 0, \tag{2.20}$$

$$\ddot{\mu}(0) = -\frac{D_0^s}{d_0^s}. \tag{2.21}$$

Substituting (2.21) into (2.17) yields

$$\ddot{\beta}(0) = -(D_0^s d_0^a - D_0^a d_0^s)/(\psi_2 \phi_2 \cdot d_0^s) = -\det k/(\psi_2 \phi_2 \cdot d_0^s) \neq 0. \tag{2.22}$$

Similarly for l_a, we have

$$\ddot{\beta}(0) = \det k/(\psi_1 \phi_1 \cdot d_0^a) \neq 0. \tag{2.23}$$

(2.22) and (2.23) imply that $f_x(\varepsilon)$ has a small but nonzero eigenvalue $\beta(\varepsilon)$ for $\varepsilon \neq 0$. Notice that f_x° has $n-2$ nonzero eigenvalues, thus the zero eigenvalue of $f_x(\varepsilon)$ for $\varepsilon \neq 0$ must be simple.

Let $\psi(\varepsilon)$ and $\phi(\varepsilon)$ be the left and right null vectors of $f_x(\varepsilon)$ respectively such that

$$\psi(\varepsilon) f_x(\varepsilon) = 0, \quad \psi(\varepsilon) \in X'_\sigma \backslash \{0\}, \psi(0) = \psi_i, i = 1, 2, \tag{2.24a}$$

$$f_x(\varepsilon)\phi(\varepsilon) = 0, \quad \phi(\varepsilon) \in X_\sigma \backslash \{0\}, \phi(0) = \phi_i, i = 1, 2. \tag{2.24b}$$

(b). In order to prove $(x(\varepsilon), \phi(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon)) \in l_s$ correspond to single quadratic turning points of $f(x, \lambda, \mu) = 0$ for $\varepsilon \neq 0$, we need only to confirm that, for $\varepsilon \neq 0$,

$$\psi(\varepsilon) f_\lambda(\varepsilon) \neq 0, \quad \psi(\varepsilon)\phi(\varepsilon) \neq 0, \tag{2.25a}$$

$$d(\varepsilon) := \psi(\varepsilon) f_{xx}(\varepsilon)\phi(\varepsilon)\phi(\varepsilon) \neq 0. \tag{2.25b}$$

(2.25a) holds since $\psi_1 f_\lambda^\circ \neq 0$, $\psi_1 \phi_1 \neq 0$. We now prove (2.25b). Obviously,

$$d(0) = \psi_1 f_{xx}^\circ \phi_1 \phi_1 = 0.$$

The first derivative of $d(\varepsilon)$ with respect to ε at $\varepsilon = 0$ yields

$$\dot{d}(0) = \dot{\psi}(0) f_{xx}^\circ \phi_1 \phi_1 + \psi(0)(f_{xxx}^\circ \dot{x}(0)\phi_1\phi_1 + 2 f_{xx}^\circ \phi_1 \dot{\phi}(0)). \tag{2.26}$$

Differentiating $\psi(\varepsilon) f_x(\varepsilon) = 0$ with respect to ε at $\varepsilon = 0$ leads to

$$\dot{\psi}(0) f_x^\circ + \psi_1 f_{xx}^\circ \phi_1 = 0.$$

Recalling $\dot{x}(0) = \phi_1$, $\dot{\phi}(0) = z_1$, $f_{xx}^\circ \phi_1 \phi_1 + f_x^\circ z_1 = 0$, we obtain

$$\begin{aligned} \dot{d}(0) \ &= -\dot{\psi}(0) f_x^\circ z_1 + \psi_1 (f_{xxx}^\circ \phi_1^3 + 2 f_{xx}^\circ \phi_1 z_1) \\ &= \psi_1 (f_{xxx}^\circ \phi_1^3 + 3 f_{xx}^\circ \phi_1 z_1) = D_0^s \neq 0. \end{aligned}$$

So (2.25b) holds for $\varepsilon \neq 0$.

Next, we deal with l_a. In this case, $\psi(\varepsilon) \in X'_a, \phi(\varepsilon) \in X_a, x(\varepsilon) \in X_s$. To prove $(x(\varepsilon), \lambda(\varepsilon), \mu(\varepsilon)) \in l_a$ is pitchfork bifurcation point of $f(x, \lambda, \mu) = 0$ for $\varepsilon \neq 0$, we need only to confirm

$$\psi(\varepsilon)\phi(\varepsilon) \neq 0, \tag{2.27a}$$
$$\psi(\varepsilon)(f_{xx}(\varepsilon)v_\lambda(\varepsilon) + f_{\lambda x}(\varepsilon))\phi(\varepsilon) \neq 0 \tag{2.27b}$$

where $v_\lambda(\varepsilon)$ is defined by

$$f_x(\varepsilon)v_\lambda(\varepsilon) + f_\lambda(\varepsilon) = 0, \quad v_\lambda(\varepsilon) \in X_s. \tag{2.28}$$

(2.27a) is obviously satisfied since $\psi_2\phi_2 \neq 0$.

The difficulty here is that $v_\lambda(\varepsilon)$ does not exist for $\varepsilon = 0$. To overcome the trouble, we introduce the following system

$$S(\zeta, \varepsilon) = \begin{pmatrix} f_x(\varepsilon)v(\varepsilon) + \tau(\varepsilon)f_\lambda(\varepsilon) \\ l_1 v(\varepsilon) - 1 \end{pmatrix} = 0 \tag{2.29}$$

where $\zeta = (v(\varepsilon), \tau(\varepsilon)),\ \zeta_0 = (\phi_1, 0),\ S : X_s \times R^2 \to X_s \times R$. It is easy to show that $S(\zeta_0, 0) = 0,\ S_\zeta(\zeta_0, 0) : X_s \times R \to X_s \times R$ is regular and there exists a unique solution path $\zeta(\varepsilon) = (v(\varepsilon), \tau(\varepsilon))$ of (2.29) such that $\zeta(0) = \zeta_0$. Notice that $\tau(\varepsilon) \neq 0$ for $\varepsilon \neq 0$, otherwise $f_x(\varepsilon)$ would have an extra null vector which contradicts (a). Hence

$$v_\lambda(\varepsilon) = \frac{v(\varepsilon)}{\tau(\varepsilon)}, \quad \text{for } \varepsilon \neq 0$$

and

$$v(0) = \phi_1.$$

Denote

$$B(\varepsilon) := \psi(\varepsilon)f_{xx}(\varepsilon)v(\varepsilon)\phi(\varepsilon) + \tau(\varepsilon)\psi(\varepsilon)f_{x\lambda}(\varepsilon)\phi(\varepsilon) \tag{2.30}$$

From the following lemma 2.5, we know $\dot{B}(0) = D_0^a \neq 0$. Thus (2.27) holds for $\varepsilon \neq 0$. The proof is completed.

Lemma 2.5. *Let $B(\varepsilon)$ be defined by (2.30). Then*

$$\dot{B}(0) = \psi_2(f^\circ_{xxx}\phi_1^2\phi_2 + 2f^\circ_{xx}\phi_1 z_0 + f^\circ_{xx}\phi_2 z_1) = D_0^a.$$

Proof. A direct calculation, with $\dot{x}(0) = \phi_1,\ \dot{\phi}(0) = z_0,\ \dot{\lambda}(0) = \dot{\mu}(0) = 0$, shows that

$$\dot{B}(0) = \dot{\tau}(0)\psi_2 f^\circ_{x\lambda}\phi_2 + \dot{\psi}(0)f^\circ_{xx}\phi_1\phi_2 + \psi_2 f^\circ_{xxx}\phi_1^2\phi_2 + \psi_2 f^\circ_{xx}\phi_1 z_0 + \psi_2 f^\circ_{xx}\phi_2\dot{v}(0). \tag{2.31}$$

Since $\zeta_0 f^\circ_x + \psi_2 f^\circ_{xx}\phi_2 = 0,\ \zeta_0 f^\circ_\lambda + \psi_2 f^\circ_{\lambda x}\phi_2 = 0$, hence

$$\dot{B}(0) = -\dot{\tau}(0)\zeta_0 f^\circ_\lambda - \zeta_0 f^\circ_x \dot{v}(0) + \dot{\psi}(0)f^\circ_{xx}\phi_1\phi_2 + \psi_2(f^\circ_{xxx}\phi_1^2\phi_2 + f^\circ_{xx}\phi_1 z_0). \tag{2.32}$$

Differentiating $\psi(\varepsilon)f_x(\varepsilon) = 0$ with respect to ε at $\varepsilon = 0$ yields

$$\dot{\psi}(0)f^\circ_x + \psi_2 f^\circ_{xx}\phi_1 = 0. \tag{2.33}$$

Differentiating $f_x(\varepsilon)v(\varepsilon)+\tau(\varepsilon)f_\lambda(\varepsilon)=0$ with respect to ε at $\varepsilon=0$ leads to

$$f^\circ_{xx}\phi_1\phi_1+f^\circ_x\dot{v}(0)+\dot{\tau}(0)f^\circ_\lambda=0. \tag{2.34}$$

Thus

$$\zeta_0 f^\circ_{xx}\phi_1\phi_1+\zeta_0 f^\circ_x\dot{v}(0)+\zeta_0 f^\circ_\lambda\dot{\tau}(0)=0. \tag{2.35}$$

By (2.35), (2.33) and (2.32), we obtain

$$\begin{aligned}\dot{B}(0) &= \zeta_0 f^\circ_{xx}\phi_1\phi_1+\psi_2 f^\circ_{xxx}\phi_1\phi_1\phi_2+2\psi_2 f^\circ_{xx}\phi_1 z_0\\ &= \psi_2(f^\circ_{xxx}\phi_1^2\phi_2+2f^\circ_{xx}\phi_1 z_o+f^\circ_{xx}\phi_2 z_1)=D^a_0.\end{aligned}$$

3. Extended Systems

We introduce the following three extended systems. Their regularity at the double S-breaking cubic turning point ensures that conventional Newton's method can be applied. In the sequel, we assume that (x_0,λ_0,μ_0) is a double S-breaking cubic turning point with null vector $\phi_1\in X_s,\ \phi_2\in X_a$.
The first one is

$$H_1(y)=\begin{bmatrix} f(x,\lambda,\mu)\\ f_x\phi\\ f_x z+f_{xx}\phi\phi\\ l_1\phi-1\\ l_1 z\end{bmatrix}=0,\quad \begin{aligned}&H_1: X_s\times X_s\times X_s\times R^2\to X_s\times X_s\times X_s\times R^2,\\ &y=(x,\phi,z,\lambda,\mu),\\ &y_0=(x_0,\phi_1,z_1,\lambda_0,\mu_0).\end{aligned} \tag{3.1}$$

Then $H_1(y_0)=0$, where z_1 is defined in (2.1c) $l_1\in X'_s$ such that $l_1\phi_1-1=0,\ l_1u=0$ for $u\in X_s\backslash R\{\phi_1\}$.
The second one is

$$H_2(y)=\begin{bmatrix} f(x,\lambda,\mu)\\ f_x\phi\\ f_x z+f_{xx}\phi\phi\\ l_2\phi-1\\ l_1 z\end{bmatrix}=0,\quad \begin{aligned}&H_2: X_s\times X_a\times X_s\times R^2\to X_s\times X_a\times X_s\times R^2,\\ &y=(x,\phi,z,\lambda,\mu),\\ &y_0=(x_0,\phi_2,z_2,\lambda_0,\mu_0).\end{aligned} \tag{3.2}$$

Then $H_2(y_0)=0$, where z_2 is defined by

$$f^\circ_x z_2+f^\circ_x\phi_2\phi_2=0,\quad z_2\in X_s, l_1 z_2=0.$$

$l_2\in X'_a$ such that $l_2\phi_2-1=0,\ l_2u=0$ for $u\in X_a\backslash R\{\phi_2\}$.

The third one is

$$H_3(y) = \begin{bmatrix} f(x,\lambda,\mu) \\ f_x\phi_s + \alpha\phi_s \\ f_x\phi_a \\ f_x z + f_{xx}\phi_s\phi_a \\ l_1\phi_s - 1 \\ l_2\phi_a - 1 \\ l_2 z \end{bmatrix} = 0, \quad \begin{array}{l} H_3 : X_s \times X_s \times X_a \times X_a \times R^3 \\ \qquad \to X_s \times X_s \times X_a \times X_a \times R^3, \\ y = (x_0, \phi_s, \phi_a, z, \lambda, \mu, \alpha), \\ y_0 = (x_0, \phi_1, \phi_2, z_0, \lambda_0, \mu_0, 0). \end{array} \tag{3.3}$$

Then $H_3(y_0) = 0$, where z_0 is defined in (2.2c).

The following theorems describe the regularity of the extended systems.

Theorem 3.1. *Assume* (2.8) *and* $D_0^s \neq 0$. *Then* $H_1(y) = 0$ *is regular at* $y_0 = (x_0, \phi_1, z_1, \lambda_0, \mu_0)$.

Theorem 3.2. *Assume* (2.8) *and* $D_0^{as} := \psi_1 f_{xxx}^\circ \phi_1\phi_2\phi_2 + 2\psi_1 f_{xx}^\circ \phi_2 z_0 + \psi_1 f_{xx}^\circ \phi_1 z_2 \neq 0$. *Then* $H_2(y) = 0$ *is regular at* $y_0 = (x_0, \phi_2, z_2, \lambda_0, \mu_0)$.

Theorem 3.3. *Assume* (2.8) *and* $D_0^a \neq 0$. *Then* $H_3(y) = 0$ *is regular at* $y_0 = (x_0, \phi_1, \phi_2, z_1, \lambda_0, \mu_0, 0)$.

We only prove Theorem 3.1. The proofs of Theorem 3.2 and Theorem 3.3 are similar.

Proof of Theorem 3.1.

We consider

$$DH_1^\circ \cdot Y = W \tag{3.4}$$

where DH_1° denote the Jacobian of $H_1(y)$ at y_0, $W = (w_1, w_2, w_3, \alpha, \beta) \in X_s \times X_s \times X_s \times R^2$, $Y = (y_1, y_2, y_3, \lambda, \mu) \in X_s \times X_s \times X_s \times R^2$. Expanding (3.4) yields

$$f_x^\circ y_1 + \lambda f_\lambda^\circ + \mu f_\mu^\circ = w_1, \tag{3.5a}$$

$$f_{xx}^\circ \phi_1 y_1 + f_x^\circ y_2 + \lambda f_{\lambda x}^\circ \phi_1 + \mu f_{\mu x}^\circ \phi_1 = w_2, \tag{3.5b}$$

$$\begin{aligned}(f_{xx}^\circ z_1 + f_{xxx}^\circ \phi_1\phi_1) y_1 + 2 f_{xx}^\circ \phi_1 y_2 + f_x^\circ y_3 + \lambda(f_{\lambda x}^\circ z_1 + f_{\lambda xx}^\circ \phi_1\phi_1) \\ + \mu(f_{\mu x}^\circ z_1 + f_{\mu xx}^\circ \phi_1\phi_1) = w_3,\end{aligned} \tag{3.5c}$$

$$l_1 y_2 = \alpha, \tag{3.5d}$$

$$l_1 y_3 = \beta. \tag{3.5e}$$

Multiplying (3.5a) by ζ_1 and (3.5b) by ψ_1 and using $\zeta_1 f_x^\circ + \psi_1 f_{xx}^\circ \phi_1 = 0$, $\zeta_1 f_\lambda + \psi_1 f_{\lambda x}^\circ \phi_1 = 0$ yields

$$\mu \cdot d_0^s = \zeta_1 w_1 + \psi_1 w_2. \tag{3.6}$$

Since $d_0^s \neq 0$ we can uniquely determine $\mu = (\zeta_1 w_1 + \psi_1 w_2)/d_0^s$. Substituting μ into (3.5a) and multiplying it by ψ_1 yields $\lambda = \psi_1(w_1 - \mu f_\mu^\circ)/\psi_1 f_\lambda^\circ$. Then we may assume that $y_1 = c_1\phi_1 + \tilde{y}_1$, $\tilde{y}_1 \in X_s$ is uniquely determined by $f_x^\circ \tilde{y}_1 = w_1 - \lambda f_\lambda^\circ - \mu f_\mu^\circ$, $l_1\tilde{y}_1 = 0$. Substituting y_1, λ and μ into (3.5b), we derive

$$c_1 f_{xx}^\circ \phi_1\phi_1 + f_x^\circ y_2 = \bar{w}_2 \tag{3.7}$$

where $\bar{w}_2 = w_2 - f^\circ_{xx}\phi_1\tilde{y}_1 - \lambda f^\circ_{\lambda x}\phi_1 - \mu f^\circ_{\mu x}\phi_1$.
From (3.7) and (3.5d), we may assume that $y_2 = \alpha\phi_1 + c_1 z_1 + \tilde{y}_2,\ \tilde{y}_2 \in X_s$ is uniquely determined by $f^\circ_x\tilde{y}_2 = \bar{w}_2\ l_1\tilde{y}_2 = 0$. Substituting $y_1,\ y_2,\ \lambda,\ \mu$ into (3.5c) we have

$$c_1(f^\circ_{xxx}\phi_1^3 + 3f^\circ_{xx}\phi_1 z_1) + \alpha f^\circ_{xx}\phi_1\phi_1 + f^\circ_x y_3 = \bar{w}_3 \tag{3.8}$$

where $\bar{w}_3 = w_3 - \lambda(f^\circ_{\lambda x}z_1 + f^\circ_{\lambda xx}\phi_1\phi_1) - \mu(f^\circ_{\mu x}z_1 + f^\circ_{\mu xx}\phi_1\phi_1) - (f^\circ_{xx}z_1 + f^\circ_{xx}\phi_1\phi_1)\tilde{y}_1 - 2f^\circ_{xx}\phi_1\tilde{y}_2$.
Since $D^s_0 \neq 0,\ c_1 = \psi_1\bar{w}_3/D^s_0$. Substituting c_1 into (3.8), we derive

$$f^\circ_x y_3 = \bar{w}_3 - c_1(f^\circ_{xxx}\phi_1^3 + 3f^\circ_{xx}\phi_1 z_1).$$

Together with (3.5e), we may determine y_3 uniquely.

From the preceding procedure, we can easily show that if $W = 0$ then $Y = 0$. Applying the open mapping theorem, theorem 3.1 is completed.

To compute the double S-breaking cubic turning point, we first restrict our attention in $X_s \times X_s \times R^2$ and use the extended system $F_1(x, \phi, \lambda, \mu) = 0$ to get the quadratic turning points, which are on l_s, by Newton's method. When Newton's method does not work for $F_1 = 0$ for some μ_0, we turn to use system (3.1) to find some high singular point (x_0, λ_0, μ_0). Just as shown theoretically, (3.1) can be solved by Newton's method practically due to its regularity at the double S-breaking cubic turning point. Similarly, we use $F_2(x, \phi, \lambda, \mu) = 0$ to obtain pitchfork bifurcation points, which are on l_a, by Newton's method near (x_0, λ_0, μ_0) and expect to find the signal that Newton's method does not solve $F_2 = 0$. If it is so, we use (3.2) to find the solution (x_1, λ_1, μ_1). It should holds that $x_0 = x_1,\ \lambda_0 = \lambda_1,\ \mu_0 = \mu_1$ at double S-breaking cubic turning point. We finally solve (3.3) to make sure that (x_0, λ_0, μ_0) is a double S-breaking cubic turning of $f(x, \lambda, \mu) = 0$, namely, the solution of (3.3) should be consistent with those of (3.1) and (3.2). We will give a numerical example to show the procedure in section 4.

4. Numerical Examples

Example 4.1. Let $h : R^n \times R^2 \to R^n$ possess a complex analytic extension H

$$H : C^n \times R^2 \to C^n, H(\bar{z}, \lambda, \mu) = \overline{H(z, \lambda, \mu)}, \quad z \in C^n, (\lambda, \mu) \in R^2.$$

Identifying $z = u + iv \in C^n$ with $x = (u, v) \in R^{2n}$, H is transformed into $g : R^{2n} \times R^2 \to R^{2n}$, where

$$g(u, v, \lambda, \mu) = \begin{pmatrix} g^r(u, v, \lambda, \mu) \\ g^i(u, v, \lambda, \mu) \end{pmatrix} = \frac{1}{2}\begin{pmatrix} H(u + iv, \lambda, \mu) + H(u - iv, \lambda, \mu) \\ -i(H(u + iv, \lambda, \mu) - H(u - iv, \lambda, \mu)) \end{pmatrix}. \tag{4.1}$$

Then $g(u, v, \lambda, \mu)$ satisfies $\mathcal{Z}_2$-symmetry, with $S = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$, since

$$g^r(u, -v, \lambda, \mu) = g^r(u, v, \lambda, \mu), \quad g^i(u, -v, \lambda, \mu) = -g^i(u, v, \lambda, \mu).$$

It is easy to check that

$$X_s = \{(u,0) \mid u \in R^n\}, \quad X_a = \{(0,v) \mid v \in R^n\}.$$

It can be shown that, $(u_0, 0, \lambda_0, \mu_0)$ is a double S-breaking cubic turning point if and only if (u_0, λ_0, μ_0) is a simple cubic turning point of $h(u, \lambda, \mu) = 0$ ([1]).

For example, consider $n = 2$, and let $h(u, \lambda, \mu) = \begin{bmatrix} u_1^3 - \lambda + \mu e^{u_1} + 1 \\ (\mu+1)u_2 + u_2^3, \end{bmatrix}$. where $u = (u_1, u_2)$, $(0,0,1,0)$ is a simple cubic turning point of $h(u, \lambda, \mu) = 0$ with respect to λ. Consider the complexification of h.

$$H(x, \lambda, \mu) = \begin{bmatrix} u_1^3 - 3u_1v_1^2 - \lambda + \mu e^{u_1}\cos v_1 + 1 \\ (\mu+1)u_2 + u_2^3 - 3u_2v_2^2 \\ 3u_1^2v_1 - v_1^3 + \mu e^{u_1}\sin v_1 \\ (\mu+1)v_2 - v_2^3 + 3u_2^2v_2 \end{bmatrix} = 0 \tag{4.2}$$

where $x = (u_1, u_2, v_1, v_2)$. Let $x = (u_1, u_2, 0, 0) \in X_s$, $\phi_s = (t_1, t_2, 0, 0) \in X_s$, $\phi_a = (0, 0, t_3, t_4) \in X_a$, $z = (0, 0, z_1, z_2) \in X_a$. Then (3.3) can be written in the form

$$H_3(y) = \begin{bmatrix} u_1^3 - \lambda + \mu e^{u_1} + 1 \\ (\mu+1)u_2 + u_2^3 \\ (3u_1^2 + \mu e^{u_1})t_1 + \alpha t_1 \\ (\mu + 1 + 3u_2^2)t_1 + \alpha t_2 \\ (3u_1^2 + \mu e^{u_1})t_3 \\ (\mu + 1 + 3u_2^2)t_4 \\ (3u_1^2 + \mu e^{u_1})z_1 + (6u_1 + \mu e^{u_1})t_1t_3 \\ (\mu + 1 + 3u_2^2)z_2 + 6u_2t_2t_4 \\ t_1 - 1 \\ t_3 - 1 \\ z_1 \end{bmatrix} = 0$$

where $y = (u_1, u_2, t_1, t_2, t_3, t_4, z_1, z_2, \lambda, \mu, \alpha)$. Newton's method is applied. The numerical results are shown in Table 4.1.

Table 4.1

iteration	u_1	u_2	λ	μ	α	$\|\delta y\|$
0	0.500000	0.500000	1.500000	0.500000	0.200000	
1	-0.249999	-0.063287	2.062497	1.284792	0.000000	0.784E+00
2	-0.025000	-0.038281	1.176562	-0.096473	-0.000001	0.308E+00
3	-0.000000	0.000018	1.000000	-0.000001	-0.000000	0.332E-01
4	0.000000	0.000000	1.000000	-0.000000	0.000000	0.140E-03
5	0.000000	0.000000	1.000000	0.000000	0.000000	0.116E-09

Example 4.2. Consider the following two-point boundary value problem

$$\begin{cases} x'' + 4\pi^2\lambda x + (x - \lambda\cos 2\pi t)^3 + 100\pi\mu\cos 2\pi t + 4\mu x = 0, \ 0 < t < 1, \\ x(0) = x(1), \quad x'(0) = x'(1). \end{cases} \tag{4.3}$$

Let

$$X = \{x \in C^2(0,1) | x(0) = x(1), x'(0) = x'(1)\}.$$

It is easy to check that (4.3) is Z_2-symmetric with $S : Sx(t) = x(1-t)$. Thus X is splitting into:

$$X = X_s \oplus X_a$$

where

$$X_s = \{x \in X | x(1-t) = x(t), x'(0) = x'(1) = 0\}$$
$$X_a = \{x \in X | x(1-t) = -x(t), x(0) = x(1) = 0\}$$

In order to discretize (4.3) we use the central differences on the mesh points $x_j = jh$ $(j = 1, \cdots, N-1)$, where $Nh = 1$, and we use the following to discretize $x'(0)$,

$$2 \cdot \frac{x_1 - x_0}{h} - \frac{x_2 - x_0}{2h}.$$

Similarly to $x'(1)$, we take $N = 60$.

First we use the extended system $F_1(x, \phi, \lambda, \mu) = 0$ restricted on $X_s \times X_s \times R^2$ (cf (3.13)). We can get the following quadratic turning points of $f(x, \lambda, \mu) = 0$ with respect to λ, which are on l_s, as follows by varying μ:

Table 4.2

μ	λ	$x(0)$	$x(\frac{1}{2})$
0.06	0.765439	2.766062	-2.766062
0.04	0.832624	2.522726	-2.522726
0.02	0.906216	2.170575	-2.170575
0.01	0.947741	1.889519	-1.889519
0.005	0.971052	1.668516	-1.668516

when $\mu = 0.0$, the system $F_1 = 0$ is not solved by Newton's method. Thus, we turn to solve the system (3.1) by taking the last row of Table 4.2 as initial estimate. We get the following solution:

Table 4.3

μ	λ	$x(0)$	$x(\frac{1}{2})$
0.000000	0.999268	0.999229	-0.999229

Similarly, the system $F_2(x, \phi, \lambda, \mu) = 0$ restricted on $X_s \times X_a \times R^2$ is used to get the following pitchfork bifurcation points of $f(x, \lambda, \mu) = 0$ with respect to λ, which are on l_a, by varying μ:

Table 4.4

μ	λ	$x(0)$	$x(\frac{1}{2})$
0.03	0.606641	5.171305	-5.171305
0.02	0.799543	4.037104	-4.037104
0.01	0.911529	3.049549	-3.049549
0.005	0.957533	2.428533	-2.428533

At $\mu = 0.0$, Newton's method also doesn't work for $F_2 = 0$. The system (3.2) is applied by taking the last row of Table 4.4 as the initial guess and the following solution is obtained:

Table 4.5

μ	λ	$x(0)$	$x(\frac{1}{2})$
0.000023	0.999084	0.999197	-0.999197

Using the value in Table 4.3 and Table 4.5 as the initial guess for the system (3.3), we can get the following solution by Newton's method:

Table 4.6

μ	λ	$x(0)$	$x(\frac{1}{2})$
0.000023	0.999084	0.999016	-0.999016

Table 4.3, 4.5, 4.6 may be regarded as the same point, that is the double S-breaking cubic turning point of $f(x, \lambda, \mu) = 0$ with respect to λ under considering the perturbation of discretization. In fact, we can check that $(x, \lambda, \mu) = (\cos 2\pi t, 1, 0)$ is a double S-breaking cubic turning point of (4.1) with $\phi_1 = \cos 2\pi t, \phi_2 = \sin 2\pi t$.

References

[1] E.L. Allgower, Bifurcation arising in the calculation of critical points via homotopy methods, *ISNM70*, 1984, 15–28.

[2] M. Golubitsky, I. Stewart and D. Schaeffer, Singularities and Groups in Bifurcation Theory, Vol.II, Springer-Verlag, 1985.

[3] Ma Ya-nan, Projective approximation of double limit points for nonlinear problems, *J. Comput. Math.*, **8** : 3 (1990), 261–275.

[4] A. Spence and B. Werner, Non-simple turning points and cusps, *IMA J. Numer. Anal.*, 2 (1982), 413–427.

[5] B. Werner, Regular systems for bifurcation points with underlying symmetries, *ISNM70*, (1984), 562–574.

[6] B. Werner, and A. Spence, The computation of symmetry-breaking bifurcation points, *SIAM J. Numer. Anal.*, **21** : 2 (1984), 388-399.

[7] Wu Wei, Wu Bai-sheng, Li Rong-hua, Bifurcation of singular points near a double fold point in Z_2-symmetric nonlinear equations, *Numerical Mathematics, a Journal of Chinese Universities.* **2** : 1 (1993), 101-115.

[8] Yang Zhong-hua, A direct method for pitchfork bifurcation points, *J. Comput. Mathe.*, **9** : 2 (1991), 149-153.