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LONG TIME ASYMPTOTIC BEHAVIOR OF SOLUTION OF DIFFERENCE SCHEME FOR A SEMILINEAR PARABOLIC EQUATION (II)*1)

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Abstract

In this paper we prove the solution of explicit difference scheme for a semilinear parabolic equation converges to the solution of difference scheme for the relevant nonlinear stationary problem as $t \to \infty$. For nonlinear parabolic problem, we obtain the long time asymptotic behavior of its discrete solution which is analogous to that of its continuous solution. For simplicity, we discuss one-dimensional problem.

Key words: Asymptotic behavior, Explicit difference scheme, Semilinear parabolic equation.

1. Introduction

Let $\Omega = (0,l)$, $f(x) \in H^1(\Omega)$, $u_0(x) \in H^2(\Omega) \cap H_0^1(\Omega)$, $\phi(u) = u^3$, we consider the following initial-boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} - \phi(u) + f(x) \quad \text{in} \quad \Omega \times R_+ \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

By the usual approach[1–4] we can get the global existence of the solution of (1.1), furthermore, the solution of (1.1) converges to the solution of the following stationary problem (1.2) as $t \to \infty$.

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} - \phi(u) + f(x) = 0 \quad in \quad \Omega \\ u(0,t) = u(l,t) = 0. \end{cases} \tag{1.2}$$

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In [6], [7], the authors considered the explicit scheme for (1.1) as $f(x) = 0$ and only the estimate in L_2 for discrete solution was obtained.

In this paper we prove that the solution of explicit difference scheme for (1.1) converges to the solution of difference scheme for (1.2) as $t \to \infty$.

2. Finite Difference Scheme

The domain Ω is divided into small segments by points $x_j = jh$ $(j = 0, 1, \cdots, J)$, where $Jh = l$, J is an integer and h is the stepsize. Let Δt be time stepsize. For any function $w(x,t)$ we denote the values $w(jh, n\Delta t)$ by w_j^n $(0 \le j \le J,\ n = 0, 1, 2, \cdots)$ and denote the discrete function $w_j^n (0 \le j \le J,\ n = 0, 1, 2, \cdots)$ by w_h^n. We introduce the following notations: $\Delta_+ w_j^n = w_{j+1}^n - w_j^n$ $(0 \le j \le J-1, n = 0, 1, 2, \cdots)$ and $\Delta_- w_j^n = w_j^n - w_{j-1}^n$ $(1 \le j \le J, n = 0, 1, 2, \cdots)$. We denote the discrete function $\frac{\Delta_+ w_j^n}{h}$ $(0 \le j \le J-1, n = 0, 1, 2, \cdots)$ by δw_h^n. Similarly, the discrete function $\frac{\Delta_+^2 w_j^n}{h^2}$ $(0 \le j \le J-2, n = 0, 1, 2, \cdots)$ is denoted by $\delta^2 w_h^n$.

Denote the scalar product of two discrete functions u_h^n and v_h^m by $(u_h^n, v_h^m) = \sum_{j=0}^{J} u_j^n v_j^m h$.

For $2 \ge k \ge 0$, define discrete norms $\|\delta^k w_h^n\|_p = \Big(\sum_{j=0}^{J-k} \Big|\frac{\Delta_+^k w_j^n}{h^k}\Big|^p h\Big)^{\frac{1}{p}}$, $+\infty > p > 1$ and $\|\delta^k w_h^n\|_\infty = \max_{j=0,1,\cdots,J-k} \Big|\frac{\Delta_+^k w_j^n}{h^k}\Big|$.

The difference equation associate with (1.1) is:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} = \frac{\Delta_+ \Delta_- u_j^n}{h^2} - \phi(u_j^n) + f_j \tag{2.1}$$

for $j = 1, \cdots, J-1$ and $n = 1, 2, \cdots\cdots$, where $f_j = f(x_j), j = 1, \cdots, J-1$,

The boundary condition of (2.1) is of the form $u_0^n = u_J^n = 0$.

The discrete form corresponding to (1.2) is:

$$\begin{aligned} &\frac{\Delta_+ \Delta_- u_j^*}{h^2} - \phi(u_j^*) + f_j = 0, \quad 0 < j < J \\ &u_0^* = u_J^* = 0 \end{aligned} \tag{2.2}$$

Let the discrete function u_h^n and u_h^* be the solution of difference equation (2.1) and (2.2) respectively. For $n = 0, 1, 2, \cdots$, the discrete function $v_h^n = \{v_j^n \mid j = 0, 1, \cdots, J\}$ is defined as $v_j^n = u_j^n - u_j^* (j = 0, 1, \cdots, J)$. Then v_h^n satisfies

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} = \frac{\Delta_+ \Delta_- v_j^n}{h^2} - [(u_j^n)^3 - (u_j^*)^3] \tag{2.3}$$

for $j = 1, \cdots, J-1$ and $n = 0, 1, 2, \cdots$ Obviously, $v_0^n = v_J^n = 0, n = 0, 1, 2, \cdots$

3. Preliminary Results

Lemma 1. *For any discrete function $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$ satisfying the homogeneous discrete boundary condition $u_0 = u_J = 0$, we have $\|u_h\|_2 \le k_1\|\delta u_h\|_2$, $\|\delta u_h\|_2 \le k_1\|\delta^2 u_h\|_2$, where k_1 is a constant independent of u_h and h.*

Proof. The first inequality is from [5], since

$$\sum_{j=0}^{J-1} (\Delta_+ u_j)^2 = -\sum_{j=1}^{J-1} u_j \Delta_+ \Delta_- u_j,$$

we can get the second inequality. ■

By [5], we have the following Lemma 2:

Lemma 2. *For any discrete function $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$, there is $\|\delta^k u_h\|_\infty \le k_2 \|u_h\|_2^{1-\frac{2k+1}{2n}} (\|\delta^n u_h\|_2 + \|u_h\|_2)^{\frac{2k+1}{2n}}$, where $0 \le k < n$ and k_2 is a constant independent of u_h and h.*

Lemma 3. *Let the discrete function $u_h^* = \{u_j^* \mid j = 0, 1, \cdots J\}$ be the solution of the difference equation (2.2), there are*

$$\|\delta^2 u_h^*\|_2 \le k_3,$$
$$\|\delta u_h^*\|_\infty \le k_4, \quad \|u_h^*\|_\infty \le k_5,$$

where k_3, k_4, k_5 are constants independent of h.

Proof. From (2.2) it follows that

$$\sum_{j=1}^{J-1} \Big(\frac{\Delta_+\Delta_- u_j^*}{h^2}\Big)^2 h - \sum_{j=1}^{J-1} \frac{\Delta_+\Delta_- u_j^*}{h^2} (u_j^*)^3 h + \sum_{j=1}^{J-1} f_j \frac{\Delta_+\Delta_- u_j^*}{h^2} h = 0.$$

Since

$$\begin{aligned}\sum_{j=1}^{J-1} (u_j^*)^3 \frac{\Delta_+\Delta_- u_j^*}{h^2} h &= -\sum_{j=1}^{J-1} [(u_{j+1}^*)^3 - (u_j)^3] \frac{u_{j+1}^* - u_j^*}{h^2} h \\ &= -\sum_{j=0}^{J-1} (u_{j+1}^* - u_j^*)^2 \frac{(u_{j+1}^*)^2 + u_{j+1}^* u_j^* + (u_j^*)^2}{h^2} h \le 0,\end{aligned}$$

we have

$$\sum_{j=1}^{J-1} \frac{\Delta_+\Delta_- u_j^*}{h^2}^2 h \le \sum_{j=1}^{J-1} f_j^2 h \tag{3.1}$$

By (3.1) and the previous Lemmas, we complete the proof. ■

Lemma 4. *For any discrete function $u_h = \{u_j \mid j = 0, 1, \cdots, J\}$ satisfying the homogeneous discrete boundary condition $u_0 = u_J = 0$, we have*

$$\|u_h\|_\infty^2 \le \frac{4}{h} \|u_h\|_2^2.$$

Proof. By [5],

$$\|u_h\|_\infty^2 = \max_j \mid u_j \mid^2 \le 2\|u_h\|_2 \|\delta u_h\|_2,$$

it is obvious that

$$\|\delta u_h\|_2^2 \le \frac{4}{h^2}\|u_h\|_2^2,$$

which implies the lemma. ■

Lemma 5. *Let the discrete function u_h^n and u_h^* be the solution of difference equation* (2.1) *and* (2.2) *respectively. For given $\epsilon \in (0,1), \epsilon_0 \in (0,1)$, if $\Delta t, h$ satisfy*

$$\frac{2(1+\epsilon)\Delta t}{h^2} \le 1 - \epsilon_0, \tag{3.2}$$

there exist positive constants k_6 and α independent of $h, n, \Delta t$ such that $\|u_h^n - u_h^*\|_2^2 \le k_6 e^{-\alpha n \Delta t}$.

Proof. Similar to [6] and [7].

Lemma 6. *Let the discrete function u_h^n be the solution of difference equation* (2.1)*. If $\Delta t, h$ satisfy* (3.2)*, there exists constant $k_7 > 0$ independent of $h, n, \Delta t$ such that $\|u_h^n\|_\infty \le k_7$.*

Proof. Define the discrete function $w_h^n, n = 0, 1, 2, \cdots$ such that

$$u_j^n = w_j^n + a x_j (l - x_j),$$

where $a \ge \dfrac{\|f\|_\infty}{2}$. It is evident that

$$\frac{w_j^{n+1} - w_j^n}{\Delta t} = \frac{\Delta_+ \Delta_- w_j^n}{h^2} - 2a - (u_j^n)^2 (w_j^n + a x_j (l - x_j)) + f_j,$$

this inequality is equivalent to

$$\begin{aligned} w_j^{n+1} =& \frac{\Delta t}{h^2}(w_{j+1}^n + w_{j-1}^n) + \Big(1 - \frac{2\Delta t}{h^2}\Big) w_j^n - 2a\Delta t + f_j \Delta t - (u_j^n)^2 (w_j^n + a x_j (l - x_j))\Delta t \\ =& \frac{\Delta t}{h^2}(w_{j+1}^n + w_{j-1}^n) + \Big(1 - \frac{2\Delta t}{h^2} - (u_j^n)^2 \Delta t\Big) w_j^n \\ & - 2a\Delta t + f_j \Delta t - (u_j^n)^2 a x_j (l - x_j)\Delta t. \end{aligned} \tag{3.3}$$

By Lemma 4,

$$(u_j^n)^2 \le 2\Big(\frac{4}{h}\|v_h^n\|_2^2 + k_5^2\Big),$$

then if $\Delta t, h$ satisfy (3.2), from Lemma 5,

$$1 - \frac{2\Delta t}{h^2} - (u_j^n)^2 \Delta t \ge 0. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$w_j^{n+1} \le (1 - (u_j^n)^2 \Delta t) \max\{w_{j-1}^n, w_j^n, w_{j+1}^n\}. \tag{3.5}$$

The inequality (3.5) yields

$$\max_{1\le j\le J-1} w_j^{n+1} \le \begin{cases} \max\limits_{1\le j\le J-1} w_j^n, & \text{when } \max\limits_{1\le j\le J-1} w_j^n \ge 0, \\ 0, & \text{when } \max\limits_{1\le j\le J-1} w_j^n < 0, \end{cases}. \tag{3.6}$$

By (3.6), there is a constant T_1 independent $\Delta t, h, n$ such that

$$\max_{1\le j\le J-1} w_j^n \le T_1. \tag{3.7}$$

Similarly, there is a constant T_2 independent $\Delta t, h, n$ such that

$$\min_{1\le j\le J-1} w_j^n \ge T_2, \tag{3.8}$$

the Lemma follows from (3.7) and (3,8). ■

A simple computation shows that

Lemma 7. *Suppose the sequence* $\{a_n\}$ *satisfies*

$$a_{n+1} \le e^{-c_1\Delta t} a_n + c_2 e^{-c_3(n+1)\Delta t}\Delta t,$$

where $a_n \ge 0, \forall n \in N, c_i > 0, i = 1,2,3,$ *then there exist* $c_4 > 0, \sigma > 0$ *such that* $a_n \le c_4 e^{-\sigma n\Delta t}$.

4. Asymptotic Behavior of Explicit Difference Solution

In this section, we intend to study the asymptotic behavior of solution of (2.1). By difference equation (2.3), we have

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + 2\Delta t\|\delta^2 v_h^n\|_2^2 = 2\Delta t\sum_{j=1}^{J-1}\Big((u_j^n)^3 - (u_j^8)^3\Big)\frac{\Delta_+\Delta_- v_j^n}{h^2}h + \|\delta(v_h^{n+1} - v_h^n)\|_2^2$$

From Lemma 1 it follows that there exists $\theta > 0$ such that

$$\begin{aligned}\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \Big(2 - \frac{\epsilon+\epsilon_0}{2(1+\epsilon)}\Big)\Delta t\|\delta^2 v_h^n\|_2^2 + \theta\Delta t\|\delta v_h^n\|_2^2 \\ \le 2\Delta t\sum_{j=1}^{J-1}((u_j^n)^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^n}{h^2}h + \|\delta(v_h^{n+1} - v_h^n)\|_2^2\end{aligned}$$

Notice that $\|\delta(v_h^{n+1} - v_h^n)\|_2^2 \le \frac{4\Delta t^2}{h^2}\Big\|\frac{v_h^{n+1} - v_h^n}{\Delta t}\Big\|_2^2$, if $\Delta t, h$ satisfy (3.2), we have

$$\begin{aligned}&\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \Big(2 - \frac{\epsilon+\epsilon_0}{2(1+\epsilon)}\Big)\Delta t\|\delta^2 v_h^n\|_2^2 + \theta\Delta t\|\delta v_h^n\|_2^2 \\ &\le 2\Delta t\sum_{j=1}^{J-1}((u_j^n)^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^n}{h^2}h + \frac{4\Delta t^2}{h^2}[\|\delta^2 v_h^n\|_2^2 \\ &\quad + \sum_{j=1}^{J-1}(v_j^n)^2\Big[H(u_j^n, u_j^*)]^2 h - 2\sum_{j=1}^{J-1}((u_j^n)^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^n}{h^2}h\Big] \\ &\le 2\Delta t\sum_{j=1}^{J-1}((u_j^n)^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^n}{h^2}h + \frac{2(1-\epsilon_0)}{1+\epsilon}\Delta t[\|\delta^2 v_h^n\|_2^2 \\ &\quad + \sum_{j=1}^{J-1}(v_j^n)^2\Big[H(u_j^n, u_j^*)\Big]^2 h - 2\sum_{j=1}^{J-1}((u_j^n)^3 - (u_j^*)^3)\frac{\Delta_+\Delta_- v_j^n}{h^2}h\Big]\end{aligned}$$

$$\leq \Big(2 - \frac{\epsilon + \epsilon_0}{1+\epsilon}\Big)\Delta t \|\delta^2 v_h^n\|_2^2 + C\Delta t \sum_{j=1}^{J-1} (v_j^n)^2 [H(u_j^n, u_j^*)]^2 h. \tag{4.1}$$

By Lemma 3 and Lemma 6, $H(u_j^n, u_j^*) \leq 2\|u_h^n\|_\infty^2 + 2\|u_h^*\|_\infty^2 \leq 2(k_7^2 + k_5^2)$, then by (4.1), there exists constant μ independent of $h, n, \Delta t$ such that

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \frac{\epsilon + \epsilon_0}{2(1+\epsilon)}\Delta t \|\delta^2 v_h^n\|_2^2 + \theta \Delta t \|\delta v_h^n\|_2^2 \leq \mu \Delta t \|v_h^n\|_2^2. \tag{4.2}$$

From (4.2) and Lemma 5, there is constant ρ independent of $h, n, \Delta t$ such that

$$\|\delta v_h^{n+1}\|_2^2 - \|\delta v_h^n\|_2^2 + \theta \Delta t \|\delta v_h^n\|_2^2 \leq \rho \Delta t e^{-\alpha n \Delta t}. \tag{4.3}$$

Therefore by Lemma 7, we have

Theorem 1. *Let the discrete function u_h^n and u_h^* be the solution of difference equation* (2.1) *and* (2.2) *respectively. If $\Delta t, h$ satisfy* (3.2)*, there exist constants $M_1 > 0, \beta > 0$ independent of $h, n, \Delta t$ such that $\|\delta(u_h^n - u_h^*)\|_2^2 \leq M_1 e^{-\beta n \Delta t}$.*

By (4.2), it suffices to show that from Theorem 1:

Theorem 2. *Let the discrete function u_h^n and u_h^* be the solution of difference equation* (2.1) *and* (2.2) *respectively. If $\Delta t, h$ satisfy* (3.2)*, for any positive integer s, there exist constants $M_2 > 0, \lambda > 0$ independent of $h, n, \Delta t$ such that*

$$\sum_{i=0}^{s} \|\delta^2 (u_h^{n+i} - u_h^*)\|_2^2 \Delta t \leq M_2 e^{-\lambda n \Delta t}.$$

Remark. Let u^* be the solution of (1.2), $\phi_h = \{\phi_j \mid j = 0, 1, \cdots, J\}$ be the discrete function satisfies $\phi_j = u^*(x_j), j = 0.1, \cdots, J$. By the well-known energy method, there is $C > 0$ such that $\|\delta(u_h^* - \phi_h)\|_2 \leq Ch^2$.

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