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MULTIGRID METHODS FOR MORLEY ELEMENT ON NONNESTED MESHES*[1)]

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Abstract

In this paper, we consider some multigrid algorithms for the biharmonic problem discretized by Morley element on nonnested meshes. Through taking the averages of the nodal variables we construct an intergrid transfer operator that satisfies a certain stable approximation property. The so-called regularity-approximation assumption is then established. Optimal convergence properties of the W-cycle and a uniform condition number estimate for the variable V-cycle preconditioner are presented. This technique is applicable to other nonconforming plate elements.

Key words: Multigrid method, Morley element, Nonnested meshes.

1. Introduction

We consider some multigrid algorithms for the biharmonic equation discretized by Morley element on nonnested meshes. To define a multigrid algorithm, certain intergrid transfer operator has to be constructed. Through taking the averages of the nodal variables, we construct an intergrid transfer operator for Morley element on nonnested meshes that satisfies a certain stable approximation property which plays a key role in multigrid methods for nonconforming plate elements on nonnested meshes. The so-called regularity-approximation assumption is established by using the stable approximation property of the intergrid transfer operator. Optimal convergence properties of the W-cycle and a uniform condition number estimate for the variable V-cycle preconditioner are obtained by applying the abstract theory of Bramble, Pasciak and Xu [2]. This technique is applicable to other nonconforming plate elements.

There are some earlier papers on multigrid methods for nonconforming plate elements. Peisker and Braess [6] considered the $W-$cycle for the Morley element. Brenner [3] studied the W-cycle for Morley element through defining the intergrid transfer operator by taking the averages of the nodal variables and simplified the algorithms and analysis. Shi, Yu and Xie [8] studied the W-cycle for Bergan's energy-orthogonal plate

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element through defining the intergrid transfer operator by taking a linear combination of the nodal parameters of the same coarse grid element. Recently, Bramble [1] discussed variable V-cycle preconditioner for Morley element. All these papers consider the case when the triangulations are nested.

The paper is organized as follows. In section 2, we briefly describe the Morley approximation of the biharmonic Dirichlet problem. In section 3, we define an intergrid transfer operator and establish a certain stable approximation property of the intergrid transfer operator using a direct technique [9]. In section 4, we describe the multigrid methods, and establish the optimal convergence properties of the W-cycle and a uniform condition number estimate for the variable V-cycle preconditioner for Morley element on nonnested meshes.

2. Morley Element Approximation

We consider the biharmonic problem in Ω with Dirichlet boundary conditions $\Delta^2 u = f$, in Ω and $u = \dfrac{\partial u}{\partial n} = 0$, on $\partial\Omega$, where Ω is a convex polygon in R^2, $f \in H^{-l}(l = 0, 1)$.

The variational form of the problem is: Find $u \in H_0^2(\Omega)$ such that

$$a(u, v) = (f, v), \ \forall v \in H_0^2(\Omega), \tag{2.1}$$

where

$$a(u, v) = \sum_{|\alpha|=2} \int_\Omega D^\alpha u D^\alpha v dx, \ (f, v) = \int_\Omega f v dx.$$

Let $\{\Gamma_k\}, k \geq 1$, be a family of quasi-uniform triangulations of Ω. Let $h_k = \max\{\text{diam}\tau;\ \tau \in \Gamma_k\}$. We allow nonnested triangulations; however, we assume that the mesh parameters h_k satisfy $0 < \gamma_1 \leq h_{k+1}/h_k \leq \gamma_2 < 1$, where $\gamma_i (i = 1, 2)$ are constants independent of k. From this assumption we see that for $\tau \in \Gamma_k$, the number of elements $\{\tau' \in \Gamma_{k-1}$ or $\tau' \in \Gamma_{k+1};\ \bar{\tau}' \cap \tau \neq \phi\}$ is finite and is independent of k. Let V_k be Morley element space with respect to Γ_k [4,7] such that

a) for each triangle $\tau \in \Gamma_k$, $u|_\tau$ is a quadratic polynomial,

b) u is continuous at vertices and vanishes at vertices along $\partial\Omega$,

c) the normal derivative $\dfrac{\partial u}{\partial n}$ is continuous at the midpoints of each $\tau \in \Gamma_k$ and vanishes at midpoints along $\partial\Omega$.

The finite element method of the problem (2.1) is: Find $u_k \in V_k$ such that

$$a_k(u_k, v_k) = (f, v), \ \forall v \in V_k, \tag{2.2}$$

where

$$a_k(u, v) = \sum_{\tau\in\Gamma_k} \sum_{|\alpha|=2} \int_\tau D^\alpha u D^\alpha v dx.$$

Denote the induced norm $\|u\|_{2,h_k} = (a_k(u, u))^{1/2}$. Let Π_k be the nodal interpolation operator of Morley element from $H^3(\Omega) \cap H_0^2(\Omega)$ onto V_k. The following estimate for the interpolation error is known (cf.[4, 7]):

$$\|w - \Pi_k w\|_{2,h_k} \leq C h_k |w|_{H^3(\Omega)} \tag{2.3}$$

for all $w \in H^3(\Omega) \cap H_0^2(\Omega)$. Through this paper we let C (with or without subscripts) be a generic positive constant independent of the mesh parameter k. The following error estimate of Morley element is known [7]

$$\|u-u_k\|_{2,h_k} \le Ch_k(\|u\|_{3,\Omega}+h_k\|f\|_{0,\Omega}), \tag{2.4}$$

where and from now on $\|u\|_{i,\Omega}=\|u\|_{H^i(\Omega)}$.

3. Intergrid Transfer Operator

The intergrid transfer operator from a coarse grid to fine grid plays an important role in the analysis of multigrid methods.

For Morley element on nested meshes, Brenner [3] has defined an intergrid transfer operator by taking averages of the nodal parameters between two adjacent elements. For Bergan's energy-orthogonal plate element, Shi, Yu and Xie [8] defined an intergrid transfer operator by taking a linear combination of the nodal parameters of the same coarse grid element. For Morley element on nonnested meshes, we now define an intergrid transfer operator $I_k : V_{k-1} \longrightarrow V_k$ as follows.

For $v \in V_{k-1}$, $I_k v \in V_k$ is defined so that

a) if p is a vertex of Γ_k which is also a vertex of Γ_{k-1} or in the interior of $\tau \in \Gamma_{k-1}$, then $(I_k v)(p) = v(p)$;

b) for other vertices p of Γ_k, v may have jumps at p and $I_k v$ takes the average of all values of v at p;

c) if m is a midpoint of an edge of Γ_k which is in the interior of $\tau \in \Gamma_{k-1}$, then $\dfrac{\partial(I_k v)}{\partial n}(m) = \dfrac{\partial v}{\partial n}(m)$;

d) for other midpoints m associated with Γ_k, $\dfrac{\partial v}{\partial n}$ may have jumps and $\dfrac{\partial(I_k v)}{\partial n}(m)$ takes the average value of $\dfrac{\partial(v)}{\partial n}$ at m.

Our analysis is based on the three properties of the intergrid transfer operator I_k as follows

$$\|I_k v\|_{2,h_k} \le C\|v\|_{2,h_{k-1}}, \quad \forall v \in V_{k-1}, \tag{3.1}$$

$$\|u_k - I_k u_{k-1}\|_{2,h_k} \le Ch_k(\|u\|_{3,\Omega}+h_k\|f\|_0), \tag{3.2}$$

and

$$\|I^{k-1}v - I^k I_k v\|_{1,\Omega} \le Ch_k\|v\|_{2,h_{k-1}}, \quad \forall v \in V_{k-1}, \tag{3.3}$$

where u_k and u_{k-1} are Morley approximations to the solution u of (2.1) on Γ_k and Γ_{k-1}, respectively. $I^{k-1}v$ refers to the Γ_{k-1}-linear interpolation of v and $I^k I_k v$ is the Γ_k-linear interpolation of $I_k v$.

Brenner [3] proved (3.1)–(3.3) for Morley element on nested meshes. We will use a direct technique (cf.[9]) to prove that (3.1)–(3.3) are still valid on nonnested meshes.

Lemma 1. *Let G be the interior of the union of two adjacent triangles τ_1 and τ_2 in $\in \Gamma_{k-1}$. Let p be an arbitrary point on the common edge $\overline{p_1p_2}$ (cf. Figure 1). For $v \in V_{k-1}$, let $v_i = v|_{\tau_i}$. Then for $\forall v \in V_{k-1}$,*

$$\begin{cases} |v_1(p)-v_2(p)| \le Ch_{k-1}(|v|_{H^2(\tau_1)}+|v|_{H^2(\tau_2)}), \\ |\nabla v_1(p)-\nabla v_2(p)| \le C(|v|_{H^2(\tau_1)}+|v|_{H^2(\tau_2)}). \end{cases} \tag{3.4}$$

Proof. Using inverse estimates and the theory of discontinuous finite element in Feng [5] yields (3.4).

Lemma 2. *Given $w \in H^3(G)$, let w_1 (respectively w_2) be the Γ_{k-1}-Morley interpolation of w on τ_1 (respectively τ_2), i.e. $w_i = (\Pi_{k-1}w)|_{\tau_i}$ $(i = 1, 2)$. There exists a positive constant C such that for all $w \in H^3(\Omega)$*

$$\begin{cases} |w_1(p) - w_2(p)| \le Ch_{k-1}^2 |w|_{H^3(G)}, \\ |\nabla w_1(p) - \nabla w_2(p)| \le Ch_{k-1}|w|_{H^3(G)}. \end{cases} \tag{3.5}$$

Proof. Since $w \in H^3(G) \sqsubseteq C^1(\bar{G})$, (3.5) follows from standard interpolation error estimates(cf.[4]).

Lemma 3. (3.1) *holds.*

Proof. Let $\tau = \triangle p_1p_2p_3 \in \Gamma_k$. The essential step is to establish the estimate

$$|I_k v|^2_{H^2(\tau)} \le C \sum_{\substack{\bar{\tau}' \cap \bar{\tau} \ne \phi \\ \tau' \in \Gamma_{k-1}}} |v|^2_{2,\tau'}, \quad \forall v \in V_{k-1}. \tag{3.6}$$

It is easy to see that

$$|I_k v|^2_{H^2(\tau)} \le C \sum_{i=1}^{3} [\partial_n(I_k v)(m_i) - \partial_n(I^k I_k v)(m_i)]^2, \tag{3.7}$$

where m_i, $i = 1, 2, 3$, are the midpoints of the edges of the element τ.

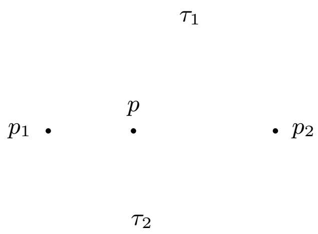

Figure 1

Figure 2

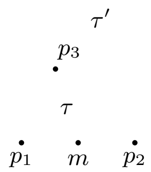

Figure 3

First we consider the case when τ belongs completely to a single $\tau' \in \Gamma_{k-1}$. In this case, if $\partial\tau \cap \partial\tau' = \phi$ (cf. Figure 2), then $I_k v = v$ and (3.6) holds. If $\partial\tau \cap \partial\tau' \ne \phi$, then there exists at least an edge of $\tau \subset \partial\tau \cap \partial\tau'$, say $\overline{p_1p_2}$ (cf. Figure 3), or $\partial\tau \cap \partial\tau' = \{p_1\}$ or $\partial\tau \cap \partial\tau' = \{p_1, p_2\}$ or $\partial\tau \cap \partial\tau' = \{p_1, p_2, p_3\}$ (cf. Figure 4).

Set $w = v|_\tau$. We first assume that there exists at least an edge of $\tau \subset \partial\tau \cap \partial\tau'$. Let m be the midpoint of an edge $\overline{p_1p_2}$ of τ (cf. Figure 3), where $\overline{p_1p_2}$ is an arbitrary common edge of two triangles τ and τ' belonging to Γ_{k-1}. Then from the definition of the operator $I_k v$, (3.4) in Lemma 1, the mean value theorem, quasi-uniform property of the triangulations, and an inverse estimate we have

$$\begin{aligned} |\partial_n(I_k v)(m) - \partial_n(I^k I_k v)(m)| &= |\overline{\partial_n(v)}(m) - \partial_n(I^k I_k v)(m)| \\ &\le C \sum_{\substack{\bar{\tau}' \cap \bar{\tau} \ne \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'} + |\partial_n w(p_1) - \partial_n(I^k I_k v)(p_1)|, \end{aligned} \tag{3.8}$$

where and from now on

$$\overline{\partial_n(v)}(m)=\frac{1}{n_m}\sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau'\in\Gamma_{k-1}}}\partial_n v|_{\tau'}(m),\ n_m \text{ is the number of } \{\tau'\in\Gamma_{k-1},\bar\tau'\cap\bar\tau\neq\phi\}.$$

Now we consider the second term on the right-hand side of (3.8). Using the mean value theorem and (3.4) in Lemma 1 yields

$$\begin{aligned}|\partial_{p_1p_2}w(p_1)-\partial_{p_1p_2}(I^kI_kv)(p_1)|&=\Big|\partial_{p_1p_2}w(p_1)-\frac{I_kv(p_2)-I_kv(p_1)}{|p_1p_2|}\Big|\\&\le\Big|\partial_{p_1p_2}w(p_1)-\frac{w(p_2)-w(p_1)}{|p_1p_2|}\Big|+\frac{|(I_kv(p_2)-w(p_2))-(I_kv(p_1)-w(p_1)|}{|p_1p_2|}\\&\le C\sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau'\in\Gamma_{k-1}}}|v|_{2,\tau'}.\end{aligned}$$

Similarly,

$$|\partial_{p_1p_3}w(p_1)-\partial_{p_1p_3}(I^kI_kv)(p_1)|\le C\sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau'\in\Gamma_{k-1}}}|v|_{2,\tau'}.$$

Therefore,

$$|\partial_nw(p_1)-\partial_n(I^kI_kv)(p_1)|\le C\sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau'\in\Gamma_{k-1}}}|v|_{2,\tau'}. \tag{3.9}$$

and hence

$$\begin{aligned}|\partial_n(I_kv)(m)-\partial_n(I^kI_kv)(m)|&\le C\sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau'\in\Gamma_{k-1}}}|v|_{2,\tau'}+|\partial_nw(p_1)-\partial_n(I^kI_kv)(p_1)|\\&\le C\sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau'\in\Gamma_{k-1}}}|v|_{2,\tau'}.\end{aligned}\tag{3.10}$$

Similarly, (3.10) holds for arbitrary edge midpoint m of τ belong to τ'. For the cases $\partial\tau\cap\partial\tau'=\{p_1\}$ or $\partial\tau\cap\partial\tau'=\{p_1,p_2\}$ or $\partial\tau\cap\partial\tau'=\{p_1,p_2,p_3\}$, we can discuss similarly. Therefore, (3.6) follows from (3.7), (3.8) and (3.10) in the first case.

Next we consider the case when τ does not belong completely to a single $\tau'\in\Gamma_{k-1}$.

Let $\overline{p_1p_2}$ be cut into l piecewises $p_1q_0,\cdots,q_1q_2,\cdots q_3p_2$ (cf. Figure 5x), by the coarse triangles $\tau_1,\cdots,\tau_l$ respectively, and $v(\cdot)$ is a polynomial on each piece, where $m\in\overline{q_1q_2}$, $m\in\bar\tau_{l_0}$. Set $v_i=v|_{\tau_i}$, $i=1,\cdots,l$. Let $\{\tau'\in\Gamma_{k-1};\ p_1\in\bar\tau'\}=\{\tau_1^{p_1},\cdots,\tau_{l_{p_1}}^{p_1}\}$, $\{\tau'\in\Gamma_{k-1};\ p_2\in\bar\tau'\}=\{\tau_1^{p_2},\cdots,\tau_{l_{p_2}}^{p_2}\}$, and $\{\tau'\in\Gamma_{k-1};\ m\in\bar\tau'\}=\{\tau_1^{m},\cdots,\tau_{l_m}^{m}\}$.

By the assumption on $\{\Gamma_k\}$, $l,l_{p_1},l_{p_2},l_m\le C$. Therefore, using the definition of the operator I_k, the triangle inequality and (3.4) in Lemma 1 yields

$$\begin{aligned}|\partial_nI_kv(m)-\partial_nI^kI_kv(m)|&=\Big|\overline{\partial_nv}(m)-\partial_n(I^kI_kv)(m)\Big|\\&\le\Big|\overline{\partial_nv}(m)-\partial_nv|_{\tau_{l_0}}(m)\Big|+\Big|\partial_nv|_{\tau_{l_0}}(m)-\partial_n(I^kI_kv)(m)\Big|\end{aligned}$$

$$\leq \left|\partial_n v|_{\tau_1}(p_1) - \partial_n v|_{\tau_{l_0}}(m)\right| + \left|\partial_n v|_{\tau_1}(p_1) - \partial_n (I^k I_k v)(p_1)\right|$$
$$+ C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'} \equiv I_1 + I_2 + C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'}, \tag{3.11}$$

where $m \in \tau_{l_0}$.

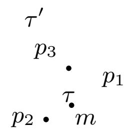

Figure 4

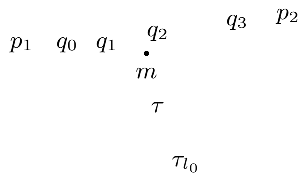

Figure 5

We now estimate I_1. Using the triangle inequality, (3.4) in Lemma 1, the mean value theorem, and inverse estimates yields

$$I_1 \leq |\partial_n v_1(p_1) - \partial_n v_1(q_0)| + |\partial_n v_1(q_0) - \partial_n v_2(q_0)|$$
$$+ \cdots + |\partial_n v_{l_0}(q_1) - \partial_n v_{l_0}(m)| \leq C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'}. \tag{3.12}$$

For I_2, similarly we have

$$\left|\partial_{p_1p_2} v_1(p_1) - \partial_{p_1p_2}(I^k I_k v(p_1)\right|$$
$$\leq \left|\partial_{p_1p_2} v_1(p_1) - \frac{1}{|p_1p_2|}\left(\frac{1}{l_{p_2}}\sum_{j=1}^{l_{p_2}} v|_{\tau_j^{p_2}}(p_2) - \frac{1}{l_{p_1}}\sum_{j=1}^{l_{p_1}} v|_{\tau_j^{p_1}}(p_1)\right)\right|$$
$$\leq \left|\partial_{p_1p_2} v_1(p_1) - \frac{(v_l(p_2) - v_1(p_1))}{|p_1p_2|}\right| + C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'}$$
$$\leq \left|\partial_{p_1p_2} v_1(p_1) - \frac{(v_l(p_2) - v_l(q_3)) + (v_l(q_3) - v_{l-1}(q_3)) + \cdots + (v_1(q_0) - v_1(p_1))}{|p_1p_2|}\right|$$
$$+ C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'} = |\partial_{p_1p_2} v_1(p_1) - (t_1 \partial_{p_1p_2} v_1(\xi_1) + \cdots + t_l \partial_{p_1p_2} v_l(\xi_l)|$$
$$+ C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'} \leq C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'}, \tag{3.13}$$

where $\xi_1 \in \overline{p_1q_0}, \cdots, \xi_l \in \overline{q_3p_2}, \cdots$, and $0 \leq t_i \leq 1$, $\sum t_i = 1$. Similarly,

$$|\partial_{p_1p_3} v|_1(p_1) - \partial_{p_1p_3}(I^k I_k v(p_1)| \leq C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'}. \tag{3.14}$$

Combining (3.13) with (3.14) yields

$$I_2 \le C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|^2_{2,\tau'}. \tag{3.15}$$

(3.6) follows from (3.7), (3.11), (3.12) and (3.15) for the second case.

Summing (3.6) over τ in Γ_k and noting that the number of repetitions, for each τ, in the summation is finite, yield (3.1).

Lemma 4. (3.2) *holds.*

Proof. By (2.3), (2.4), Lemma 1 and the method similar to Theorem 2 in [3], we can prove the Lemma.

Lemma 5. (3.3) *holds.*

Proof. For $v \in V_{k-1}$, we have

$$\|I^{k-1}v - I^k(I_k v)\|_{1,\Omega} \le \|I^{k-1}v - I^k(I^{k-1}v)\|_{1,\Omega} + \|I^k(I^{k-1}v) - I^k(I_k v)\|_{1,\Omega}. \tag{3.16}$$

Now we estimate the first term on the right-hand side of (3.16). Set $g = I^{k-1}v - I^k(I^{k-1}v)$. For $\tau = \triangle p_1p_2p_3 \in \Gamma_k$, we have

$$|g|^2_{H^1(\tau)} \le Ch_k^2 \sum_{\substack{\bar\tau \cap \bar\tau \neq \phi \\ \tau \in \Gamma_{k-1}}} |g|^2_{1,\infty,\tau'\cap\tau}. \tag{3.17}$$

For arbitrary $\tau_1, \tau_2 \in \Gamma_{k-1}$, $\partial\tau_1 \cap \partial\tau_2 \neq \phi$, $\bar\tau_1 \cap \bar\tau \neq \phi$ and $\bar\tau_2 \cap \bar\tau \neq \phi$, by using mean theorem and inverse estimates we can prove that

$$\begin{aligned} |\nabla g|_{\tau_1\cap\tau} - \nabla g|_{\tau_2\cap\tau}| &= |\nabla(I^{k-1}v)|_{\tau_1\cap\tau} - \nabla(I^{k-1}v)|_{\tau_2\cap\tau}| \\ &\le C(|v|_{2,\tau_1} + |v|_{2,\tau_2}) \le C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'}. \end{aligned} \tag{3.18}$$

For $\tau \in \Gamma_k$, set $\{\tau_1, \cdots, \tau_j, \cdots, \tau_{l_\tau}\} = \{\tau' \in \Gamma_{k-1};\ \bar\tau' \cap \bar\tau \neq \phi\}$. It follows from (3.18) that

$$|\nabla g|_{\tau_i\cap\tau} - \nabla g|_{\tau_j\cap\tau}| \le C \sum_{\substack{\bar\tau' \cap \bar\tau \neq \phi \\ \tau' \in \Gamma_{k-1}}} |v|_{2,\tau'} \quad (i, j \le l_\tau). \tag{3.19}$$

Since $g(p_1) = g(p_2) = 0$, using mean value theorem yields (cf. Figure 5)

$$\begin{cases} g(q_1) - g(p_1) = \partial_{p_1p_2} g(\xi_1)(q_1 - p_1) \\ g(q_2) - g(q_1) = \partial_{p_1p_2} g(\xi_2)(q_2 - q_1) \\ \cdots \\ g(p_2) - g(q_3) = \partial_{p_1p_2} g(\xi_l)(p_2 - q_3) \end{cases}$$

where $\xi_1, \cdots, \xi_l \in p_1q_0, \cdots, q_3p_2$ respectively. Therefore,

$$0 = \partial_{p_1p_2} g(\xi_1)t_1' + \partial_{p_1p_2} g(\xi_2)t_2' + \cdots + \partial_{p_1p_2} g(\xi_l)t_l', \tag{3.20}$$

where $\sum t_i' = 1, t_i' \ge 0$.

It follows from (3.19)–(3.20) that

$$|\partial_{p_1p_2}g|_{\tau_1\cap\tau}(p_1)| \le C \sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|_{2,\tau}, \tag{3.21}$$

and similarly,

$$|\partial_{p_1p_3}g|_{\tau^1_{p_1}\cap\tau}(p_1)| \le C \sum_{\substack{\bar\tau'\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|_{2,\tau}, \tag{3.22}$$

here the sense of τ_1 and τ' for p_1 are the same as the proof of Lemma 2.

It follows from (3.21)–(3.22) and Lemma 1 that

$$|\nabla g|_{\tau_1}(p_1)| \le \sum_{\substack{\bar\tau\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|_{2,\tau}.$$

Similarly,

$$|\nabla g|_{0,\infty,\tau} \le C \sum_{\substack{\bar\tau\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|_{2,\tau}. \tag{3.23}$$

Combining (3.23) with (3.17) yields

$$|g|^2_{H^1(\tau)} \le C \sum_{\substack{\bar\tau\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|^2_{2,\tau}. \tag{3.24}$$

Summing (3.24) over all τ in Γ_k and noting that number of repetitions, for each τ, in the summation is finite, yields

$$\|I^{k-1}v - I^kI^{k-1}v\|_1 \le Ch_k\|v\|_{2,h_{k-1}}. \tag{3.25}$$

It remains to estimate the second term on the right-hand side of (3.16).

Set $h = I^k(I^{k-1}v) - I^k(I_kv)$, then we have

$$|h|^2_{1,\tau} \le C(h(p_1) - h(p_2))^2 + (h(p_2) - h(p_3))^2) \le C(h(p_1)^2 + h(p_2)^2 + h(p_3)^2), \tag{3.26}$$

where $h(p_i) = I^{k-1}v(p_i) - I_kv(p_i)$. If p_i is a vertex of Γ_{k-1}, then $h(p_i) = 0$. If p_i is a point of the common edge of τ_1 and τ_2 which belong to Γ_{k-1}, then by Lemma 1 we have

$$\begin{aligned}|h(p_i)| &= \Big|\frac{1}{2}(v_1+v_2)(p_i) - I^{k-1}v_1(p_i)\Big| \le |v_1(p_i) - I^{k-1}v_1(p_i)| + \Big|\frac{1}{2}(v_1-v_2)(p_i)\Big|\\ &\le Ch_k(|v|_{2,\tau_1} + |v|_{2,\tau_2}) \le Ch_k \sum_{\substack{\bar\tau\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|_{2,\tau}. \end{aligned}\tag{3.27}$$

If p_i is an internal points of $\tau' \in \Gamma_{k-1}$, then

$$|h(p_i)| = |I^{k-1}v(p_i) - v(p_i)| \le Ch_k \sum_{\substack{\bar\tau\cap\bar\tau\neq\phi\\ \tau\in\Gamma_{k-1}}} |v|_{2,\infty,\tau}. \tag{3.28}$$

It follows from (3.26)-(3.28) that

$$|h|_{1,\tau}^2 \le Ch_k^2 \sum_{\substack{\bar{\tau}\cap\bar{\tau}\neq\phi \\ \tau\in\Gamma_{k-1}}} |v|_{2,\tau}^2. \tag{3.29}$$

Summing (3.29) over all τ in Γ_k yields

$$\|I^k I^{k-1}v - I^k I_k v\|_1 \le Ch_k \|v\|_{2,h_{k-1}}, \tag{3.30}$$

and hence (3.3) follows from (3.16), (3.25) and (3.30).

4. Multigrid Methods for Morley Element

Consider the discrete problem (2.2). Define $A_k : V_k \longrightarrow V_k$ by

$$(A_k u_k, v_k) = a_k(u_k, v_k), \quad \forall u_k, v_k \in V_k. \tag{4.1}$$

Let $R_k : V_k \longrightarrow V_k$ be a linear smoother and $R_k^s = R_k$ if s is odd and $R_k^{(s)} = R_k^t$ if s is even. Here R_k^t is the $(\cdot,\cdot)$ adjoint of R_k. The spaces V_{k-1} and V_k are related by the intergrid transfer operator $I_k : V_{k-1} \longrightarrow V_k$. Define projection operators $P_{k-1} : V_k \longrightarrow V_{k-1}$ and $Q_{k-1} : V_k \longrightarrow V_{k-1}$ by

$$a_{k-1}(P_{k-1}w, v) = a_k(w, I_{k-1}v) \tag{4.2}$$

and

$$(Q_{k-1}w, v) = (w, I_k v), \tag{4.3}$$

for all $v \in V_{k-1}$.

The multigrid operator $B_k : V_k \longrightarrow V_k$ is defined by induction as follows.

Algorithm Set $B_0 = A_0^{-1}$. Define $B_k g = y^{2m_k}$ in terms of B_{k-1} as follows:

(1) Set $x^0 = 0$, $q^0 = 0$ and define $x^s = x^{s-1} + R_k^{(s+m_k)}(g - A_k x^{s-1})$, $s = 1, \cdots, m_k$.

(2) Define $y^{m_k} = x^{m_k} + I_k q^p$, where q^i for $i = 1, \cdots, p$ is $q^i = q^{i-1} + B_{k-1}[Q_{k-1}(g - A_k x^{m_k}) - A_{k-1}q^{i-1}]$.

(3) Define y^s for $s = m_k + 1, \cdots, 2m_k$ by $y^s = y^{s-1} + R_k^{(s+m_k)}(g - A_k y^{s-1})$. Here m_k is the number of smoothing steps on level k. The case $p = 1$ and $p = 2$ corresponds to the V-cycle and the W-cycle, respectively.

Let Λ_k be the maximum eigenvalue of A_k. Using the estimates (3.1)–(3.3) we can prove (cf.[1]) that the regularity-approximation property [2] holds

$$|a_k((I - I_k P_{k-1})u, u)| \le C\Big(\frac{\|A_k u\|_0^2}{\Lambda_k}\Big)^{1/4}(a_k(u,u))^{3/4} \tag{A.1}$$

for the Morley element on a convex polygonal domain Ω.

Let $K_k = I - R_k A_k$ and $K_k^* = I - R_k^t A_k$ be the adjoint of K_k with respect to $a_k(\cdot,\cdot)$. Let $\bar{R}_k = (I - K_k^* K_k)A_k^{-!}$. We need the following two assumptions concerning the smoother and the number of smoothing steps.

$$C\Lambda_k^{-1}(u,u) \le (\bar{R}_k u, u), \quad \forall u \in V_k. \tag{A.2}$$

There exist β_0 and β_1, $1 < \beta_0 \leq \beta_1$ such that the smoothing steps for variable $V-$cycle satisfy

$$\beta_0 m_k \leq m_{k-1} \leq \beta_1 m_k. \tag{A.3}$$

Let δ_k or δ be the contraction number of the multigrid algorithm, that is $|a_k((I - B_kA_k)u, u)| \leq \delta_k a_k(u, u)$. A standard argument now yields the following two theorems.

Theorem 1. *If the smoother R_k satisfies $(A.2)$, and the number of smoothing steps $m_k \equiv m$ is sufficient large, but independent of k, then there exists a constant $M > 0$ such that the contraction number for $W-$cycle multigrid satisfies* $\delta \leq \dfrac{M}{M + m^{1/4}}$.

Theorem 2. *If the smoother R_k satisfies $(A.2)$ and the number of smoothing m_k satisfies $(A.3)$, then there exists a constant $M > 0$ such that the variable $V-$cycle preconditioner satisfies*

$$\frac{m_k^{1/4}}{M + m_k^{1/4}} a_k(u, u) \leq a_k(B_kA_ku, u) \leq \frac{M + m_k^{1/4}}{m_k^{1/4}} a_k(u, u).$$

Thus, the condition number of the matrix B_kA_k is unifoormly bounded, that is $Cond(B_kA_k) \leq \Big[\dfrac{M + m_k^{1/4}}{m_k^{1/4}}\Big]^2$.

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