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THE APPROXIMATIONS OF THE EXACT BOUNDARY CONDITION AT AN ARTIFICIAL BOUNDARY FOR LINEARIZED INCOMPRESSIBLE VISCOUS FLOWS*1)

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Abstract

We consider the linearized incompressible Navier-Stokes (Oseen) equations in a flat channel. A sequence of approximations to the exact boundary condition at an artificial boundary is derived. Then the original problem is reduced to a boundary value problem in a bounded domain, which is well-posed. A finite element approximation on the bounded domain is given, furthermore the error estimate of the finite element approximation is obtained. Numerical example shows that our artificial boundary conditions are very effective.

Key words: Oseen equations, Artificial boundary, Artificial boundary condition, Finite element approximation, Error estimate.

1. Introduction

Many problems arising in fluid mechanics are given in an unbounded domain, such as fluid flow around obstacles. When computing the numerical solutions of these problems, one often introduces artificial boundaries and sets up artificial boundary conditions on them. Then the original problem is reduced to a problem in a bounded computational domain. In order to limit the computational cost these boundaries must be not too far from the domain of interest. Therefore, the artificial boundary conditions must be good approximate to the "exact" boundary conditions (i.e. such that the solution of the problem in the bounded domain is equal to the solution of the original problem). Thus the accuracy of the artificial boundary conditions and the computational cost are closely related. It has often been studied during the last ten years to design artificial boundary conditions with high accuracy on a given artificial boundary for solving partial differential equations on an unbounded domain. For example, Goldstein[5], Feng[4], Han and Wu[14,15], Hagstrom and Keller[6,7], Halpern[8], Halpern and Schatzman[9], Nataf[17], Han, Lu and Bao[13], Han and Bao[11,12], Bao[1] and others have studied how to design the artificial boundary conditions for solving partial differential equations in an unbounded domain.

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In this paper we consider the linearized incompressible Navier-Stokes equations in a slip flat channel. It is an approximate problem of two-dimensional steady incompressible viscous folw around obstacles. We derived a solution which can be written in the form of Fourier series in the unbounded domain by the method of separation of variables. Then the exact and a series of approximate artificial boundary conditions are derived by the continuity of velocity and the normal stress at the artificial boundary. Therefore the original problem is reduced to a series of problems in a bounded computational domain. Particularly, a finite element approximation on the bounded domain is given, and the error estimate of the finite element approximation is obtained. Numerical example shows the effectiveness of the artificial boundary condition.

2. Oseen Equations and their Solution

Let Ω_i be an obstruction in a channel defined by $\mathbb{R}\times(0,L)$ and $\Omega=\mathbb{R}\times(0,L)\setminus\bar{\Omega}_i$. Consider the following Oseen equations:

$$a\frac{\partial u}{\partial x_1}+\nabla p=\nu\triangle u,\quad \text{in } \Omega, \tag{2.1}$$

$$\nabla\cdot u=0,\quad \text{in } \Omega, \tag{2.2}$$

with boundary conditions

$$u_2|_{x_2=0,L}=0,\ \sigma_{12}|_{x_2=0,L}=\nu\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)\Big|_{x_2=0,L}=0,\ -\infty<x_1<+\infty, \tag{2.3}$$

$$u|_{\partial\Omega_i}=0, \tag{2.4}$$

$$u(x)\to u_\infty=(a,0)^T,\quad \text{when}\quad x_1\to\pm\infty; \tag{2.5}$$

where $u=(u_1,u_2)^T$ is the velocity, p is the pressure, $\nu>0$ is the kinematic viscosity, $x=(x_1,x_2)^T$ is coordinate, $a>0$ is a constant and σ_{12} is the tangential stress on the wall. Obviously condition (2.3) is equivalent to the following condition:

$$\frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L}=u_2|_{x_2=0,L}=0. \tag{2.6}$$

Taking two constants $b<d$, such that $\Omega_i\subset(b,d)\times(0,L)$, then Ω is divided into three parts Ω_b, Ω_T and Ω_d by the artificial boundaries Γ_b and Γ_d with

$$\begin{aligned}
&\Gamma_b=\{x\in\mathbb{R}^2|\ x_1=b,\ 0\le x_2\le L\},\\
&\Gamma_d=\{x\in\mathbb{R}^2|\ x_1=d,\ 0\le x_2\le L\},\\
&\Omega_b=\{x\in\mathbb{R}^2|\ -\infty<x_1<b,\ 0<x_2<L\},\\
&\Omega_T=\{x\in\mathbb{R}^2|\ b<x_1<d,\ 0<x_2<L\}\setminus\bar{\Omega}_i,\\
&\Omega_d=\{x\in\mathbb{R}^2|\ d<x_1<+\infty,\ 0<x_2<L\}.
\end{aligned}$$

We now consider the Oseen equations on the unbounded domain Ω_d:

$$a\frac{\partial u}{\partial x_1}+\nabla p=\nu\triangle u,\quad \text{in } \Omega_d, \tag{2.7}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega_d, \tag{2.8}$$

$$\frac{\partial u_1}{\partial x_2}|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad d \leq x_1 < +\infty, \tag{2.9}$$

$$u(x) \to u_\infty, \quad \text{when } x_1 \to +\infty, \tag{2.10}$$

$$u|_{\Gamma_d} = u(d, x_2), \quad 0 \leq x_2 \leq L. \tag{2.11}$$

From (2.7)–(2.8), we have

$$\triangle\Big(\nu\triangle - a\frac{\partial}{\partial x_1}\Big)u_2 = 0. \tag{2.12}$$

Equation (2.12) with boundary condition (2.9) can be solved by the method of separation of variables. We obtain

$$u_2(x) = \sum_{m=1}^{\infty}\Big[a_m e^{-\frac{m\pi}{L}(x_1-d)} + b_m e^{\lambda^-(m)(x_1-d)}\Big]\sin\frac{m\pi x_2}{L}, \tag{2.13}$$

where

$$\lambda^-(m) = \frac{a - \sqrt{a^2 + 4\nu^2 m^2\pi^2/L^2}}{2\nu}, \quad m = 1, 2, 3, \cdots$$

Substituting (2.13) into (2.8) and (2.7) respectively, we obtain

$$u_1(x) = a + \sum_{m=1}^{\infty}\Big[a_m e^{-\frac{m\pi}{L}(x_1-d)} - \frac{m\pi}{L\lambda^-(m)}b_m e^{\lambda^-(m)(x_1-d)}\Big]\cos\frac{m\pi x_2}{L}, \tag{2.14}$$

$$p(x) = -a\sum_{m=1}^{\infty} a_m e^{-\frac{m\pi}{L}(x_1-d)}\cos\frac{m\pi x_2}{L}, \tag{2.15}$$

where we assume

$$\lim_{x_1\to+\infty} p(x) = p_\infty = 0.$$

Then (2.13)–(2.15) satisfy (2.7)–(2.10) for any constants a_1, b_1, a_2, b_2, $\cdots$ Therefore we derived a general solution of Oseen equations in the unbounded domain Ω_d.

3. The Exact Boundary Condition and its Approximations at the Artificial Boundary Γ_d

We now consider the following problem:

$$a\frac{\partial u}{\partial x_1} + \nabla p = \nu\triangle u, \quad \text{in } \Omega\setminus\bar{\Omega}_b, \tag{3.1}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega\setminus\bar{\Omega}_b, \tag{3.2}$$

$$\frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad b \leq x_1 < +\infty, \tag{3.3}$$

$$u|_{\partial\Omega_i} = 0, \tag{3.4}$$

$$u|_{\Gamma_b} = u_\infty, \tag{3.5}$$

$$u(x) \to u_\infty, \quad \text{when } x_1 \to +\infty. \tag{3.6}$$

Let $\varepsilon(u)=(\varepsilon_{ij}(u))_{2\times 2}$ and $\sigma(u,p)=(\sigma_{ij}(u,p))_{2\times 2}$ denote the rate of strain and stress tensors respectively. We have that

$$\varepsilon_{ij}(u)=\frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\Big),\quad i,j=1,2 \tag{3.7}$$

and

$$\sigma_{ij}(u,p)=-p\delta_{ij}+2\nu\varepsilon_{ij}(u),\quad i,j=1,2, \tag{3.8}$$

where δ_{ij} is the Kronecker Delta whose properties are

$$\delta_{ij}=\begin{cases}1, & i=j,\\ 0, & i\neq j.\end{cases}$$

Furthermore let $\sigma_n=(\sigma_{n_1},\sigma_{n_2})^T$ denote the normal stress on the artificial boundary Γ_d, then

$$\sigma_{n_1}=\sigma_{11}n_1+\sigma_{12}n_2=\sigma_{11}=-p+2\nu\frac{\partial u_1}{\partial x_1}\Big|\Gamma_d, \tag{3.9}$$

$$\sigma_{n_2}=\sigma_{21}n_1+\sigma_{22}n_2=\sigma_{21}=\nu\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)\Big|\Gamma_d, \tag{3.10}$$

where $n=(n_1,n_2)^T=(1,0)^T$ is the outward normal vector on Γ_d.

we now use the transmission conditions

$$u(d^-,x_2)=u(d^+,x_2), \tag{3.11}$$

$$\sigma_n(d^-,x_2)=\sigma_n(d^+,x_2) \tag{3.12}$$

to obtain the exact boundary condition and its approximations at the artificial boundary Γ_d. Substituting (2.13)–(2.15) into (3.9)–(3.10), we get

$$\sigma_{n_1}=\sum_{m=1}^{\infty}\Big[\Big(a-\frac{2\nu m\pi}{L}\Big)a_m-\frac{2\nu m\pi}{L}b_m\Big]\cos\frac{m\pi x_2}{L}, \tag{3.13}$$

$$\sigma_{n_2}=\nu\sum_{m=1}^{\infty}\Big[-\frac{2m\pi}{L}a_m+\Big(\lambda^-(m)+\frac{m^2\pi^2}{L^2\lambda^-(m)}\Big)b_m\Big]\sin\frac{m\pi x_2}{L}. \tag{3.14}$$

From (2.13)–(2.14) and (3.13)–(3.14), a computation shows:

$$\begin{aligned}\sigma_{n_1}=\sum_{m=1}^{\infty}\Big[&\frac{2\nu(-m\pi+L\lambda^-(m))}{L^2}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\\ &-\frac{2\nu m\pi(m\pi+L\lambda^-(m))}{L^3\lambda^-(m)}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\Big]\cos\frac{m\pi x_2}{L}\equiv T_1(u),\end{aligned} \tag{3.15}$$

$$\sigma_{n_2}=\sum_{m=1}^{\infty}\Big[\frac{-2\nu(m\pi+L\lambda^-(m))}{L^2}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2$$

$$+\frac{2\nu(-m\pi+L\lambda^-(m))}{L^2}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\Big]\sin\frac{m\pi x_2}{L}\equiv T_2(u) \tag{3.16}$$

and

$$\begin{aligned} u_1|_{\Gamma_d} =& a+\sum_{m=1}^{\infty}C_m\Big[\frac{a}{\nu}\int_0^L \sigma_{n_1}(d,x_2)\cos\frac{m\pi x_2}{L}dx_2+\frac{m\pi}{L}\Big(2+\frac{2m\pi}{L\lambda^-(m)} \\ &-\frac{a}{\nu\lambda^-(m)}\Big)\int_0^L \sigma_{n_2}(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\Big]\cos\frac{m\pi x_2}{L}\equiv S_1(\sigma_n), \end{aligned} \tag{3.17}$$

$$\begin{aligned} u_2|_{\Gamma_d} =& \sum_{m=1}^{\infty}C_m\Big[\Big(\frac{2m\pi}{L}+\lambda^-(m)+\frac{m^2\pi^2}{L^2\lambda^-(m)}\Big)\int_0^L \sigma_{n_1}(d,x_2)\cos\frac{m\pi x_2}{L}dx_2 \\ &+\frac{a}{\nu}\int_0^L \sigma_{n_2}(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\Big]\sin\frac{m\pi x_2}{L}\equiv S_2(\sigma_n), \end{aligned} \tag{3.18}$$

where

$$C_m=\frac{2\nu}{(aL-2\nu m\pi)\Big(a+\frac{2\nu^2m^2\pi^2}{L^2\lambda^-(m)}\Big)-\frac{4\nu^2m^2\pi^2}{L}},\qquad m=1,2,\cdots.$$

Let

$$T(u)=\begin{pmatrix}T_1(u)\\ T_2(u)\end{pmatrix},\quad S(\sigma_n)=\begin{pmatrix}S_1(\sigma_n)\\ S_2(\sigma_n)\end{pmatrix}.$$

Therefore we obtain the exact boundary condition (3.15)–(3.16) or (3.17)–(3.18) at the artificial boundary Γ_d. Then the problem (3.1)–(3.6) can be reduced to the following two problems in a bounded domain Ω_T:

Problem (I)

$$a\frac{\partial u}{\partial x_1}+\nabla p=\nu\triangle u,\quad \text{in } \Omega_T, \tag{3.19}$$

$$\nabla\cdot u=0,\quad \text{in } \Omega_T, \tag{3.20}$$

$$\frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L}=u_2|_{x_2=0,L}=0,\quad b\le x_1\le d, \tag{3.21}$$

$$u|_{\partial\Omega_i}=0, \tag{3.22}$$

$$u|_{\Gamma_b}=u_\infty, \tag{3.23}$$

$$\sigma_n=T(u). \tag{3.24}$$

Problem (II)

$$a\frac{\partial u}{\partial x_1}+\nabla p=\nu\triangle u,\quad \text{in } \Omega_T, \tag{3.25}$$

$$\nabla\cdot u=0,\quad \text{in } \Omega_T, \tag{3.26}$$

$$\frac{\partial u_1}{\partial x_2}|_{x_2=0,L}=u_2|_{x_2=0,L}=0,\quad b\le x_1\le d, \tag{3.27}$$

$$u|_{\partial\Omega_i}=0, \tag{3.28}$$

$$u|_{\Gamma_b}=u_\infty, \tag{3.29}$$

$$u|_{\Gamma_d} = S(\sigma_n), \tag{3.30}$$

$$\int_{\Gamma_d} p(d, x_2)dx_2 = 0. \tag{3.31}$$

Fortunately we can prove the following theorem.

Theorem 1. *Problem* (I) *is equivalent to Problem* (II).

Proof. Let (u, p) be a solution of problem (I), then (u, p) satisfy (3.25)–(3.29). Multiplying (3.15) and (3.16) by $\cos\frac{m\pi x_2}{L}$, $\sin\frac{m\pi x_2}{L}$ $(m = 1, 2, \cdots)$ respectively and integrating on Γ_d, we obtain

$$u_1|_{\Gamma_d} = c_0 + S_1(\sigma_n), \quad u_2|_{\Gamma_d} = S_2(\sigma_n),$$

where c_0 is a constant.

$$\begin{aligned} 0 &= \int_{\Omega_T} \nabla \cdot u dx = \int_{\partial\Omega_T} u \cdot n ds = -\int_{\Gamma_b} a dx_2 + \int_{\Gamma_d} u_1(d, x_2)dx_2 \\ &= -a|\Gamma_b| + \int_{\Gamma_d} [c_0 + S_1(\sigma_n)]dx_2 = |\Gamma_b|(c_0 - a), \end{aligned}$$

where $|\Gamma_b|$ is the length of the segment Γ_b. Then $c_0 = a$ and

$$\begin{aligned} \int_{\Gamma_d} p(d, x_2)dx_2 &= -\int_{\Gamma_d} \Big[-p(d, x_2) - 2\nu\frac{\partial u_2}{\partial x_2}(d, x_2)\Big]dx_2 \\ &= -\int_{\Gamma_d} \Big[-p(d, x_2) + 2\nu\frac{\partial u_1}{\partial x_1}(d, x_2)\Big]dx_2 = -\int_{\Gamma_d} \sigma_{n_1} dx_2 \\ &= -\int_{\Gamma_d} T_1(u)dx_2 = 0 \end{aligned}$$

Thus (u, p) is a solution of problem (II).

On the other hand, let (u, p) be a solution of problem (II). Then (u, p) satisfy (3.19)–(3.23). Multiplying (3.17) and (3.18) by $\cos\frac{m\pi x_2}{L}$ and $\sin\frac{m\pi x_2}{L}$ $(m = 1, 2, \cdots)$ respectively and integrating on Γ_d, we obtain

$$\sigma_{n_1} = c_0 + T_1(u), \quad \sigma_{n_2} = T_2(u),$$

where c_0 is a constant.

$$\begin{aligned} 0 &= \int_{\Gamma_d} \Big[-p(d, x_2) - 2\nu\frac{\partial u_2}{\partial x_2}(d, x_2)\Big]dx_2 = \int_{\Gamma_d} [-p(d, x_2) + 2\nu\frac{\partial u_1}{\partial x_1}(d, x_2)dx_2 \\ &= \int_{\Gamma_d} \sigma_{n_1} dx_2 = \int_{\Gamma_d} [c_0 + T_1(u)]dx_2 = \int_{\Gamma_d} c_0 dx_2 = c_0|\Gamma_d|. \end{aligned}$$

Thus $c_0 = 0$. Hence (u, p) is a solution of problem (I). The proof is completed.

Let

$$T_1^N(u) = \sum_{m=1}^{N} \Big[\frac{2\nu(-m\pi + L\lambda^-(m))}{L^2} \int_0^L u_1(d, x_2)\cos\frac{m\pi x_2}{L}dx_2$$

$$- \frac{2\nu m\pi(m\pi + L\lambda^-(m))}{L^3\lambda^-(m)} \int_0^L u_2(d, x_2) \sin\frac{m\pi x_2}{L} dx_2 \Big] \cos\frac{m\pi x_2}{L}, \tag{3.32}$$

$$T_2^N(u) = \sum_{m=1}^{N} \Big[\frac{-2\nu(m\pi + L\lambda^-(m))}{L^2} \int_0^L u_1(d, x_2) \cos\frac{m\pi x_2}{L} dx_2$$

$$+ \frac{2\nu(-m\pi + L\lambda^-(m))}{L^2} \int_0^L u_2(d, x_2) \sin\frac{m\pi x_2}{L} dx_2 \Big] \sin\frac{m\pi x_2}{L}, \tag{3.33}$$

$$T^N(u) = \begin{pmatrix} T_1^N(u) \\ T_2^N(u) \end{pmatrix}.$$

Then we get a sequence of approximate boundary conditions at the artificial boundary Γ_d.

$$\sigma_n = T^N(u), \quad N = 0, 1, 2, \cdots \tag{3.34}$$

Hence the original problem (3.1)–(3.6) is reduced to the following problem on the bounded domain Ω_T approximately for $N = 0, 1, 2, \cdots$

$$a\frac{\partial u}{\partial x_1} + \nabla p = \nu \triangle u, \quad \text{in } \Omega_T, \tag{3.35}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega_T, \tag{3.36}$$

$$\frac{\partial u_1}{\partial x_2}\Big|_{x_2=0,L} = u_2|_{x_2=0,L} = 0, \quad b \le x_1 \le d, \tag{3.37}$$

$$u|_{\partial\Omega_i} = 0, \tag{3.38}$$

$$u|_{\Gamma_b} = u_\infty, \tag{3.39}$$

$$\sigma_n = T^N(u). \tag{3.40}$$

In the following section we shall show that the boundary value problems (3.19)–(3.24) and (3.35)–(3.40) are well-posed.

4. The solutions of the problems (3.19)–(3.24) and (3.35)–(3.40)

Let $H^m(\Omega_T)$ and $H^s(\Gamma_d)$ denote the usual Sobolev spaces on the domain Ω_T and the boundary Γ_d, with integer m and real number s. Furthermore let

$$\begin{aligned}
\Gamma_1 =& \{x \in \mathbb{R}^2 |\ x_2 = 0, \quad b \le x_1 \le d\} \cup \{x \in \mathbb{R}^2 |\ x_2 = L, \quad b \le x_1 \le d\}, \\
\Gamma_i =& \partial\Omega_i, \\
V =& \{u \in H^1(\Omega_T) \times H^1(\Omega_T) |\ u|_{\Gamma_b \cup \Gamma_i} = 0, \quad u_2|_{\Gamma_1} = 0\}
\end{aligned}$$

with norm $\|u\|_V^2 = \|u_1\|_{1,2,\Omega_T}^2 + \|u_2\|_{1,2,\Omega_T}^2$,

$$\begin{aligned}
W =& L^2(\Omega_T) \quad \text{ with norm } \|q\|_W = \|q\|_{L^2(\Omega_T)}, \\
M =& \{u \in H^1(\Omega_T) \times H^1(\Omega_T) | \quad u|_{\Gamma_i} = 0, \quad u|_{\Gamma_b} = u_\infty, \quad u_2|_{\Gamma_1} = 0\}.
\end{aligned}$$

Then the boundary value problem (3.19)–(3.24) is equivalent to the following variational problem:

$$\text{Find} \quad (u, p) \in M \times W, \quad \text{such that}$$

$$A(u,v)+A_0(u,v)+A_1(u,v)+B(v,p)=0, \quad \forall v\in V, \tag{4.1}$$
$$B(u,q)=0, \quad \forall q\in W; \tag{4.2}$$

where

$$A(u,v)=2\nu\int_{\Omega_T}\sum_{i,j=1}^{2}\varepsilon_{ij}(u)\cdot\varepsilon_{ij}(v)dx\equiv 2\nu\int_{\Omega_T}\varepsilon(u):\varepsilon(v)dx,$$

$$A_0(u,v)=a\int_{\Omega_T}v\frac{\partial u}{\partial x_1}dx,$$

$$\begin{aligned}A_1(u,v)=&-\int_{\Gamma_d}\sigma_n\cdot v dx_2=-\int_{\Gamma_d}T(u)\cdot v dx_2\\
=&\sum_{m=1}^{\infty}\Big[\frac{2\nu(m\pi-L\lambda^-(m))}{L^2}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\int_0^L v_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\\
&+\frac{2\nu m\pi(m\pi+L\lambda^-(m))}{L^3\lambda^-(m)}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\int_0^L v_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\\
&+\frac{2\nu(m\pi+L\lambda^-(m))}{L^2}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\int_0^L v_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\\
&+\frac{2\nu(m\pi-L\lambda^-(m))}{L^2}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\int_0^L v_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\Big],\end{aligned}$$

$$B(u,q)=-\int_{\Omega_T}q\nabla\cdot u dx.$$

Furthermore let

$$\begin{aligned}A_1^N(u,v)=&-\int_{\Gamma_d}T^N(u)\cdot v dx_2\\
=&\sum_{m=1}^{N}\Big[\frac{2\nu(m\pi-L\lambda^-(m))}{L^2}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\int_0^L v_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\\
&+\frac{2\nu m\pi(m\pi+L\lambda^-(m))}{L^3\lambda^-(m)}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\int_0^L v_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\\
&+\frac{2\nu(m\pi+L\lambda^-(m))}{L^2}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2\int_0^L v_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\\
&+\frac{2\nu(m\pi-L\lambda^-(m))}{L^2}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\int_0^L v_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2\Big].\end{aligned}$$

Then the problem (3.35)–(3.40) is equivalent to the following variational problem:

$$\text{Find}\quad (u_N,p_N)\in M\times W,\quad \text{such that}$$
$$A(u_N,v)+A_0(u_N,v)+A_1^N(u_N,v)+B(v,p_N)=0, \quad \forall v\in V, \tag{4.3}$$
$$B(u_N,q)=0, \quad \forall q\in W. \tag{4.4}$$

From Körn's inequality[16], we know

Lemma 1. *The bilinear form $A(u,v)$ is symmetric, bounded and coercive on $V\times V$, namely there are two positive constants α_0 and β_0 such that*

$$|A(u,v)| \le \alpha_0\|u\|_V \cdot \|v\|_V, \quad \forall u,v\in V,$$
$$A(u,u) \ge \beta_0\|u\|_V^2, \quad \forall u\in V.$$

Lemma 2. *The bilinear form $B(u,q)$ is bounded on $V\times W$ and satisfies the Babuška-Brezzi $(B-B)$ condition*[3], *namely there are positive constants α_1 and β_1, such that*

$$|B(u,q)| \le \alpha_1\|u\|_V \cdot \|q\|_W, \quad \forall u\in V,\ q\in W,$$
$$\sup_{u\in V\backslash\{0\}} \frac{B(u,q)}{\|u\|_V} \ge \beta_1\|q\|_W,\ \forall q\in W.$$

Lemma 3. *The bilinear forms $A_0(u,v)+A_1(u,v)$ and $A_0(u,v)+A_1^N(u,v)$ are bounded on $V\times V$, i. e. there is a constant $\alpha_2>0$, such that*

$$|A_0(u,v)+A_1(u,v)| \le \alpha_2\|u\|_V\cdot\|v\|_V, \quad \forall u,v\in V, \tag{4.5}$$
$$|A_0(u,v)+A_1^N(u,v)| \le \alpha_2\|u\|_V\cdot\|v\|_V, \quad \forall u,v\in V, \tag{4.6}$$

Furthermore

$$A_0(u,u)+A_1(u,u)\ge 0, \quad \forall u\in V,$$
$$A_0(u,u)+A_1^N(u,u)\ge 0, \quad \forall u\in V,\ N=0,1,2,\cdots.$$

Proof. For any $u,v\in V$, we know that $u_1|_{\Gamma_d}$ and $v_1|_{\Gamma_d}$ belong to $H^{\frac12}(\Gamma_d)$, $u_2|_{\Gamma_d}$ and $v_2|_{\Gamma_d}$ belong to $H_0^{\frac12}(\Gamma_d)$, Suppose

$$u_1(d,x_2)=\frac{a_0}{2}+\sum_{m=1}^{\infty} a_m\cos\frac{m\pi x_2}{L}, \quad a_m=\frac{2}{L}\int_0^L u_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2,$$
$$u_2(d,x_2)=\sum_{m=1}^{\infty} b_m\sin\frac{m\pi x_2}{L}, \qquad b_m=\frac{2}{L}\int_0^L u_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2,$$
$$v_1(d,x_2)=\frac{\tilde a_0}{2}+\sum_{m=1}^{\infty} \tilde a_m\cos\frac{m\pi x_2}{L}, \qquad \tilde a_m=\frac{2}{L}\int_0^L v_1(d,x_2)\cos\frac{m\pi x_2}{L}dx_2,$$
$$v_2(d,x_2)=\sum_{m=1}^{\infty} \tilde b_m\sin\frac{m\pi x_2}{L}, \qquad \tilde b_m=\frac{2}{L}\int_0^L v_2(d,x_2)\sin\frac{m\pi x_2}{L}dx_2.$$

Then by the trace theorem, there is a constant $\alpha_3>0$, such that

$$\sqrt{\sum_{m=1}^{\infty} m(a_m^2+b_m^2)} \le \alpha_3\|u\|_V, \quad \sqrt{\sum_{m=1}^{\infty} m(\tilde a_m^2+\tilde b_m^2)} \le \alpha_3\|v\|_V.$$

A computation shows that

$$A_1(u,v)=\sum_{m=1}^{\infty}\Big[\frac{\nu(m\pi-L\lambda^-(m))a_m\tilde a_m}{2}+\frac{\nu m\pi(m\pi+L\lambda^-(m))b_m\tilde a_m}{2L\lambda^-(m)}$$

$$+\frac{\nu(m\pi+L\lambda^-(m))a_m\tilde{b}_m}{2}+\frac{\nu(m\pi-L\lambda^-(m))b_m\tilde{b}_m}{2}\Big].$$

Since $0<-L\lambda^-(m)\le m\pi$, $\forall m\in\mathbb{N}$, and $\lim\limits_{m\to+\infty}\dfrac{m\pi}{-L\lambda^-(m)}=1$, we know that there is a constant $c_1>0$, such that

$$\begin{aligned}|A_1(u,v)|\le& c_1\sum_{m=1}^{\infty}m(|a_m\tilde{a}_m|+|b_m\tilde{a}_m|+|a_m\tilde{b}_m|+|b_m\tilde{b}_m|)\\ \le& c_1\sqrt{\sum_{m=1}^{\infty}m(a_m^2+b_m^2)}\cdot\sqrt{\sum_{m=1}^{\infty}m(\tilde{a}_m^2+\tilde{b}_m^2)}\le c_2\|u\|_V\cdot\|v\|_V,\end{aligned}$$

where $c_2=c_1\cdot\alpha_3^2$. Thus the inequality (4.5) holds. Furthermore

$$\begin{aligned}A_1(u,u)=&\frac{\nu}{2}\sum_{m=1}^{\infty}\Big[(m\pi-L\lambda^-(m))(a_m^2+b_m^2)+\frac{(m\pi+L\lambda^-(m))^2}{L\lambda^-(m)}a_mb_m\Big]\\ \ge&\frac{\nu}{4}\sum_{m=1}^{\infty}\Big[2m\pi-2L\lambda^-(m)+\frac{(m\pi+L\lambda^-(m))^2}{L\lambda^-(m)}\Big](a_m^2+b_m^2)\\ =&\sum_{m=1}^{\infty}\Big(m\pi\nu-\frac{aL}{4}\Big)(a_m^2+b_m^2)\\ \ge&-\frac{a}{2}\sum_{m=1}^{\infty}\frac{L}{2}(a_m^2+b_m^2)\ge-\frac{a}{2}\int_{\Gamma_d}u(d,x_2)\cdot u(d,x_2)dx_2.\end{aligned}$$

Therefore

$$A_0(u,u)+A_1(u,u)\ge a\int_{\Omega_T}u\cdot\frac{\partial u}{\partial x_1}dx-\frac{a}{2}\int_{\Gamma_d}u\cdot u dx_2=0,\quad\forall u\in V.$$

Similarly for $A_0(u,v)+A_1^N(u,v)$, we obtain

$$\begin{aligned}&|A_0(u,v)+A_1^N(u,v)|\le\alpha_2\|u\|_V\cdot\|v\|_V,\quad\forall u,v\in V,\\ &A_0(u,u)+A_1^N(u,u)\ge0,\quad\forall u\in V.\end{aligned}$$

Furthermore if u is a solution of the problem (4.1)–(4.2) and $u|_{\Gamma_d}\in H^2(\Gamma_d)\times H^2(\Gamma_d)$. Then $u_1|_{\Gamma_d}\in\{w\in H^2(0,L),\ \frac{\partial w}{\partial z}|_{z=0,L}=0\}$ and $u_2|_{\Gamma_d}\in\{w\in H^2(0,L),\ w|_{z=0,L}=0\}$. Thus we have that

$$\sqrt{\sum_{m=1}^{\infty}m^4(a_m^2+b_m^2)}\le\alpha_4\|u\|_{2,\Gamma_d},$$

where α_4 is a constant. Hence

$$\begin{aligned}|A_1(u,v)-A_1^N(u,v)|=&\Big|\sum_{m=N+1}^{\infty}\Big[\frac{\nu(m\pi-L\lambda^-(m))a_m\tilde{a}_m}{2}+\frac{\nu m\pi(m\pi+L\lambda^-(m))b_m\tilde{a}_m}{2L\lambda^-(m)}\\ &+\frac{\nu(m\pi+L\lambda^-(m))a_m\tilde{b}_m}{2}+\frac{\nu(m\pi-L\lambda^-(m))b_m\tilde{b}_m}{2}\Big]\Big|\end{aligned}$$

$$
\begin{aligned}
&\leq c_1 \sqrt{\sum_{m=N+1}^{\infty} m(a_m^2+b_m^2)} \cdot \sqrt{\sum_{m=N+1}^{\infty} m(\tilde{a}_m^2+\tilde{b}_m^2)} \\
&\leq \frac{c_2}{(N+1)^{3/2}} \sqrt{\sum_{m=N+1}^{\infty} m^4(a_m^2+b_m^2)} \cdot \sqrt{\sum_{m=N+1}^{\infty} m(\tilde{a}_m^2+\tilde{b}_m^2)} \\
&\leq \frac{c_2}{(N+1)^{3/2}} \sqrt{\sum_{m=N+1}^{\infty} m^4(a_m^2+b_m^2)} \cdot \|v\|_{\frac{1}{2},\Gamma_d},
\end{aligned}
$$

where c_2 is a constant. Let

$$
\begin{aligned}
d_0 &= \max_{x\in\bar{\Omega}_i}\{x_1\}, \\
\Gamma_{d_0} &= \{x \in \mathbb{R}^2 |\ x_1 = d_0,\ 0 \leq x_2 \leq L\}.
\end{aligned}
$$

Assume that

$$
\begin{aligned}
u_1(d_0,x_2) &= \frac{\bar{a}_0}{2} + \sum_{m=1}^{\infty} \bar{a}_m \cos\frac{m\pi x_2}{L},\ \bar{a}_m = \frac{2}{L}\int_0^L u_1(d_0,x_2)\cos\frac{m\pi x_2}{L}dx_2, \\
u_2(d_0,x_2) &= \sum_{m=1}^{\infty} \bar{b}_m \sin\frac{m\pi x_2}{L},\ \bar{b}_m = \frac{2}{L}\int_0^L u_2(d_0,x_2)\sin\frac{m\pi x_2}{L}dx_2.
\end{aligned}
$$

By the equalities (2.13)–(2.14), we obtain

$$
\begin{aligned}
a_m =& \frac{1}{L\lambda^-(m)+m\pi}\Big\{[m\pi e^{\lambda^-(m)(d-d_0)} + L\lambda^-(m)e^{-\frac{m\pi}{L}(d-d_0)}]\bar{a}_m \\
&+ m\pi[e^{-\frac{m\pi}{L}(d-d_0)} - e^{\lambda^-(m)(d-d_0)}]\bar{b}_m\Big\}, \\
b_m =& \frac{1}{L\lambda^-(m)+m\pi}\Big\{L\lambda^-(m)[e^{-\frac{m\pi}{L}(d-d_0)} - e^{\lambda^-(m)(d-d_0)}]\bar{a}_m \\
&+ [m\pi e^{-\frac{m\pi}{L}(d-d_0)} + L\lambda^-(m)e^{\lambda^-(m)(d-d_0)}]\bar{b}_m\Big\},
\end{aligned}
$$

Thus there exist constants $c_3 > 0$ and $\lambda_N = O(N)$ as $N \to +\infty$, such that

$$
a_m^2 + b_m^2 \leq c_3 e^{-\lambda_N^2(d-d_0)^2}(\bar{a}_m^2+\bar{b}_m^2), \quad \forall m \geq N+1.
$$

Therefore

$$
\begin{aligned}
|A_1(u,v) - A_1^N(u,v)| \leq& \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}} \sqrt{\sum_{m=N+1}^{\infty} m^4(\bar{a}_m^2+\bar{b}_m^2)} \cdot \|v\|_{\frac{1}{2},\Gamma_d} \\
\leq& \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}} \cdot \|v\|_{\frac{1}{2},\Gamma_d} \quad \forall d \geq d_0,
\end{aligned}
$$

where c is a constant. Hence we obtain the following estimate:

Lemma 4. *If u is a solution of the problem* (4.1)-(4.2) *and* $u|_{\Gamma_{d_0}} \in H^2(\Gamma_{d_0}) \times H^2(\Gamma_{d_0})$, *then the following estimate holds:*

$$|A_1(u,v) - A_1^N(u,v)| \le \frac{c}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}} \cdot \|v\|_{\frac{1}{2},\Gamma_d}, \ \forall d \ge d_0, \ \forall v \in V, \quad (4.7)$$

where c is a constant independent of N, u, v, p.

Theorem 2. *The variational problem* (4.1)–(4.2) *has a unique solution* $(u,p) \in M \times W$ *and the problem* (4.3)–(4.4) *has a unique solution* $(u_N, p_N) \in M \times W$ *for* $N = 0, 1, 2, \cdots$ *Furthermore we have the following error estimate if* $u|_{\Gamma_{d_0}} \in H^2(\Gamma_{d_0}) \times H^2(\Gamma_{d_0})$:

$$\|u - u_N\|_V + \|p - p_N\|_W \le \frac{c}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}}. \quad (4.8)$$

Proof. By Lemma 1 and 4, we know that $A(u,v) + A_0(u,v) + A_1(u,v)$ and $A(u,v) + A_0(u,v) + A_1^N(u,v)$ are two bounded and coercive bilinear forms on $V \times V$. By Lemma 2, we know that $B(u,q)$ is a bounded bilinear form on $V \times W$, and satisfies the B-B condition. From the Brezzi Theorem [2], we obtain that the problem (4.1)–(4.2) has a unique solution $(u,p) \in M \times W$ and the problem (4.3)–(4.4) has a unique solution $(u_N, p_N) \in M \times W$.

Let $e_u = u - u_N$, $e_p = p - p_N$, then (e_u, e_p) satisfy

$$A(e_u,v) + A_0(e_u,v) + A_1^N(e_u,v) + B(v,e_p) = A_1^N(u,v) - A_1(u,v), \quad \forall v \in V, \quad (4.9)$$

$$B(e_u,q) = 0, \quad \forall q \in W. \quad (4.10)$$

Taking $v = e_u$ in (4.9) and $q = e_p$ in (4.10), we obtain

$$\begin{aligned}
\beta_0 \|e_u\|_V^2 \le & A(e_u,e_u) \le A(e_u,e_u) + A_0(e_u,e_u) + A_1^N(e_u,e_u) \\
= & A_1^N(u,e_u) - A_1(u,e_u) \le \frac{c}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}} \cdot \|e_u\|_{\frac{1}{2},\Gamma_d} \\
\le & \frac{c}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}} \cdot \|e_u\|_V,
\end{aligned}$$

where c is a constant, which has different meaning in different place. Thus

$$\begin{aligned}
\|e_u\| \le & \frac{c}{\beta_0 (N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}}. \\
B(v,e_p) = & A_1^N(u,v) - A_1(u,v) - A(e_u,v) - A_0(e_u,v) - A_1^N(e_u,v) \\
\le & \left[\frac{c}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}} + (\alpha_0 + \alpha_2) \|e_u\|_V \right] \cdot \|v\|_V.
\end{aligned}$$

Then

$$\begin{aligned}
\|e_p\|_W = & \|p - p_N\|_W \le \frac{1}{\beta_1} \sup_{v \in V \setminus \{0\}} \frac{B(v,e_p)}{\|v\|_V} \\
\le & \frac{1}{\beta_1} \left[\frac{c}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}} + (\alpha_0 + \alpha_2) \|e_u\|_V \right] \\
\le & \frac{\bar{c}}{(N+1)^{3/2} e^{\lambda_N(d-d_0)}} \|u\|_{2,\Gamma_{d_0}},
\end{aligned}$$

where $\bar{c} = \frac{c}{\beta_1}[1 + \frac{\alpha_0 + \alpha_2}{\beta_0}]$. Then the inequality (4.8) follows immediately.

5. The Finite Element Approximation of the Problem (4.3)-(4.4)

Let $\mathcal{T}_h$ be a regular partition of the domain Ω_T and suppose V_h and W_h are finite element subspaces of V and W. Particularly, we also assume they are the optimal choice. Then V_h and W_h should satisfy the following conditions[10]

a). The errors $\inf\limits_{v\in V_h}\|u-v\|_V$ and $\inf\limits_{q\in W_h}\|p-q\|_W$ have the same order in h, i. e. there is a constant α, such that

$$\inf_{v\in V_h}\|u-v\|_V\le \alpha h^m|u|_{m+1,2,\Omega_T},\qquad \inf_{q\in W_h}\|p-q\|_W\le \alpha h^m|p|_{m,2,\Omega_T}. \tag{5.1}$$

b). There exists a constant β independent of h, such that

$$\sup_{v\in V_h\backslash\{0\}}\frac{B(v,q)}{\|v\|_V}\ge \beta\|q\|_W,\quad \forall q\in W_h. \tag{5.2}$$

Let M_h be a subset of M, which satisfies $V_h=\{u_h-v_h|\ \forall u_h,\, v_h\in M_h\}$. Consider the finite element approximation of the problem (4.3)–(4.4):

$$\text{Find}\ \ (u_N^h,p_N^h)\in M_h\times W_h,\quad \text{such that}$$

$$A(u_N^h,v)+A_0(u_N^h,v)+A_1^N(u_N^h,v)+B(v,p_N^h)=0,\quad \forall v\in V_h, \tag{5.3}$$

$$B(u_N^h,q)=0,\quad \forall q\in W_h. \tag{5.4}$$

Theorem 3. *The problem* (5.3)–(5.4) *has a unique solution* $(u_N^h,p_N^h)\in M_h\times W_h$.

The proof of this theorem is similiar to the proof of theorem 2. It is omitted here.

Theorem 4. *Let* (u,p) *be the solution of the problem* (4.1)–(4.2) *and* (u_N^h,p_N^h) *be the solution of the problem* (5.3)–(5.4). *Suppose* $u\in H^{m+1}(\Omega_T)\times H^{m+1}(\Omega_T)$, $u|_{\Gamma_{d_0}}\in H^2(\Gamma_{d_0})\times H^2(\Gamma_{d_0})$, $p\in H^m(\Omega_T)$. *Then we have the following error estimate:*

$$\begin{aligned}\|u-u_N^h\|_V+\|p-p_N^h\|_W\le& ch^m[|u|_{m+1,2,\Omega_T}+|p|_{m,2,\Omega_T}]\\&+\frac{\bar c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}},\end{aligned} \tag{5.5}$$

where c, $\bar c$ *independent of* h, u, p, N.

Proof. Let $e_u^h=u-u_N^h$, $e_p^h=p-p_N^h$. Then from the equalities (4.1)–(4.2) and (5.3)–(5.4), (e_u^h,p_u^h) satisf is

$$\begin{aligned}&A(e_u^h,v)+A_0(e_u^h,v)+A_1^N(e_u^h,v)+B(v,e_p^h)\\=&A_1^N(u,v)-A_1(u,v),\quad \forall v\in V_h,\end{aligned} \tag{5.6}$$

$$B(e_u^h,q)=0,\quad \forall q\in W_h. \tag{5.7}$$

Then we have that

$$\begin{aligned}\beta_0\|u_N^h-u_0-v\|_V^2\le& A(u_N^h-u_0-v,u_N^h-u_0-v)\le A(u_N^h-u_0-v,u_N^h-u_0-v)\\&+A_0(u_N^h-u_0-v,u_N^h-u_0-v)+A_1^N(u_N^h-u_0-v,u_N^h-u_0-v)\\=&A(u-u_0-v,u_N^h-u_0-v)+A_0(u-u_0-v,u_N^h-u_0-v)\end{aligned}$$

$$
\begin{aligned}
&+ A_1^N(u-u_0-v, u_N^h-u_0-v) - A(e_u^h, u_N^h-u_0-v) \\
&- A_0(e_u^h, u_N^h-u_0-v) - A_1^N(e_u^h, u_N^h-u_0-v) \\
=&A(u-u_0-v, u_N^h-u_0-v) + A_0(u-u_0-v, u_N^h-u_0-v) \\
&+ A_1^N(u-u_0-v, u_N^h-u_0-v) + A_1(u, u_N^h-u_0-v) \\
&- A_1^N(u, u_N^h-u_0-v) + B(u_N^h-u_0-v, p-q) \\
\le&(\alpha_0+\alpha_2)\|u-u_0-v\|_V \cdot \|u_N^h-u_0-v\|_V \\
&+ \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}} \cdot \|u_N^h-u_0-v\|_V \\
&+ \alpha_1\|p-q\|_W \cdot \|u_N^h-u_0-v\|, \quad \forall v \in \tilde{V}_h, \quad \forall q \in W_h,
\end{aligned}
$$

where $u_0 \in M_h$, $\tilde{V}_h = \{v_h \in V_h | \ B(v_h, q) = B(-u_0, q), \forall q \in W_h\}$. Thus

$$
\begin{aligned}
\|u_N^h-u_0-v\|_V \le&\frac{1}{\beta_0}[(\alpha_0+\alpha_2)\|u-u_0-v\|_V \\
&+ \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}} + \alpha_1\|p-q\|_W]. \\
\|e_u^h\|_V \le&\|u-u_0-v\|_V + \|v+u_0-u_N^h\|_V \\
\le&\frac{1}{\beta_0}\Big[(\beta_0+\alpha_0+\alpha_2)\|u-u_0-v\|_V + \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}} \\
&+ \alpha_1\|p-q\|_W\Big] \quad \forall v \in \tilde{V}_h, \quad q \in W_h.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\|e_u^h\|_V \le&\Big(\frac{\alpha_0+\alpha_2}{\beta_0}+1\Big)\inf_{v\in\tilde{V}_h}\|u-u_0-v\|_V + \frac{\alpha_1}{\beta_0}\inf_{q\in W_h}\|p-q\|_W \\
&+ \frac{c}{\beta_0(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}} \\
\le&ch^m[|u|_{m+1,2,\Omega_T} + |p|_{m,2,\Omega_T}] + \frac{c}{\beta_0(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}}
\end{aligned}
$$

In order to estimate the error $\|p-p_N^h\|_W$, we consider

$$
\begin{aligned}
B(v, p_N^h-q) =&B(v, p-q) - B(v, e_p^h) \\
=&B(v, p-q) + A(e_u^h, v) + A_0(e_u^h, v) + A_1^N(e_u^h, v) + A_1(u, v) - A_1^N(u, v) \\
\le&\alpha_1\|v\|_V \cdot \|p-q\|_W + (\alpha_0+\alpha_2)\|e_u^h\|_V \cdot \|v\|_V \\
&+ \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}} \cdot \|v\|_V.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|e_p^h\|_W \le&\|p-q\|_W + \|q-p_N^h\|_W \le \|p-q\|_W + \frac{1}{\beta_1}\sup_{v\in V_h\setminus\{0\}}\frac{B(v, p_N^h-q)}{\|v\|_V} \\
\le&\|p-q\|_W + \frac{1}{\beta_1}\Big[\alpha_1\|p-q\|_W + (\alpha_0+\alpha_2)\|e_u^h\|_V + \frac{c}{(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}}\Big]
\end{aligned}
$$

$$\leq\Big(1+\frac{\alpha_1}{\beta_1}\Big)\|p-q\|_W+\frac{\alpha_0+\alpha_2}{\beta_1}\|e_u^h\|_V+\frac{c}{\beta_1(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}},\ \forall q\in W_h.$$

Thus

$$\begin{aligned}\|e_p^h\|_W\leq&\Big(1+\frac{\alpha_1}{\beta_1}\Big)\inf_{q\in W_h}\|p-q\|_W+\frac{\alpha_0+\alpha_2}{\beta_1}\|e_u^h\|_V+\frac{c}{\beta_1(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}}\\ \leq& ch^m[|u|_{m+1,2,\Omega_T}+|p|_{m,2,\Omega_T}]+\frac{c}{\beta_1(N+1)^{3/2}e^{\lambda_N(d-d_0)}}\|u\|_{2,\Gamma_{d_0}}.\end{aligned}$$

Then the inequality (5.5) is proved.

6. Numerical Implementation and Example

For the sake of simplicity, let $\mathcal{T}_h$ be a rectangle partition of Ω_T, with $\Omega_T=\cup_{K\in\mathcal{T}_h}K$, where K is a rectangle.

For each rectangle $K\in\mathcal{T}_h$, connected the mid-points of the opposite sides of K, then each rectangle K is divided into four smaller rectangles. Let $\mathcal{T}_{\hat{h}}$ denote this new partition. Therefore let $V_h=\{v\in V|\ v|_K$ is a bilinear polynomial, $\forall K\in\mathcal{T}_{\hat{h}}\}$, $W_h=\{p\in W|\ p|_K$ is constant, $\forall K\in\mathcal{T}_h\}$, $M_h=\{v\in M|\ v|_K$ is a bilinear polynomial, $\forall K\in\mathcal{T}_{\hat{h}}\}$. Then V_h and W_h satisfy the B-B condition and the following approximate property[18]: $\inf_{v\in V_h}\|u-v\|_V\leq ch|u|_{2,2,\Omega_T}$ and $\inf_{q\in W_h}\|p-q\|_W\leq ch|p|_{1,2,\Omega_T}$. We use this finite element approximation to solve the following example.

Example The effect of the artificial boundary conditions for Oseen equations.

Suppose that the unbounded domain $\Omega=\{x\in\mathbb{R}^2|\ b<x_1<+\infty,\ 0<x_2<L\}$. Let

$$\begin{aligned}u_1(x)&=a+\sum_{m=1}^{\infty}\Big[a_me^{-\frac{m\pi}{L}(x_1-b)}-\frac{m\pi}{L\lambda^-(m)}b_me^{\lambda^-(m)(x_1-b)}\Big]\cos\frac{m\pi x_2}{L},\\ u_2(x)&=\sum_{m=1}^{\infty}\Big[a_me^{-\frac{m\pi}{L}(x_1-b)}+b_me^{\lambda^-(m)(x_1-b)}\Big]\sin\frac{m\pi x_2}{L},\\ p(x)&=-a\sum_{m=1}^{\infty}a_me^{-\frac{m\pi}{L}(x_1-b)}\cos\frac{m\pi x_2}{L};\end{aligned}$$

where

$$a_m=\frac{4L^2[1-(-1)^m]}{[m\pi+L\lambda^-(m)]m^2\pi^2},\quad b_m=\frac{4L^3\lambda^-(m)[1-(-1)^m]}{[m\pi+L\lambda^-(m)]m^3\pi^3},\ m=1,2\cdots$$

Then (u,p) is the unique solution of the following boundary value problem:

$$\begin{aligned}&a\frac{\partial u}{\partial x_1}+\nabla p=\nu\triangle u,\quad \text{in }\Omega,\quad \nabla\cdot u=0,\quad \text{in }\Omega,\\ &\frac{\partial u_1}{\partial x_2}|_{x_2=0,L}=u_2|_{x_2=0,L}=0,\quad b\leq x_1<+\infty,\\ &u_1|_{\Gamma_b}=a,\quad u_2|_{\Gamma_b}=x_2(L-x_2),\quad u\to u_\infty,\quad \text{when }x_1\to+\infty.\end{aligned}$$

We take $\Gamma_d = \{x \in \mathbb{R}^2 |\ x_1 = d,\ 0 \le x_2 \le L\}$ and then consider the finite element approximation of the above problem in the bounded domain $\Omega_T = \{x|\ b < x_1 < d,\ 0 < x_2 < L\}$. We also take $b = 0$, $d = 1$, $L = 1$, $\nu = 1$ and $a = 1.0$.

Three meshes were used in computation. Figure 1 shows the partition $\mathcal{T}_h$ for mesh A. Mesh B was generated by divided each rectangle in mesh A into four small rectangles. And mesh C was similarly generated from mesh B. Bilinear finite element approximation to u and constant finite element approximation to p were used in computation. Table 1 shows the maximum errors $u - u_N^h$ and $p - p_N^h$ over the mesh points when $N = 5$. We can see from the table that the convergence is fast and the rate is higher than linear. Tables 2-4 show the maximum errors of $u - u_N^h$ and $p - p_N^h$ for mesh A, B and C when $N = 0, 1, 3, 5$. As we can see from the tables, the artificial boundary conditions are very effective and $N = 1$ is good enough for mesh A, B and C, this because the meshes are too coarse and the error we used is maximum error in the domain.

Table 1. Maximum error when $N = 5$

mesh	A	B	C
$\max \lvert u_1 - u_{1N}^h \rvert$	3.849E−2	1.645E−2	5.855E−3
$\max \lvert u_2 - u_{2N}^h \rvert$	2.600E−2	8.446E−3	2.323E−3
$\max \lvert p - p_N^h \rvert$	3.074E-2	1.419E−2	5.847E−3

Table 2. Maximum error for mesh A

N	0	1	3	5
$\max \lvert u_1 - u_{1N}^h \rvert$	3.273E−2	3.847E−2	3.847E−2	3.849E−2
$\max \lvert u_2 - u_{2N}^h \rvert$	2.242E−2	2.594E−2	2.594E−2	2.600E−2
$\max \lvert p - p_N^h \rvert$	1.459E−1	3.235E−2	3.235E−2	3.074E−2

Fig. 1 Mesh A

Fig. 2

Fig. 3

Table 3 Maximum error for mesh B

N	0	1	3	5
$\max \lvert u_1 - u_{1N}^h \rvert$	3.140E−2	1.645E−2	1.645E−2	1.645E−2
$\max \lvert u_2 - u_{2N}^h \rvert$	1.905E−2	8.446E−3	8.446E−3	8.446E−3
$\max \lvert p - p_N^h \rvert$	2.384E−1	1.419E−2	1.419E−2	1.419E−2

Table 4. Maximum error for mesh C

N	0	1	3	5
$\max\|u_1 - u_{1N}^h\|$	3.450E−2	5.855E−3	5.855E−3	5.855E−3
$\max\|u_2 - u_{2N}^h\|$	1.965E−2	2.323E−3	2.323E−3	2.323E−3
$\max\|p - p_N^h\|$	3.000E−1	5.848E−3	5.848E−3	5.847E−3

Fig. 4

Fig. 5

Figures 2-5 show the relative error of u at outflow boundary Γ_d for meshes B and C. Then the effect of N is shown for meshes B and C. As shown in the Figures, $N = 3$ gives good approximation and therefore in computations very few terms in the bilinear form $A_1^N(u, v)$ are needed in order to get good accuracy.

The example shows that the artificial boundary condition presented in this paper is very effective. Furthermore this approach can be applied to problems of two dimensional incompressible viscous flow around obstacles.

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