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THE STABILITY OF THE θ-METHODS FOR DELAY DIFFERENTIAL EQUATIONS*

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Abstract

This paper deals with the stability analysis of numerical methods for the solution of delay differential equations. We focus on the behaviour of three θ-methods in the solution of the linear test equation $u'(t) = A(t)u(t) + B(t)u(\tau(t))$ with $\tau(t)$ and $A(t), B(t)$ continuous matrix functions. The stability regions for the three θ-methods are determined.

Key words: Delay differential equations, Numerical solution, Stability, θ-methods.

1. Introduction

1.1. The three θ-methods

We deal with the numerical solution of the initial value problem:

$$\begin{cases} u'(t) = f(t, u(t), u(\tau(t))), & t > t_0, \\ u(t) = u_0(t), & t \le t_0. \end{cases} \tag{1.1}$$

Here f, u_0, τ denote given functions with $\tau(t) \le t$, whereas $u(t)$ is unknown (for $t > t_0$). With the so-called one-leg θ-method, linear θ-method and new θ-method, one can compute approximations u_n to $u(t)$ at the gridpoint $t_n = t_0 + nh$, where $h > 0$ denotes the stepsize and $n = 1, 2, 3, \cdots$.

The one-leg θ-method was considered in [1, 2, 3, 4]

$$\begin{aligned} &u_{n+1} = u_n + hf(\theta t_{n+1} + (1-\theta)t_n, \theta u_{n+1} + (1-\theta)u_n, \\ &u^h(\tau(\theta t_{n+1} + (1-\theta)t_n))), \quad n \ge 0 \end{aligned} \tag{1.2a}$$

where θ is a parameter, with $0 \le \theta \le 1$ specifying the method.

Further we define $u^h(t)$ as follows:

$$\begin{aligned} &u^h(t) = u_0(t), \quad t \le t_0, \\ &u^h(t) = \frac{t_{n+1} - t}{h} u_n + \frac{t - t_n}{h} u_{n+1}, \quad t \in (t_n, t_{n+1}], \quad n \ge 0. \end{aligned}$$

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The linear θ-method to problem of type (1.1) gives rise to the following formula

$$u_{n+1} = u_n + h\{\theta f(t_{n+1}, u_{n+1}, u^h(\tau(t_{n+1}))) + (1-\theta) f(t_n, u_n, u^h(\tau(t_n)))\}, \quad n \geq 0, \tag{(1.2b)}$$

which was considered in [1, 2, 4–7].

Finally,we consider the new θ-method as follows:

$$\begin{aligned} u_{n+1} = u_n + hf(&\theta t_{n+1} + (1-\theta)t_n, \theta u_{n+1} + (1-\theta)u_n, \\ &\theta u^h(\tau(t_{n+1})) + (1-\theta)u^h(\tau(t_n))), \quad n \geq 0, \end{aligned} \tag{1.2c}$$

which was considered in [1].

1.2. The test problem

Consider the test problem

$$\begin{cases} u'(t) = A(t)u(t) + B(t)u(\tau(t)), & t \geq t_0, \\ u(t) = u_0(t), & t \leq t_0. \end{cases} \tag{1.3}$$

Here $A, B : [t_0, \infty) \to C^{d\times d}$ $(d \geq 1)$, $t - \tau(t) \geq \tau_0$ $(t \geq t_0)$, τ_0 is a positive constant, $u_0(t)$ is a known complex function for $t \leq t_0$.

Applying (1.2a), (1.2b), (1.2c) to (1.3) we have the following recurrence relations:

$$\begin{aligned} (I - \theta x(t_{n+\theta}))u_{n+1} =& (I + (1-\theta)x(t_{n+\theta}))u_n + \delta(t_{n+\theta})y(t_{n+\theta})u_{n-m(t_{n+\theta})+1} \\ &+ (1-\delta(t_{n+\theta}))y(t_{n+\theta})u_{n-m(t_{n+\theta})}, \quad (n \geq m), \end{aligned} \tag{1.4a}$$

Here

$$\begin{aligned} &\delta(t_{n+\theta}) = \frac{\tau(t_{n+\theta})}{h} - r(t_{n+\theta}), \\ &r(t_{n+\theta}) = \Big[\frac{\tau(t_{n+\theta})}{h}\Big], \quad \delta(t_{n+\theta}) \in [0,1), \\ &m(t_{n+\theta}) = n - r(t_{n+\theta}), t_{n+\theta} = t_n + \theta h, \\ &x(t) = hA(t), \quad y(t) = hB(t). \end{aligned}$$

$$\begin{aligned} (I - \theta x(t_{n+1}))u_{n+1} =& (I + (1-\theta)x(t_n))u_n + \theta y(t_{n+1})(\delta(t_{n+1})u_{n+2-m(t_{n+1})} \\ &+ (1-\delta(t_{n+1}))u_{n+1-m(t_{n+1})}) + (1-\theta)y(t_n)(\delta(t_n)u_{n+1-m(t_n)} \\ &+ (1-\delta(t_n))u_{n-m(t_n)}), \quad n \geq m \end{aligned} \tag{1.4b}$$

and

$$\begin{aligned} (I - \theta x(t_{n+\theta}))u_{n+1} =& (I + (1-\theta)x(t_{n+\theta}))u_n + \theta y(t_{n+\theta})(\delta(t_{n+1})u_{n+2-m(t_{n+1})} \\ &+ (1-\delta(t_{n+1}))u_{n+1-m(t_{n+1})}) + (1-\theta)y(t_{n+\theta})(\delta(t_n)u_{n+1-m(t_n)} \\ &+ (1-\delta(t_n))u_{n-m(t_n)}), \quad n \geq m. \end{aligned} \tag{1.4c}$$

Here, $\delta(t) = \dfrac{\tau(t)}{h} - r(t), r(t) = \Big[\dfrac{\tau(t)}{h}\Big]$, $0 \leq \delta(t) < 1$, $m(t) = \dfrac{t}{h} - r(t)$.

In what follows we give our basic definition of the stability.

Definition 1.1. *For all $\delta(t) \in [0,1)$ and x, y be given complex $d \times d$-matrice functions. A method is called stable at (x,y) if and only if any application of the method to problem* (1.3) *satisfies*

(I) *the matrix $I - \theta x(t) - \delta(t)\theta y(t)$ is invertible whenever $t \geq t_0, \theta \in [0,1]$.*

(II) *the method yields approximations u_n to $u(t_n)$ $(n = 1, 2, \cdots)$.*

such that

$$\|u_n\| \leq \max_{t \leq t_0} \|u_0(t)\| \quad (n = 1, 2, \cdots)$$

whenever t_0, τ, h, A, B, u_0 are given with $A(t) = \dfrac{x(t)}{h}$, $B(t) = \dfrac{y(t)}{h}$ and $m(t)$ is nonnegative integer $(t \geq t_0)$.

Definition 1.2. *The set consisting all pairs (x, y) at which a numerical method is stable is called stability region.*

For the one-leg θ-method we denote the stability region by S_θ and for the linear θ-method by $\tilde{S}_\theta$ and for the new θ-method by $\hat{S}_\theta$ respectively.

In the literature,several authors have dealt with the scalar case ($d = 1$) of test equation (1.3) in order to arrive at conclusions about the stability of numerical methods for delay differential equations (cf. [3, 9]). From these investigations, a complete characterzation for the set G_θ of all pairs of complex numbers (x, y) at which precesses (1.4) is stable can easily be obtained[9]. Further, the question has been studied whether or not, for given θ, the condition $H \subset G_\theta$ is fulfilled, where $H = \{(x(t), y(t)) | x(t) \in C^d, y(t) \in C^d, \|y(t)\| \leq -\mu(x(t))\}$, $\|\cdot\|$ is a given norm and $\mu(\cdot)$ is the corresponding logarithmic norm[8]. With the scalar case ($d = 1$), [3, 9] considered the test equation:

$$\begin{cases} u'(t) = a(t)u(t) + b(t)u(t-\tau), & t \geq t_0, \\ u(t) = u_0(t), & t \leq t_0. \end{cases} \tag{1.5}$$

Here $\tau > 0$ is constant, $a(t), b(t)$ are complex function $(t \geq t_0)$. It is known[3] that

$$|b(t)| \leq -\mathrm{Re}(a(t)) \Rightarrow |u(t)| \leq \max_{t \leq t_0} |u_0(t)|.$$

The general case of test equation (1.5) seems not to have been studied in the literature so far. In this paper we shall consider the test equation (1.3) with the general case of (1.5), i.e. the test equation (1.3) with arbitrary dimension $d (\geq 1)$ and arbitrary delay function $\tau(t)$ with $t - \tau(t) \geq \tau_0 > 0$.

1.3. Scope of our paper

The main purpose of the present paper is to determine all sets $S_\theta, \tilde{S}_\theta, \hat{S}_\theta$. In Section 2 we derive a complete characterization for the set of all pairs of complex $d \times d$-matrices $(x(t), y(t))$ at which process (1.4) is stable.

In Section 3, we obtain a criterion on the matrices $A(t), B(t)$ such that all exact solution $u(t)$ to test equation (1.3) satisfy $\|u(t)\| \leq \max_{t \leq t_0} \|u_0(t)\|$ for $t \geq t_0$. This generalizes the criterion of [3], which dealt with the case where $d = 1$.

In Section 4 we make the comparision of the three θ-methods and prove all three θ-methods are PN-stable if and only if $\theta = 1$.

2. Stability Regions of the θ-Methods

In this section we shall determine the sets $S_\theta, \tilde{S}_\theta$ and $\hat{S}_\theta$.

For a given matrix norm induced by an inner product, we denote

$$\alpha(\xi) = [I - \theta x(\xi)]^{-1}[I + (1-\theta)x(\xi)], \qquad \beta(\xi) = [I - \theta x(\xi)]^{-1} y(\xi),$$
$$M_\theta(\xi) = \|\alpha(\xi)\| + \|\beta(\xi)\|, \qquad \sigma(\xi,\eta) = [I - \theta x(\eta)]^{-1}[I + (1-\theta)x(\xi)],$$
$$\gamma(\xi,\eta) = [I - \theta x(\eta)]^{-1} y(\xi), \quad \tilde{M}_\theta(\xi,\eta) = \|\sigma(\xi,\eta)\| + \theta\|\beta(\eta)\| + (1-\theta)\|\gamma(\xi,\eta)\|.$$

We have the following theorem:

Theorem 2.1. *Let $d=1$, then the θ-methods are stable at (x,y) if and only if*

a) *$M_\theta(\xi) \le 1$, for all $\xi \ge t_0$ (if $\theta = 0$) and for all $\xi > t_0$ (if $\theta \ne 0$).*

b) *$\tilde{M}_\theta\ (\xi,\eta) \le 1$, for all $\eta > \xi \ge t_0$.*

c) *$M_\theta(\xi) \le 1$, for all $\xi \ge t_0$ (if $\theta = 0$) and for all $\xi > t_0$ (if $\theta \ne 0$).*

for one-leg θ-method, linear θ-method, new θ-method respectively.

We shall prove Theorem 2.1 only for the one-leg θ-method, since the technique of the proof for other two θ-method is completely analogous to that for the one-leg θ-method.

To prove Theorem 2.1, we shall need the following Lemma:

Lemma 2.1. *Let $d \ge 1, \theta, \xi, \eta$ be given, it hold*

i) *If $M_\theta(\xi) \le 1$, then*

$$\|[I - \theta x(\xi) - \delta y(\xi)]^{-1}[I + (1-\theta)x(\xi)]\| + (1-\delta(\xi))\|[I - \theta x(\xi) - \delta y(\xi)]^{-1} y(\xi)\| \le 1$$

for all $\delta(\xi)$ with $0 \le \delta(\xi) < 1$, $\xi \ge t_0$.

ii) *If $\tilde{M}_\theta\ (\xi,\eta) \le 1$, then*

$$\begin{aligned} &\|[I - \theta x(\eta) - \theta\delta(\eta)y(\eta)]^{-1}[I + (1-\theta)x(\xi)]\| \\ &\quad + (1-\delta(\eta))\theta\|[I - \theta x(\eta) - \delta(\eta)\theta y(\eta)]^{-1} y(\eta)\| \\ &\quad + (1-\theta)\|[I - \theta x(\eta) - \theta\delta(\eta)y(\eta)]^{-1} y(\xi)\| \le 1 \end{aligned}$$

for all $\delta(\eta)$ with $0 \le \delta(\eta) < 1$, $t \ge t_0$.

Proof. i) It is sufficient to prove

$$\begin{aligned} &\|[I - \theta x(\xi) - \delta(\xi)y(\xi)]^{-1}[I + (1-\theta)x(\xi)]\| + \|[I - \theta x(\xi) - \delta(\xi)\theta y(\xi)]^{-1} y(\xi)\| \\ &\quad \le 1 + \delta(\xi)\|[I - \theta x(\xi) - \delta(\xi)y(\xi)]^{-1} y(\xi)\|. \end{aligned}$$

It is easy to see from $M_\theta(\xi) \le 1$ that

$$\begin{aligned} &\|[I - \theta x(\xi) - \delta(\xi)y(\xi)]^{-1}[I + (1-\theta)x(\xi)]\| + \|[I - \theta x(\xi) - \delta(\xi)\theta y(\xi)]^{-1} y(\xi)\| \\ &\quad \le \|[I - \theta x(\xi) - \delta(\xi)y(\xi)]^{-1}[I - \theta x(\xi)]\| = \|I + \delta(\xi)[I - \theta x(\xi) \\ &\qquad - \delta(\xi)y(\xi)]^{-1} y(\xi)\| \le 1 + \delta(\xi)\|[I - \theta x(\xi) - \delta(\xi)y(\xi)]^{-1} y(\xi)\|. \end{aligned}$$

ii) The inequality is equivalent to

$$\|[I - \theta x(\eta) - \theta\delta(\eta)y(\eta)]^{-1}[I + (1-\theta)x(\xi)]\| + \theta\|[I - \theta x(\eta) - \delta(\eta)\theta y(\eta)]^{-1} y(\eta)\|$$

$$
\begin{aligned}
&+(1-\theta)\|[I-\theta x(\eta)-\theta\delta(\eta)y(\eta)]^{-1}y(\xi)\| \\
\le& 1+\delta(\eta)\theta\|[I-\theta x(\eta)-\delta(\eta)y(\eta)]^{-1}y(\eta)\|.
\end{aligned}
$$

Then from $\tilde{M}_\theta\ (\xi,\eta)\le 1$, it can be obtained that

$$
\begin{aligned}
\|[I-\theta x(\eta)-\theta\delta(\eta)y(\eta)]^{-1}&[I+(1-\theta)x(\xi)]\|+\theta\|[I-\theta x(\eta)-\delta(\eta)\theta y(\eta)]^{-1}y(\eta)\| \\
&+(1-\theta)\|[I-\theta x(\eta)-\theta\delta(\eta)y(\eta)]^{-1}y(\xi)\| \\
\le&\|[I-\theta x(\eta)-\theta\delta(\eta)y(\eta)]^{-1}[I-\theta x(\eta)]\|=\|I+\delta(\eta)\theta[I-\theta x(\eta) \\
&-\theta\delta(\eta)y(\eta)]^{-1}y(\eta)\|\le 1+\delta(\eta)\theta\|[I-\theta x(\eta)-\delta(\eta)y(\eta)]^{-1}y(\eta)\|. \ \square
\end{aligned}
$$

The proof of Theorem 2.1

(1) Assume that (a) holds. We obtain from (1.4a) and Lemma 2.1

$$
\|u_{n+1}\|\le\max(\|u_n\|,\|u_{r(t_{n+\theta})+1}\|,\|u_{r(t_{n+\theta})}\|)\quad(\text{if } r(t_{n+\theta})<n). \tag{2.1}
$$

and

$$
\|u_{n+1}\|\le\|u_n\|,\quad \text{if } r(t_{n+\theta})=n \tag{2.2}
$$

which implies by induction that

$$
\|u_{n+1}\|\le\max_{t\le t_0}\|u_0(t)\|.
$$

(2) Assume that there exists a $\xi\ge t_0$ (if $\theta=0$) or $\xi>t_0$ (if $\theta\ne 0$) such that $M_\theta(\xi)>1$, we shall prove that the one-leg θ-method is not stable.

let $t_\theta=\xi$, $\tau>\xi$,

$$
\begin{aligned}
&h=\begin{cases}\dfrac{\xi-t_0}{\theta}, & \theta\ne 0\\ \text{arbitrarily choosen}, & \theta=0\end{cases}\\
&A(t)=\frac{x(t)}{h},\quad B(t)=\frac{y(t)}{h},\\
&u_0(t_0)=\begin{cases}e^{-\arg(\alpha(\xi))}, & \alpha(\xi)\ne 0;\\ 1, & \alpha(\xi)=0\end{cases}\\
&u_0(t_\theta-\tau)=\begin{cases}e^{-\arg(\beta(\xi))}, & \beta(\xi)\ne 0;\\ 1, & \beta(\xi)=0\end{cases}
\end{aligned}
$$

such that $u_0(t)$ is continuous with $\max\limits_{-\tau\le t\le 0}|u_0|=1$.

Applying the one-leg θ-method (1.4a) with above h to the equation

$$
\begin{cases}u'(t)=A(t)u(t)+B(t)u(t-\tau), & t\ge 0;\\ u(t)=u_0(t), & -\tau\le t\le 0.\end{cases}
$$

we obtain

$$
u_1=\alpha(\xi)u_0+\beta(\xi)u(t_\theta-\tau)=|\alpha(\xi)|+|\beta(\xi)|=M_\theta(\xi)>1=\max_{-\tau\le t\le 0}|u_0(t)|.
$$

This prove that the one-leg θ-method is not stable, hence the assumption doesn't hold. $\square$

Remark 2.1. For $d > 1$, we don't know whether there are two vector $x, y \in C^d$ such that

$$\|\alpha(\xi)x + \beta(\xi)y\| > \max\{\|x\| + \|y\|\}. \tag{2.3}$$

If (2.3) holds, the sufficiency of Theorem 2.1 can be proved.

It is easy to obtain the following stetement from Theorm 2.1.

Corollary 2.1. *$S_\theta = \hat{S}_\theta = \{(x,y)\colon M_\theta(\xi) \le 1$, for all $\xi \ge t_0$ (if $\theta = 0$) and for all $\xi > t_0$ (if $\theta \ne 0$)$\}$, $\tilde{S}_\theta = \{(x,y)\colon \tilde{M}_\theta\ (\xi,\eta) \le 1$, for all $\eta > \xi \ge t_0\}$, if $d = 1$.*

3. The Stability of Test Equation (1.3)

Consider the following nonlinear equations:

$$\begin{aligned} &y'(t) = f(t, y(t), y(\tau(t))), \quad t \ge t_0 \\ &y(t) = \Phi(t), \quad t \le t_0 \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} &z'(t) = f(t, z(t), z(\tau(t))), \quad t \ge t_0 \\ &z(t) = p(t), \quad t \le t_0 \end{aligned} \tag{3.2}$$

where $f : [t_0, +\infty) \times C^d \times C^d \to C^d$, $y, z : R \to C^d$, $t - \tau(t) \ge \tau_0 > 0$, τ_0 is constant.

Before giving our stability criterion on test equation (1.3), we introduce the following Lemmas.

Lemma 3.1. *Assume that the delay function $\tau(t)$ is continuous and that there exists $\langle\cdot,\cdot\rangle$, an inner product on C^d, such that*

$$\gamma(t) \le -\sigma(t) \text{ for every } t \ge t_0$$

where

$$\sigma(t) := \sup_{z, y_1, y_2 \in C^d,\ y_1 \ne y_2} \frac{Re(\langle f(t, y_1, z) - f(t, y_2, z), y_1 - y_2\rangle)}{\|y_1 - y_2\|^2}, \tag{3.3}$$

$$\gamma(t) := \sup_{y, z_1, z_2 \in C^d,\ z_1 \ne z_2} \frac{\|f(t, y, z_1) - f(t, y, z_2)\|}{\|z_1 - z_2\|} \tag{3.4}$$

and $\|x\|^2 = \langle x, x\rangle$ for every $x \in C^d$. Then $\|y(t) - z(t)\| \le \max_{t \le t_0} \|\Phi(t) - p(t)\|$ for every $t \ge t_0$.

Proof. See [3]. $\square$

Lemma 3.2. *Let $A \in C^{d\times d}$, then*

(1) $\mu(A) = \sup_{\xi \in c^d, \xi \ne 0} Re\ \Big(\dfrac{\langle A\xi, \xi\rangle}{\|\xi\|^2}\Big) = \sup_{\xi \in c^d, \xi \ne 0} \dfrac{1}{2}\Big[\dfrac{\langle A\xi, \xi\rangle + \overline{\langle A\xi, \xi\rangle}}{\|\xi\|^2}\Big]$,

(2) $\max\{\mu(A), -\mu(A)\} \le \dfrac{1}{\|A^{-1}\|}$, *if A is nonsingular, where $\mu(\cdot)$ is logarithmic norm under a given inner product $\langle\cdot,\cdot\rangle$.*

Proof. To refer [8]. □

Combining Lemma 3.1 and Lemma 3.2 we have

Theorem 3.1. *Consider the delay differential equation:*

$$\begin{cases} u'(t) = A(t)u(t) + B(t)u(\tau(t)), & t \geq t_0, \\ u(t) = u_0(t), & t \leq t_0 \end{cases} \tag{3.5}$$

where $A(t), B(t)$ are complex matrix function and $t - \tau(t) \geq \tau_0 > 0$, $\tau(t)$ is continuous function. If

$$\|B(t)\| \leq -\mu(A(t)), \quad t \geq t_0, \tag{3.6}$$

then

$$\|u(t)\| \leq \max_{t \leq t_0} \|u_0\|,$$

where $\|\cdot\|$ is a norm induced by an inner product $\langle\cdot,\cdot\rangle$ and $\mu(\cdot)$ is corresponding logarithmic norm.

Proof. Observe that

$$\sigma(t) = \mu(A(t)), \quad \gamma(t) = \|B(t)\|$$

in Lemma 3.1,then the theorem is proved. □

Remark 3.1. Theorem 3.3 holds when applied it to the pantograph equation:

$$\begin{cases} U'(t) = A(t)U(t) + B(t)U(qt), & t > 0, \\ U(0) = U_0 \end{cases} \tag{3.7}$$

here $q \in (0,1)$. Observe that there is not a constant τ_0 such that $t - qt \geq \tau_0 > 0$ $(t > 0)$, but (3.7) can be transformed to the case (3.5) by introducing a transformation in the following way. Let $x(t) = U(e^t)$, for $t \geq \lg q$, then $x(t)$ satisfies the following initial value problem:

$$\begin{cases} x'(t) = A(t)e^t x(t) + B(t)e^t x(t + \lg q), & t > 0 \\ x(t) = U(e^t), & t \in [\lg q, 0]. \end{cases} \tag{3.8}$$

Hence, all results in our paper hold readily for the equation (3.7).

4. Comparision of the Three θ-Methods

In view of Theorem 3.3 it is natural to consider the following definition:

Definition 4.1. *A numerical method is called PN-stable if H is contained in the numerical method stability region, where*

$$H = \{(x,y) | x(t) \in C^d, y(t) \in C^d, \|y(t)\| \leq -\mu(x(t))\}. \tag{4.1}$$

It is easy to obtain the following conclusion from Corollary 2.2.

Lemma 4.1. $\tilde{S}_\theta \subset S_\theta = \hat{S}_\theta$ *for all* $\theta \in [0,1]$ *and* $d = 1$.

Proof. The proof can be obtained by noting that

$$\tilde{M}_\theta(\xi,\eta) = M_\theta(\xi) \text{ holds if } \xi = \eta.$$

Lemma 3.2. *None of the three θ-methods is PN-stable if $\theta \in [0,1)$ and $d \geq 1$.*

Proof. We only give the proof for one-leg θ-method. Without generality,we only consider the special case ($d = 1$) of (1.3).

Let x, y be real continuous functions with $x(t) = y(t) < -\dfrac{h}{1-\theta}$ and $h = 1$. Obviously we have $(x, y) \in H$. But for any $\xi \geq 0$,

$$M_\theta(\xi) = \Big|\frac{1+(1-\theta)x(\xi)}{1-\theta x(\xi)}\Big| + \Big|\frac{x(\xi)}{1-\theta x(\xi)}\Big| = -\frac{2x(\xi)}{1-\theta x(\xi)} - 1 > 1,$$

which implies $(x, y) \notin S_\theta$.

It can be seen that the three θ-methods are identical with $\theta = 0$ and $\theta = 1$. □

Theorem 4.1. *All three θ-methods are PN-stable if and only if $\theta = 1$.*

Proof. The "only if" part can be justified by Lemma 4.2. We only give the proof for the "if" part for the one-leg θ-method. The proofs for the other two θ-methods are analogous.

Let $\theta = 1$, then

$$M_\theta(\xi) = \|(I - x(\xi))^{-1}\| + \|(I - x(\xi))^{-1} y(\xi)\|.$$

It is easy to obtain

$$\begin{aligned} \|y(\xi)\| \leq -\mu(x(\xi)) &\Rightarrow 1 + \|y(\xi)\| \leq \mu(I - x(\xi)) \\ &\Rightarrow 1 + \|y(\xi)\| \leq \frac{1}{\|(I - x(\xi))^{-1}\|} \Rightarrow M_\theta(\xi) \leq 1 \end{aligned}$$

In the above proof, we have used Lemma 3.2. Then the "if" part is proved. □

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