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THE RELAXING SCHEMES FOR HAMILTON-JACOBI EQUATIONS*[1)]

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Abstract

Hamilton-Jacobi equation appears frequently in applications, e.g., in differential games and control theory, and is closely related to hyperbolic conservation laws[3, 4, 12]. This is helpful in the design of difference approximations for Hamilton-Jacobi equation and hyperbolic conservation laws. In this paper we present the relaxing system for Hamilton-Jacobi equations in arbitrary space dimensions, and high resolution relaxing schemes for Hamilton-Jacobi equation, based on using the local relaxation approximation. The schemes are numerically tested on a variety of 1D and 2D problems, including a problem related to optimal control problem. High-order accuracy in smooth regions, good resolution of discontinuities, and convergence to viscosity solutions are observed.

Key words: The relaxing scheme, The relaxing systems, Hamilton-Jacobi equation, Hyperbolic conservation laws.

1. Introduction

We are interested in the numerical approximation of viscosity solution of the following first-order Hamilton-Jacobi equation

$$\phi_t + H(\phi_{x_1}, \phi_{x_2}, ..., \phi_{x_d}) = 0, \tag{1.1}$$

with initial data $\phi(x, 0) = \phi_0(x)$. It is well known that the solutions to problem (1.1) typically are continuous (typically they are locally Lipschitz continuous) but with discontinuous derivatives, even though the initial data $\phi_0 \in C^\infty$. The nonuniqueness of such solutions to (1.1) also necessitates the introduction of the notions of entropy conditions and viscosity solutions, to pick out a unique practically relevant solution (We refer the reader to [1] for details). Therefore, the numerical schemes for solving (1.1) are expected to have: (i) higher order accuracy; (ii) no spurious oscillations in the presence of discontinuous derivatives.

Hamilton-Jacobi equation is often encountered in applications, e.g., in differential games and control theory, and are closely related to a hyperbolic conservation laws[3, 4, 5, 13]

$$\frac{\partial u}{\partial t} + \sum_{i=1}^{d} \frac{\partial f_i(u)}{\partial x_i} = 0. \tag{1.2}$$

In fact, for the case $d = 1$, (1.1) is equivalent to (1.2) if let $u = \phi_x$ [3]. For $d > 1$, this direct correspondence disappears, but in some sense we can still think about (1.1) as (1.2) "integrated once". This is helpful in the design of difference approximations. For example, successful

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numerical methodology for equation (1.2) should be applicable to equation (1.1)[13], and numerical schemes for equation (1.2) can also be designed from equation (1.1)[3, 18]. Crandal and Lions in [5] discussed an important class of numerical schemes for (1.1), the class of monotone schemes. They also proved convergence of monotone schemes to the viscosity solutions of (1.1). Unfortunately, monotone schemes are at most first order accuracy. In [13] Osher and Shu generalized ENO schemes for (1.2) to (1.1). Computational results have shown good accuracy in regions of smoothness and sharp resolution of discontinuities in the derivatives are obtained. However, implementation of ENO schemes seem to be more inconvenient.

In this paper we will present a class of high resolution relaxing schemes for Hamilton-Jacobi equation, based on using the local relaxation approximation [8, 15, 16, 17]. The relaxing scheme is obtained in the following way: Firstly a linear hyperbolic system with a stiff source term is constructed to approximate the original equation (1.1) with a small dissipative correction. Then this linear hyperbolic system can be solved easily by underresolved stable numerical discretizations. The main advantage of the schemes is to use neither nonlinear or linear Riemann solvers spatially nor nonlinear system of algebraic equations solvers temporally. Moreover, there is no exact or numerical integration in current schemes. The schemes are numerically tested on a variety of 1D and 2D problems, including a problem related to control optimization. High-order accuracy in smooth regions, good resolution of discontinuities in the derivatives, and convergence to viscosity solutions are also shown.

The paper is organized as follows. In section 2, the relaxing systems with a stiff source term are introduced to approximate the equation (1.1). In section 3, the relaxing schemes are constructed. The schemes are shown to have correct asymptotic limit as $\epsilon \to 0^+$. In section 4, some numerical tests are presented on a variety of 1D and 2D problems, including a problem related to control optimization. We conclude the paper with a few remarks in section 5.

We point out here that when the first version of our preprint was completed, Prof. Jin kindly informed the author that in an independent work [private communication], he and Xin considered the numerical passages from systems of conservation laws to Hamilton-Jacobi equations. From their note [9], it is clear that, besides totally different techniques, the results of the two works for relaxing schemes for Hamilton-Jacobi equation are different. Our result is also valid for Hamilton-Jacobi equation in arbitrary space dimensions.

2. The Relaxing Systems for Hamilton-Jacobi Equations

In this section we introduce the relaxing system with a stiff source, to approximate the equation (1.1). For the sake of simplicity in the presentation, we will focus on the single equation. First we consider Hamilton-Jacobi equation in one space variable.

To approximate the equation (1.1)($d = 1$), we can introduce a linear system with a stiff source term (hereafter called the *relaxing system*) as follows:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \\ &\frac{\partial v}{\partial t} + a\frac{\partial u}{\partial x} = -\frac{1}{\epsilon}(v - H(u)), \end{aligned} \tag{2.1}$$

where the small positive parameter ϵ is the relaxation rate, and a is a positive constant satisfying

$$| H'(u) | \leq \sqrt{a}, \text{ for all } u \in \mathcal{R}. \tag{2.2}$$

Remark. Here we can also consider the more general $a(x, t)$ instead of the above constant a. The results in this paper are not limited by the above constant a.

System (2.1) is equivalent to the one-dimensional perturbed equation

$$\phi_t + H(\phi_x) = \epsilon(a\phi_{xx} - \phi_{tt}),$$

if we take

$$v = -\phi_t, \ u = \phi_x.$$

By applying the Chapman-Enskog expansion, we can also derive the following first order approximation

$$v = H(u) - \epsilon\{a - [H'(u)]^2\}\frac{\partial u}{\partial x}, \tag{2.3}$$

$$\frac{\partial u}{\partial t} + \frac{\partial H(u)}{\partial x} = \epsilon\frac{\partial}{\partial x}(\{a - [H'(u)]^2\}\frac{\partial u}{\partial x}). \tag{2.4}$$

It is clear that the above second equation (2.4) is dissipative under condition (2.2), which is referred to as the *subcharacteristic condition* by Liu in [10].

In the following, we choose the special initial condition for the relaxing system (2.1) as follows:

$$u(x,0) = u_0(x) \equiv \partial\phi_0(x)/\partial x,$$
$$v(x,0) = v_0(x) \equiv H(u_0(x)).$$

The aim is to avoid the initial layer introduced by the relaxing system (2.1). In doing so the state is already in equilibrium initially. On the other hand, to avoid any new boundary layers in solving boundary value problems, we can also impose the boundary conditions for u and v that are consistent to the local equilibrium.

Then we consider Hamilton-Jacobi equation in two space variables, i.e.

$$\phi_t + H(\phi_{x_1}, \phi_{x_2}) = 0.$$

In a similar way for Hamilton-Jacobi equation in one space variable, one can introduce a *relaxing system* as follows:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial v}{\partial x} = 0, \\ &\frac{\partial w}{\partial t} + \frac{\partial v}{\partial y} = 0, \\ &\frac{\partial v}{\partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial w}{\partial y} = -\frac{1}{\epsilon}(v - H(u,w)), \end{aligned} \tag{2.5}$$

where the relaxation rate ϵ is a small positive parameter, and a and b is two positive constants. System (2.5) is equivalent to the following two-dimensional perturbed equation

$$\phi_t + H(\phi_x, \phi_y) = \epsilon(a\phi_{xx} + b\phi_{yy} - \phi_{tt}),$$

if we take

$$v = -\phi_t, u = \phi_x, w = \phi_y.$$

The special initial condition for the relaxing system (2.5) can be considered as

$$u(x,y,0) = u_0(x,y) \equiv \partial\phi_0(x,y)/\partial x,$$
$$w(x,y,0) = w_0(x,y) \equiv \partial\phi_0(x,y)/\partial y,$$
$$v(x,y,0) = v_0(x,y) \equiv H(u_0(x,y), w_0(x,y)).$$

Relaxation systems are important in many physical situations, for example, in gases not in thermodynamic equilibrium, kinetic theory, chromatography, river flows, and traffic flows etc.. The relaxation limit for systems of conservation laws with a stiff source term was first studied by Liu in [11]). Convergence of solutions of the general relaxing systems are considered later in [1, 2]. In this paper we are concerned with construction of the relaxing schemes for Hamilton-Jacobi equation from the relaxing systems(2.1) or (2.5).

3. The Relaxing Schemes for Hamilton-Jacobi Equations

Based on the above relaxing systems, we can consider construction of the relaxing schemes for Hamilton-Jacobi equation. In this section we will only do for Cauchy problems of 1D equation.

Introduce the spatial grid points $x_j,\ j \in \mathcal{Z}$ with the uniform mesh width $\Delta x = x_{j+1} - x_j$, i.e. Δx is a constant, and denote by $w_j(t)$ the approximate point value of $w(x,t)$ at $x = x_j$. The discrete time level are spaced uniformly with the step $\Delta t = t^{n+1} - t^n$ for $n \in \mathcal{Z}^+ \cup \{0\}$. In the following we will assume $\lambda = \frac{\Delta t}{\Delta x}$ a constant. The numerical relaxing approximations are obtained by discreting the system (2.1), for which it is convenient to treat the spatial and time discretization separately .

I. The spatial discretizations

A spatial discretization to (2.1) in conservation form can be written as

$$\begin{aligned} &\frac{\partial}{\partial t}u_j + \frac{1}{\Delta x}(v_{j+1/2} - v_{j-1/2}) = 0, \\ &\frac{\partial}{\partial t}v_j + \frac{a}{\Delta x}(u_{j+1/2} - u_{j-1/2}) = -\frac{1}{\epsilon}(v_j - H(u_j)), \end{aligned} \tag{3.1}$$

where the numerical flux $u_{j+1/2}$ and $v_{j+1/2}$ will be defined in four ways specified below.

Scheme I:(First order central scheme)[15, 16] The numerical flux in (3.1) is defined as:

$$\begin{aligned} v_{j+1/2} &= \frac{1}{2}(v_{j+1} + v_j) - \frac{1}{2\lambda}(u_{j+1} - u_j), \\ u_{j+1/2} &= \frac{1}{2}(u_{j+1} + u_j) - \frac{1}{2a\lambda}(v_{j+1} - v_j). \end{aligned} \tag{3.2}$$

Scheme II:(Second order MUSCL-type central scheme)[15, 16] The numerical flux in (3.1) is defined as:

$$\begin{aligned} v_{j+1/2} &= \frac{1}{2}(v^R + v^L) - \frac{1}{2\lambda}(u^R - u^L), \\ u_{j+1/2} &= \frac{1}{2}(u^R + u^L) - \frac{1}{2a\lambda}(v^R - v^L), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} v^L &= v_j + \frac{1}{2}\phi(r_j)\Delta v_{j+1/2}, \\ v^R &= v_{j+1} - \frac{1}{2}\phi(\frac{1}{r_{j+1}})\Delta v_{j+1/2}, \\ u^L &= u_j + \frac{1}{2}\phi(s_j)\Delta u_{j+1/2}, \\ u^R &= u_{j+1} - \frac{1}{2}\phi(\frac{1}{s_{j+1}})\Delta u_{j+1/2}, \\ r_j &= \frac{v_j - v_{j-1}}{v_{j+1} - v_j},\ s_j = \frac{u_j - u_{j-1}}{u_{j+1} - u_j}, \end{aligned}$$

and $\phi(r)$ is some symmetric limiters [18].

On the other hand, we also consider the upwind approximation to (2.1). For the sake of simplicity in the presentation, define $w^+ = v + \sqrt{a}u$ and $w^- = v - \sqrt{a}u$, which imply $v = \frac{1}{2}(w^+ + w^-)$ and $u = \frac{1}{2\sqrt{a}}(w^+ - w^-)$.

Scheme III:(First order upwind scheme)[8] The numerical flux in (3.1) is defined as:

$$w^+_{j+1/2} = w^+_j,\ w^-_{j+1/2} = w^-_{j+1}. \tag{3.4}$$

Scheme IV:(Second order MUSCL-type upwind scheme)[8] The numerical flux in (3.1) is defined as:

$$\begin{aligned} w^+_{j+1/2} &= w^+_j + \frac{1}{2}\phi(r^+_j)(w^+_{j+1} - w^+_j) = w^+_j + \frac{1}{2}\phi(\frac{1}{r^+_j})(w^+_j - w^+_{j-1}), \\ w^-_{j+1/2} &= w^-_{j+1} - \frac{1}{2}\phi(r^-_{j+1})(w^-_{j+2} - w^-_{j+1}) = w^-_j - \frac{1}{2}\phi(\frac{1}{r^-_{j+1}})(w^-_{j+1} - w^-_j), \end{aligned} \tag{3.5}$$

where

$$r_j^{\pm} = \frac{w_j^{\pm} - w_{j-1}^{\pm}}{w_{j+1}^{\pm} - w_j^{\pm}}.$$

Remark. (1) One simple choice of limiters is the so-called minmod limiters

$$\phi(r) = \max(0, \min(1, r)). \tag{3.6}$$

A sharper limiter was introduced by van Leer[18] as

$$\phi(r) = (\mid r \mid + r)/(1 + \mid r \mid). \tag{3.7}$$

(2) In fact, the above spatial discretizations (3.2) and (3.3) are using the Lax-Friedrichs type central difference without using linear or nonlinear Riemann solvers. Moreover, the central and upwind schemes have been shown to be TVD(total variation diminishing) and be of the similar relaxed form (see [8, 15] for details) in the zero relaxation limit.

II. The time discretizations

Numerical schemes for stiff relaxing systems such as (2.1) were studied in [10]. Proper implicit time discretizations should be taken to overcome the stability constraints brought by the stiff source. A simple way is to keep the convection terms explicit and the stiff source terms implicit. Since the source terms in form (2.1) is linear in the variable v, we can avoid to solve nonlinear systems of algebraic equation. As in [15], a general second order Runge-Kutta splitting scheme to (2.1) can be given

$$\begin{aligned}
&\bar{u}_j = u_j^n, && \bar{v}_j = v_j^n - \frac{\Delta t}{\epsilon}(\bar{v}_j - f(\bar{u}_j)),\\
&u_j^{(1)} = \bar{u}_j - \lambda\Delta_+\bar{v}_{j-1/2}, && v_j^{(1)} = \bar{v}_j - a\lambda\Delta_+\bar{u}_{j-1/2},\\
&\bar{\bar{u}}_j = u_j^{(1)}, && \bar{\bar{v}}_j = v_j^{(1)} + \alpha\frac{\Delta t}{\epsilon}(\bar{\bar{v}}_j - f(\bar{\bar{u}}_j)) + \beta\frac{\Delta t}{\epsilon}(\bar{v}_j - f(\bar{u}_j)),\\
&u_j^{(2)} = \bar{\bar{u}}_j - \lambda\Delta_+\bar{\bar{v}}_{j-1/2}, && v_j^{(2)} = \bar{\bar{v}}_j - a\lambda\Delta_+\bar{\bar{u}}_{j-1/2},\\
&u_j^{n+1} = \frac{1}{2}(u_j^n + u_j^{(2)}), && v_j^{n+1} = \frac{1}{2}(v_j^n + v_j^{(2)}),
\end{aligned} \tag{3.8}$$

where two parameters α and β should satisfy the consistent condition: $\alpha + \beta = -1$. For example, one can take $\alpha = +1,\ \beta = -2$ as in [8] or $\alpha = -1,\ \beta = 0$ as in [15, 16].

In this section we have presented the numerical discretization for the relaxing systems (2.1), which can be directly generalized to the relaxing systems in several space variables. To form a numerical algorithm for Hamilton-Jacobi equation, we have the following relaxing schemes for (1.1):

Step 1. Assume that $\phi^n(x)$ is given, then compute $u^n(x) = \phi_x^n$ and $v^n(x) = -\phi_t^n \equiv H(\phi_x^n)$.

Step 2. Use time discrete scheme (3.8) and spatial difference scheme (3.2) (or (3.3), (3.4), (3.5)) to evolve $u^{n+1}(x)$ and $v^{n+1}(x)$.

Step 3. Compute ϕ^{n+1} by using the following second–order accurate scheme

$$\phi^{n+1} = \phi^n - \frac{1}{2}\Delta t(v^{n+1} + v^n).$$

4. Numerical Results

In this section, we will present numerical examples which demonstrate the performance of our family of high resolution relaxing schemes for Hamilton-Jacobi equations. We experiment with the following members:

(1) The second order central relaxing scheme (3.3) with minmod limiters (3.6) and the second order Runge-Kutta splitting scheme (3.8). This scheme is referred to as CRS2M.

(2) The second order central relaxing scheme (3.3) with van Leer's limiters (3.7) and the second order Runge-Kutta splitting scheme (3.8). It is referred to as CRS2V.

(3) The second order upwind relaxing scheme (3.3) with minmod limiters (3.6) and the second order Runge-Kutta splitting scheme (3.8), which is referred to as URS2M.

(4) The second order upwind relaxing scheme (3.3) with van Leer's limiters (3.7) and the second order Runge-Kutta splitting scheme (3.8). This scheme is referred to as URS2V.

On the other hand, we will also present numerical results with the first order monotone Lax-Friedrichs scheme [5, 13] to compare it with the results obtained by our relaxing schemes. A 2D Lax-Friedrichs scheme can be given as

$$\begin{aligned}
&\phi_{j,k}^{n+1} = \phi_{j,k}^n - \Delta t \bar{H}\Big(\frac{\Delta_+^x \phi_{j,k}^n}{\Delta x}, \frac{\Delta_-^x \phi_{j,k}^n}{\Delta x}, \frac{\Delta_+^y \phi_{j,k}^n}{\Delta y}, \frac{\Delta_-^y \phi_{j,k}^n}{\Delta y}\Big),\\
&\Delta_-^x \phi_{j,k}^n = \pm(\phi_{j\pm1,k} - \phi_{j,k}), \Delta_-^y \phi_{j,k}^n = \pm(\phi_{j,k\pm1} - \phi_{j,k}),\\
&\bar{H}(a,b,c,d) = H\Big(\frac{a+b}{2}, \frac{c+d}{2}\Big) - \frac{1}{2}(a-b) - \frac{1}{2}(c-d).
\end{aligned} \tag{4.1}$$

I. One dimensional problem

Example 1. We use the above relaxing schemes for solving

$$\begin{cases} \phi_t + H(\phi_x) = 0, \quad -1 \le x < 1, \\ \phi(x,0) = -\cos(\pi x), \end{cases} \tag{4.2}$$

with a convex Hamiltonian H (Burgers' equation):

$$H(u) = \frac{1}{2}(u+\alpha)^2. \tag{4.3}$$

Here we take $\alpha = 1$ and present numerical solutions (in diamonds etc.) and the exact solutions (in solid lines) in Fig.1 and Fig.2. The results are computed to $t = \frac{1}{2\pi^2}$ (when the solution is still smooth) and to $t = \frac{3}{2\pi^2}$ (when the solution has a discontinuous derivative). In Fig.1 and Figs.2a-2b, numerical solutions is obtained with 20 grid points. In Fig.2c, the result is given at $t = \frac{3}{2\pi^2}$ with CRS2M scheme under 80 grid points. In all the tests we always use CFL=0.5, $a = 1.0$ and $\epsilon = 10^{-8}$ for the relaxing schemes, and use CFL=0.2 for Lax-Friedrichs scheme.

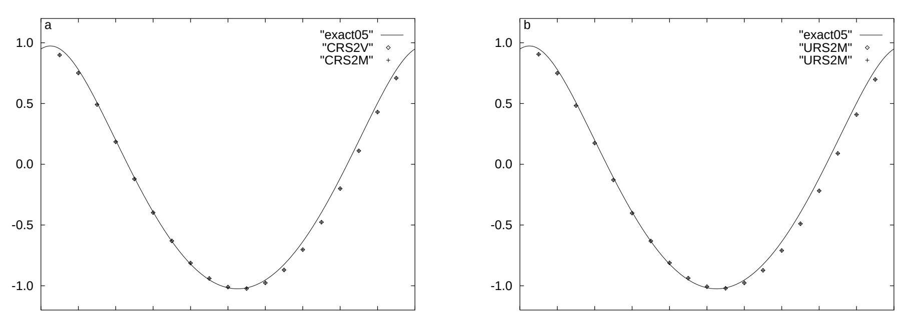

Fig.1. Problem (4.2) and (4.3), $t = \frac{1}{2\pi^2}$.

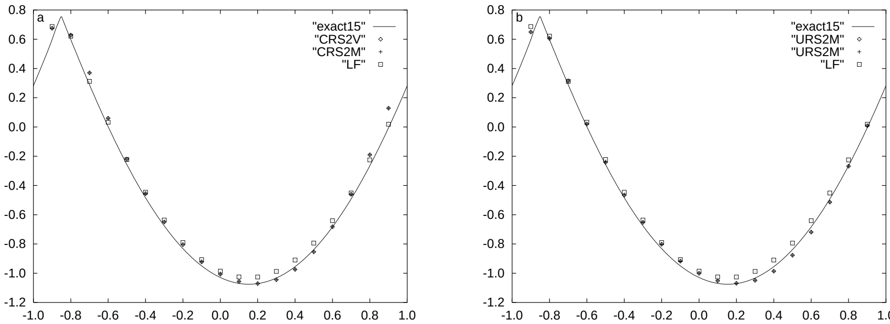

Fig.2. Problem (4.2) and (4.3), $t = \frac{3}{2\pi^2}$.

Example 2. We solve (4.2) with a nonconvex Hamiltonian H:

$$H(u) = -\cos(u + \alpha). \tag{4.4}$$

The results shown in Fig.3 are obtained at $t = \frac{3}{2\pi^2}$. In Figs.3a–3b, the results is obtained with 20 grid points. In Fig.3c, the result is given at $t = \frac{3}{2\pi^2}$ with CRS2M scheme under 80 grid points. In all the tests, parameters α,CFL, a and ϵ are taken as in **Example 1.**

The relaxing schemes capture important structures on the fine grid and the results are comparable to other high resolution schemes in [13].

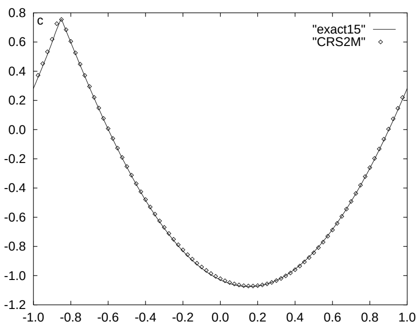

Fig.2–Continued.

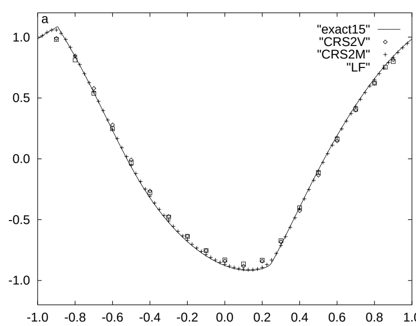

Fig.3 Problem (4.2) and (4.4), $t = \frac{3}{2\pi^2}$.

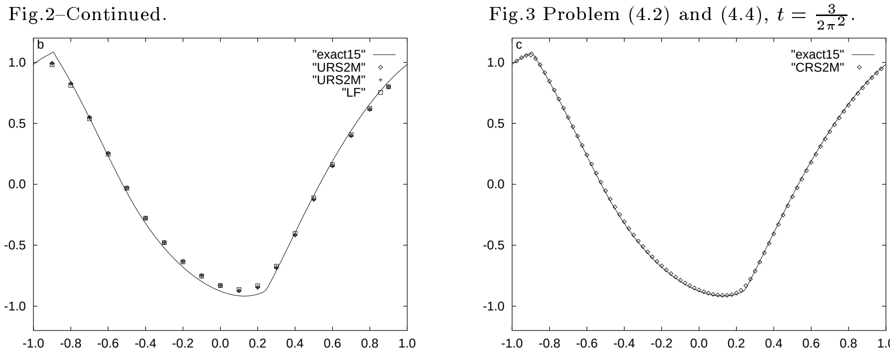

Fig.3–Continued.

II. Two dimensional problem

Example 3. We solve

$$\begin{aligned} &\phi_t + \frac{1}{2}(u + v + \alpha)^2 = 0, \quad -2 \le x, y < 2, \\ &\phi(x, y, 0) = -cos(\pi(x + y)/2). \end{aligned} \tag{4.5}$$

The results are shown in Fig.4 at $t = \frac{3}{2\pi^2}$ by the CRS2M scheme and LF scheme. The computational domain is divided into 40×40 grid points with $\Delta y = \Delta x = 1/20$. In this test we take $\alpha = 1$,CFL=0.5, $a = 1.0$ and $\epsilon = 10^{-8}$ for the relaxing schemes, and take CFL=0.2 for Lax-Friedrichs scheme.

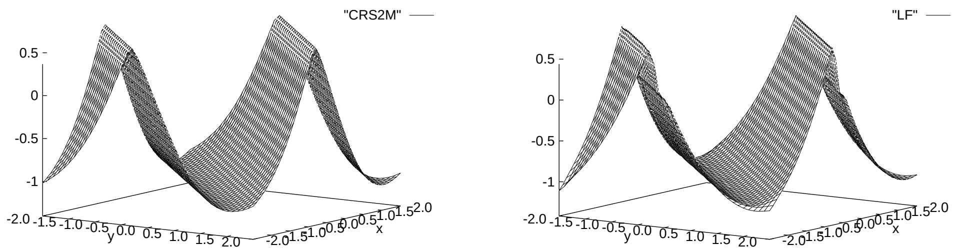

Fig.4. Problem (4.5), $t = \frac{3}{2\pi^2}$.

Example 4. We solve

$$\begin{aligned} &\phi_t - \cos(u + v + \alpha) = 0, \quad -2 \le x, y < 2, \\ &\phi(x, y, 0) = -\cos(\pi(x + y)/2). \end{aligned} \tag{4.6}$$

The results are plotted in Fig.5 at $t = \frac{3}{2\pi^2}$ by the CRS2M scheme and LF scheme with 40×40 grid points. In this test, parameters CFL, a,α and ϵ are taken as in **Example 3**.

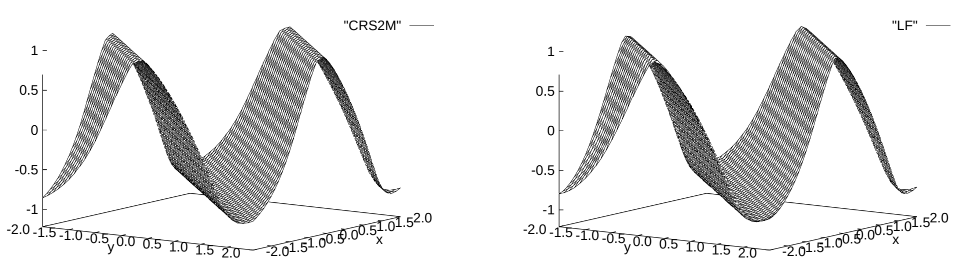

Fig.5. Problem (4.6), $t = \frac{3}{2\pi^2}$.

Example 5. We solve a two dimensional Riemann problem

$$\begin{aligned} &\phi_t + \sin(\phi_x + \phi_y) = 0, \quad |x|, \ |y| < \infty, \\ &\phi(x, y, 0) = \pi(| \, y \, | - | \, x \, |), \end{aligned} \tag{4.7}$$

with 80×80 grid points by the CRS2M scheme and LF scheme. In this test we use CFL=0.5, $a = 2.0$ and $\epsilon = 10^{-8}$ for the relaxing schemes, and use CFL=0.2 for Lax-Friedrichs scheme. The results at $t = 1$ are presented in Fig.6.

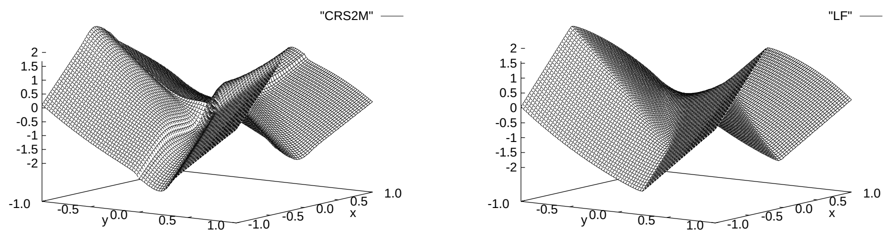

Fig.6. Problem (4.7), t=1.

Example 6. We solve a problem related to control optimal cost determination

$$\begin{aligned}&\phi_t - \sin(y)\phi_x + (\sin(x) + \text{sign}(\phi_y))\phi_y - \frac{1}{2}\sin^2(y) - (1 - \cos(x)) = 0, \\ &\phi(x, y, 0) = 0, \quad -\pi \le x,\ y < \pi,\end{aligned} \tag{4.8}$$

assuming periodicity, with 40×40 grid points by the CRS2M scheme and LF scheme. In this test we use CFL=0.5, $a = 2.0$ and $\epsilon = 10^{-8}$ for the relaxing schemes, and use CFL=0.2 for Lax-Friedrichs scheme. The results are shown in Fig.7. One sees that CRS2M scheme has sharper discontinuities (in derivative) resolution than first order LF scheme.

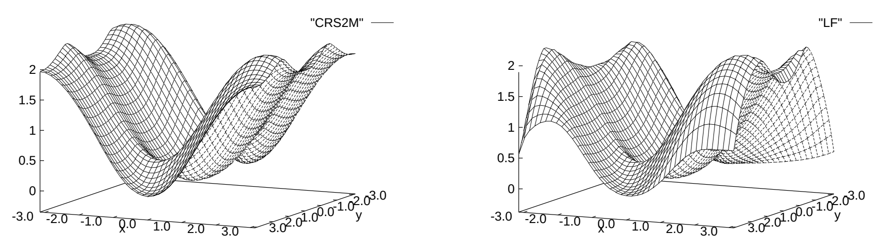

Fig.7. Problem (4.8), t=1.

5. Conclusions

In this paper we have presented the relaxing system for Hamilton-Jacobi equation in arbitrary space dimensions, and the relaxing schemes for Hamilton-Jacobi equations, based on using the local relaxation approximation. Our schemes are obtained without using linear or nonlinear Riemann solvers, and from a framework, which differs substantially in both concept and methodology from one as in [12]. Implementation of the relaxing schemes is more simply and convenient. Numerical results indicate high-order accuracy in smooth regions, good resolution of discontinuities in the derivatives, and convergence to viscosity solutions.

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