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ON BANDLIMITED SCALING FUNCTION*[1)]

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Abstract

This paper discuss band–limited scaling function, especially on the interval band case and three interval bands case, its relationship to oversampling property and weakly translation invariance are also studied. At the end, we propose an open problem.

Key words: Scaling function, Oversampling property, Weakly translation invariance, Aliasing error.

1. Introduction

A MRA of $L^2(R)$ is an increasing family of subspaces $\{V_m\}_{m\in Z}$, with
1) $V_m \subset V_{m+1}$,
2) $f(x) \in V_m$ if and only if $f(2x) \in V_{m+1}$,
3) $\bigcup_m V_m = L^2(R)$ and $\bigcap_m V_m = \{0\}$,
4) There exists a function $\varphi(x) \in V_0$(called **scaling function** of MRA $\{V_m\}_m$) such that $\{\varphi(x-m)\}_m$ forms a Riesz basis of V_0.

An **orthogonal MRA** is a MRA with $\{\varphi(x-m)\}_m$ forming an orthogonal Riesz basis of V_0. Sometimes, φ (or a MRA $\{V_m\}_m$) is said to be **orthonormal** if $\{\varphi(x-m)\}_m$ is an orthonormal Riesz basis of V_0.

Clearly, $\{\varphi(2^jx-m)\}_{m\in Z}$forms the basis of V_j. Let $W_0 = V_1 \ominus V_0$ be the direct complement of V_0 in V_1, $\psi(x) = \sum d_k\varphi(2x-k)$ (called wavelet) such that $\{\psi(x-n)\}_{n\in Z}$ forms a Riesz basis of W_0. $\varphi(x) = \sum c_k\varphi(2x-k)$, then we have

$$\widehat{\varphi}(\omega) = m_0(\omega/2)\widehat{\varphi}(\omega/2), \tag{1.1}$$

and

$$\widehat{\psi}(\omega) = m_1(\omega/2)\widehat{\varphi}(\omega/2), \tag{1.2}$$

where $m_0(\omega) = \frac{1}{2}\sum_k c_k e^{-ik\omega}, m_1(\omega) = \frac{1}{2}\sum_k d_k e^{-ik\omega}$ and the Fourier transform is defined by

$$\widehat{f}(\omega) = \int_R f(t)e^{-i\omega t}dt \quad \text{for} \quad f(t) \in L^2(R)\cap L^1(R). \tag{1.3}$$

From (1.1) and (1.2) we know that the scaling function play a very important role for constructing a wavelet $\psi(x)$. How to find a mild wavelet for some problems can be reduced to find a proper scaling function $\varphi(x)$.

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As a special important case, [4] and [7] have dealed with the compactly supported scaling function systems, it is to solve the following dilation equation with finite non–zero c_k

$$\varphi(x) = \sum c_k \varphi(2x - k). \tag{1.4}$$

However, many important scaling function, such as Shannon Wavelet and Meyer Wavelet [9] are not compactly supported but band limited, therefore dealing with band–limited scaling function in more general case is necessary and useful. Unfortunately, this problem is not so easy to solve as the compactly support case. Up to now, there are yet no systematic results about the problem, except some sporadic works such as [3].

In the paper, we focus on the scaling function with interval band and finite unions of interval bands. After general discussing on scaling function In section 3, a characterization of bandlimited scaling function will be given which formerly only can hold with translation invariance (see [9] or section 3). By the way, we also prove interval band scaling functions are translation invariance and have oversampling property (see [12] or section 3). Additionally, the aliasing error with oversampling property (see [J] or section 1) is estimated.

In section 4, we will discuss the scaling function with three interval bands and follow some similar results as section 3. At the end of the paper, we propose an open problem.

2. Elementary Properties of Scaling Function

The scaling functions of a MRA have many important elementary properties. We here as completion will list some propositions concerning these properties which will be applied to the following sections. Some proofs that appeared in other references will be omitted.

Proposition 2.1. *$\{V_j\}$ is an orthonormal MRA with scaling function $\varphi(t)$, $\widehat{\varphi}(\xi)$ is continuous, then*

(a) $\sum_k |\widehat{\varphi}(\omega + 2k\pi)|^2 = 1, |m_0(\omega)|^2 + |m_1(\omega)|^2 = 1 \quad a,e \quad \omega \in R,$

(b) $\widehat{\varphi}(0) = 1$ *and* $\widehat{\varphi}(2k\pi) = 0 \quad$ *for* $\quad k \neq 0,$

(c) $\sum_k \varphi(x - k) = \widehat{\varphi}(0)$ *when* $\varphi(x) \in L^1(R).$

Proof of (a), (b) and (c) are well known and can be found in many books concerning wavelet, such as [6], [8], [5] and [11].

Proposition 2.2. *$\varphi(x)$ is an orthonormal scaling function of MRA $\{V_m\}$, $supp\widehat{\varphi}(\omega) = \Omega$, then*

(a) $\Omega \subset 2\Omega,$

(b) $\bigcup_k \{\Omega + 2k\pi\} = R,$

(c) $\bigcup_j \{2^j \Omega\} = R.$

Proof. The proof is easy and also can be found in [9].

(a) follows from $\Omega = \mathrm{supp}\widehat{\varphi}(\omega) \subset \mathrm{supp}\widehat{\varphi}(\frac{\omega}{2}) = 2\Omega$. due to (1.1).

(b) follows from (a) of proposition 2.1.

(c) follows from $\overline{\bigcup_m V_m} = L^2(R)$.

Proposition 2.3. *$\varphi(t)$ is an orthonormal scaling function with $supp\widehat{\varphi}(\omega) = \Omega$, then $|\Omega| \geq 2\pi$ where $|\Omega|$ is the Lebesque measure of measurable set Ω.*

Proof. Let $R = \sum_k [2k\pi, 2(k+1)\pi)$, then $\Omega = \sum_k \Omega_k$, where $\Omega_k = \Omega \bigcap [2k\pi, 2(k+1)\pi)$. Clearly, $\bigcup_k \{\Omega_k - 2k\pi\} \subset [0, 2\pi]$. We can claim

$$|[0, 2\pi] \setminus \bigcup_k \{\Omega_k - 2k\pi\}| = 0. \tag{2.1}$$

Otherwise there must exist a measurable set $\delta \subset [0,2\pi) \setminus \bigcup_k \{\Omega_k - 2k\pi\}$, such that $|\delta| > 0$. Therefore

$$\sum_k |\widehat{\varphi}(\omega + 2k\pi)|^2 = 0 \quad \text{for each } \omega \in \delta. \tag{2.2}$$

The contradiction of (2.2) and (a) of proposition 2.1 implies that (2.1) holds.

But

$$|\bigcup_k \{\Omega_k - 2k\pi\}| \le \sum |\Omega_k - 2k\pi| = \sum |\Omega_k| = |\Omega|. \tag{2.3}$$

By (2.1) and (2.3), we conclude that $|\Omega| \ge 2\pi$.

Proposition 2.4. *Suppose that $\varphi(t)$ is a scaling function with supp$\widehat{\varphi}(\omega) = \Omega$. Let $a = \inf \Omega$ and $b = sup\Omega$, then $a \le 0$ and $b \ge 0$.*

Proof. If $a > 0$, then $2a > a$. Since there is a number sequence $\alpha_n \in \Omega$ such that $\alpha_n \to a$, we can find α_{n_0} with

$$a \le \alpha_{n_0} < 2a. \tag{2.4}$$

By (a) of proposition 2.2, we know

$$\alpha \ge 2a \quad \text{for each} \quad \alpha \in \Omega \tag{2.5}$$

(2.4) and (2.5) imply that $a \le 0$.

Similarly we can prove $b \ge 0$.

As the end of the section, we will list a result about band–limited scaling function, its proof can be found in [3] or [6].

Proposition 2.5. *Assume that $\varphi(t)$ is the scaling function of MRA $\{V_j\}$, $\widehat{\varphi}(\omega)$ is continuous and compactly support, then*

$$\widehat{\varphi}(\frac{4}{3}\pi) = 0 \quad or \quad \widehat{\varphi}(-\frac{4}{3}\pi) = 0. \tag{2.6}$$

3. Interval Band Scaling Function

Let $\varphi(t)$ be a scaling function of a MRA $\{V_m\}$ with $\widehat{\varphi}(\omega)$ being continuous, then supp$\widehat{\varphi} = \widehat{\varphi}^{-1}\{R \setminus \{0\}\}$ is an open set in R, we can express supp$\widehat{\varphi} = \sum_i \delta_i$, where δ_i is an open interval in R, but (b) of proposition 2.1 implies that there must exist a δ_{i_0} containing zero point. As a special case, if supp$\widehat{\varphi} = (a,b)$ with $a < 0$ and $b > 0$, we have following result.

Proposition 3.1. *$\varphi(t)$ is an orthonormal scaling function with supp$\widehat{\varphi}(\omega) = (a,b)$, then*

(a) $2\pi \le b - a \le \frac{8}{3}\pi, \quad \frac{b}{2} - a \le 2\pi, \quad b - \frac{a}{2} \le 2\pi,$

(b) *Let $\alpha = b - a - 2\pi$, then $|\widehat{\varphi}(\omega)| = 1$ when $\omega \in (a+\alpha, b-\alpha)$,*

(c) *$|\widehat{\varphi}(\omega)|^2 + |\widehat{\varphi}(\omega + 2\pi)|^2 = 1$ when $\omega \in (a, a+\alpha)$ or equivalently $|\widehat{\varphi}(\omega)|^2 + |\widehat{\varphi}(\omega - 2\pi)|^2 = 1$ when $\omega \in (b-\alpha, b)$.*

Proof. (A) supp$\widehat{\varphi}(\omega) = (a,b)$ implies that supp$\widehat{\varphi}(2\omega) = (\frac{a}{2}, \frac{b}{2})$.

But $\widehat{\varphi}(2\omega) = m_0(\omega)\varphi(\omega)$ due to (1.1).

We have

$$\text{supp} m_0(\omega) \supset (\frac{a}{2}, \frac{b}{2}), \tag{3.1}$$

and

$$m_0((a, \frac{a}{2}) \cup (\frac{b}{2}, b)) = 0. \tag{3.2}$$

Therefore

$$m_0((a, \frac{a}{2}) + 2\pi) = 0 \quad \text{and } m_0((\tfrac{b}{2}, b) - 2\pi) = 0 \tag{3.3}$$

due to the 2π–periodicity of $m_0(\omega)$. By (3.2) and (3.3) we know

$$a + 2\pi \geq \frac{b}{2}, \quad \frac{a}{2} + 2\pi \geq b. \tag{3.4}$$

Take summation of the first two inequalities of (3.4), we have

$$b - a \leq \frac{8}{3}\pi. \tag{3.5}$$

(3.4) and (3.5) are just (a) of the proposition.

(B) For each $\omega \in (a+\alpha, b-\alpha)$, we have $\omega + 2\pi \geq b$ and $\omega - 2\pi \leq a$, hence (a) of proposition 2.1 degenerate to $|\widehat{\varphi}(\omega)| = 1$

(C) Similarly, the (c) of proposition (3.1) follows from (a) of preposition 2.1 and (a) of proposition 3.1.

Remark 3.2. From the proof of (A) we know that the orthonormality condition of $\varphi(t)$ is not necessary for (a).

With the help of proposition 3.1, we now can give a characterization of interval band scaling function.

Proposition 3.3. *$\varphi(t)$ is an orthonormal scaling function, then supp$\widehat{\varphi}(\omega) = (a,b)$ is a bounded interval if and only if there is an $\varepsilon \in R$ with $0 \leq \varepsilon \leq \frac{\pi}{3}$ and a distribution h with supp$h = (-\varepsilon, \varepsilon)$, such that $\widehat{h}(0) = 1$ and $|\widehat{\varphi}(\omega)|^2 = h * \chi_{(-\pi,\pi)}(\omega - \alpha)$, where $\alpha = \frac{a+b}{2}$.*

Proof. "If"is obvious.

"Only if"

By translation, we assume that $\alpha = 0$. Define $h(\omega)$ as the distribution in $\mathcal{D}'$ given by the distribution derivative

$$h(\omega) = \begin{cases} \frac{d}{d\omega}|\widehat{\varphi}(w-\pi)|^2, & \omega \in (a+\pi, a+3\pi] \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

But (a) of proposition 2.1 and (a) of proposition 3.1 imply

$$|\widehat{\varphi}(\omega+\pi)|^2 + |\widehat{\varphi}(\omega-\pi)|^2 = 1 \quad \text{for } \omega \in [b-3\pi, b-\pi) \cup (a+\pi, a+3\pi]. \tag{3.7}$$

Take distribution derivative on both sides of (3.7), we have

$$\frac{d}{d\omega}|\widehat{\varphi}(\omega+\pi)|^2 = -\frac{d}{d\omega}|\widehat{\varphi}(\omega-\pi)|^2 \quad \text{for } \omega \in [b-3\pi, b-\pi) \cup (a+\pi, a+3\pi]. \tag{3.8}$$

By (a) of proposition 3.1 and (3.8), we know

$$\frac{d}{d\omega}|\widehat{\varphi}(\omega+\pi)|^2 = -\frac{d}{d\omega}|\widehat{\varphi}(\omega-\pi)|^2 = 0 \quad \omega \in [b-3\pi, a+\pi], \tag{3.9}$$

$$\frac{d}{d\omega}|\widehat{\varphi}(\omega+\pi)|^2 = -\frac{d}{d\omega}|\widehat{\varphi}(\omega-\pi)|^2 = h(\omega) \quad \omega \in (a+\pi, b-\pi), \tag{3.10}$$

and

$$\frac{d}{d\omega}|\widehat{\varphi}(\omega+\pi)|^2 = -\frac{d}{d\omega}|\widehat{\varphi}(\omega-\pi)|^2 = 0 \quad \omega \in [b-\pi, a+3\pi]. \tag{3.11}$$

Therefore

$$\frac{d}{d\omega}|\widehat{\varphi}(\omega)|^2 = \begin{cases} h(\omega+\pi) & \omega \in (a, b-2\pi) \\ 0 & \text{otherwise} \\ -h(\omega-\pi) & \omega \in (a+2\pi, b) \end{cases} \tag{3.12}$$

But (3.6) and (3.11) imply

$$\text{supp}h(\omega) = (a+\pi, b-\pi). \tag{3.13}$$

Hence

$$\text{supp}h(\omega+\pi) = (a, b-2\pi), \quad \text{supp}h(\omega-\pi) = (a+\pi, b). \tag{3.14}$$

(3.12) and (3.14) imply

$$\begin{aligned} \frac{d}{d\omega}|\widehat{\varphi}(\omega)|^2 &= h(\omega+\pi)-h(\omega-\pi) \\ &= h*(\delta(\omega+\pi)-\delta(\omega-\pi)) \\ &= h*\tfrac{d}{d\omega}\chi_{(-\pi,\pi)}(\omega), \end{aligned}$$

where $\chi_{(-\pi,\pi)}$ is the characteristic function of interval $(-\pi,\pi)$.

Furthermore, we have

$$|\widehat{\varphi}(\omega)|^2 = h(\omega)*\chi_{(-\pi,\pi)}(\omega). \tag{3.15}$$

Let $\varepsilon = \frac{b-a-2\pi}{2}$. From (3.13), we know

$$\text{supp}h(\omega) = (-\varepsilon,\varepsilon).$$

By inverse translation, we know

$$|\widehat{\varphi}(\omega)|^2 = h(\omega)*\chi_{(-\pi,\pi)}(\omega-\alpha)$$

and

$$\widehat{\varphi}(0)=1 \quad \text{imply } \widehat{h}(0)=1.$$

Remark 3.4. [9] get a similar characterization of bandlimited scaling function with additional assumption weakly translation invariance. But from proposition 3.3, the weakly translation invariance is not necessary, its proof benefit from proposition 3.1.

[10] proposed the oversampling property when he developed Shannon sampling theorem to wavelet subspaces. Furthermore [12 generalized [10] to J case. We first read the definition of oversampling property, then show a proposition about that of bandlimited scaling function.

Definition 3.5. *Let $\varphi(t)$ be the scaling function of a MRA $\{V_m\}$, with $\varphi(t) = O(|t|^{-\varepsilon-1})$ for some $\varepsilon > 0$, then $\varphi(t)$ or MRA $\{V_m\}$ are called to have* **oversampling property** *with rate $J(J \in Z^+ \cup \{0\})$ if*

$$f(t) = \sum f(\frac{n}{2^J})\varphi(2^J t - n) \quad \textit{for } f(t) \in V_0. \tag{3.16}$$

Proposition 3.6. *$\varphi(t)$ is a scaling function with interval band and $\varphi(t) = O(|t|^{-\varepsilon-1})$ for some $\varepsilon > 0$, then $\varphi(t)$ has oversampling property with rate $J(J \geq 1)$.*

Proof. [12] has showed $\varphi(t)$ had oversampling property if and only if

$$\widehat{\varphi}(\omega) = \widehat{\varphi}_J^*(\omega)\widehat{\varphi}(\omega/2^J), \tag{3.17}$$

where $\widehat{\varphi}_J^* = \sum \widehat{\varphi}(\omega + 2^{J+1}k\pi)$.

Let supp$\widehat{\varphi}(\omega) = (a,b)$, then

$$\text{supp}\widehat{\varphi}_J^* = \sum_k \{(a,b) + 2^{J+1}k\pi\} \quad \text{due to } b-a \leq \tfrac{8}{3}\pi.$$

But (a) of proposition 3.1 implies that

$$2\pi + 2^{-J}a > b \quad \text{and} \quad a > 2^{-J}b - 2\pi, \qquad \text{for } J \geq 1.$$

Therefore

$$\widehat{\varphi}(\omega) = \widehat{\varphi}^*(\omega) \quad \text{on} \quad \text{supp}\widehat{\varphi}(\tfrac{\omega}{2^J}) = 2^J(a,b). \tag{3.18}$$

(3.17) and (3.18) imply that we only need to show

$$\widehat{\varphi}(\omega) = \widehat{\varphi}(\omega)\widehat{\varphi}(\omega/2^J). \tag{3.19}$$

By (b) of proposition 3.1, we know

$$\widehat{\varphi}(\frac{\omega}{2^J}) = 1 \quad \text{on} \quad 2^J(a+\alpha, b-\alpha),$$

where $\alpha = b - a - 2\pi \le 2\pi/3$.
Since $2^J(a+\alpha, b-\alpha) \supset (a,b)$ due to $2\pi \ge b - 2^J a$ and $2\pi \ge 2^{-J}b - a$. Therefore (3.19) holds.

With the oversampling property, (3.16) holds for each $f \in V_0$, but we wish to know the aliasing error $\|e(f,t)\|$ if approximating $f \in V_1$ by (3.16), where $e(f,t) = f - \sum f(\frac{n}{2^J})\varphi(\frac{x}{2^J} - n)$, such as one can reconstruct a non–bandlimited function by means of the band pass sampling theorem [1], [2]. [10] have estimated $\|e(f,t)\|$ when $J=1$, and obtained $\|e(f,t)\| \le C\|C_f\|$, where $C_f = \sum_n < f, \psi(x-n) > e^{-in\omega}$, the following corollary will tell us more information about the constant C for $J \ge 1$.

Corollary 3.7. *$\varphi(t)$ is the orthonormal scaling function of MRA $\{V_m\}$ with interval band, then the aliasing error*

$$\|e(f,t)\| \le 2^{J/2}(\sum |b_n|^2)^{1/2}\|1 - \widehat{\varphi}^*(\omega)\|_\infty$$

for $J > 0$, where $b_n = < f, \psi(\cdot - n) >$, $\psi(x)$ is the wavelet of $\{V_m\}$ and $\widehat{\varphi}^* = \sum_k \widehat{\varphi}(\omega + 2k\pi)$.

Proof. From proposition 3.6, we only need to consider $f \in W_0$. Let

$$f(t) = \sum_n b_n \psi(t-n), \quad c_f(\omega) = \sum b_n e^{-in\omega},$$

then we have

$$\begin{aligned} \|e(f,t)\| &= \|f - \sum_n f(\frac{n}{2^J})\varphi(2^J t - n)\| \\ &= \|C_f(\omega)\widehat{\psi}(\omega) - 2^{-J}\sum_n f(\frac{n}{2^J})e^{-in\omega}\widehat{\varphi}(\frac{\omega}{2^J})\|. \end{aligned}$$

Utilizing Poisson Summation Formula, we have

$$\begin{aligned} \|e(f,t)\| &= \|C_f(\omega)m_1(\tfrac{\omega}{2})\widehat{\varphi}(\tfrac{\omega}{2}) - \sum_k \widehat{f}(\omega + 2^{J+1}k\pi)\widehat{\varphi}(\frac{\omega}{2^J})\| \\ &= \|C_f(\omega)m_1(\tfrac{\omega}{2})\prod_{j=2}^{J} m_0(\frac{\omega}{2^j})\widehat{\varphi}(\frac{\omega}{2^J}) - C_f(\omega)m_1(\frac{\omega}{2}) \\ &\cdot \prod_{j=2}^{J} m_0(\frac{\omega}{2^j})\widehat{\varphi}^*(\frac{\omega}{2^J})\widehat{\varphi}(\frac{\omega}{2^J})\| \\ &\le \|C_f(\omega)\widehat{\varphi}(\tfrac{\omega}{2^J}) - C_f(\omega)\widehat{\varphi}^*(\tfrac{\omega}{2^J})\widehat{\varphi}(\tfrac{\omega}{2^J})\| \\ &= (\sum_k \int_0^{2^{J+1}\pi} |C_f(\omega)|^2 |\widehat{\varphi}(\frac{\omega}{2^J} + 2k\pi)|^2 |1 - \widehat{\varphi}^*(\frac{\omega}{2^J})|^2)^{1/2} \\ &= (\textstyle\int_0^{2^{J+1}\pi} |C_f(\omega)|^2 |1 - \widehat{\varphi}^*(\tfrac{\omega}{2^J})|^2 \displaystyle\sum_k |\widehat{\varphi}(\frac{\omega}{2^J} + 2k\pi)|^2)^{1/2} \\ &= (\textstyle\int_0^{2^{J+1}\pi} |C_f(\omega)|^2 |1 - \widehat{\varphi}^*(\tfrac{\omega}{2^J})|^2)^{1/2}, \end{aligned}$$

the inequality due to $|m_1(\omega)|$, $|m_0(\omega)| \le 1$. Therefore

$$\|e(f,t)\| \le 2^{J/2}\|C_f\|_{L^2_{[0,2\pi]}}\|1 - \widehat{\varphi}^*(\omega)\|_\infty.$$

[9] introduced the concept of weakly translation invariance when he developed the shift invariance subspace from the sense of integer to real number and showed Meyer–type wavelets had this property. But the following theorem will tell us any scaling functions with interval band is weakly translation invariance. We still repeat the definition of weakly translation invariance here.

Definition 3.8. *Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}$, τ_h is a translation operator, $\tau_h\varphi(t) = \varphi(t-h)$ for $h \in R$. MRA $\{V_m\}$ are called to be weakly translation invariant if $\tau_h\varphi(t) \in V_1$ for each $h \in R$.*

Proposition 3.9. *$\varphi(t)$ is the scaling function of MRA $\{V_m\}$ with interval band, then $\{V_m\}_m$ is weakly translation invariant.*

Proof. Let supp$\widehat{\varphi}(\omega) = (a,b)$, then $(b-a) \leq \frac{8}{3}\pi \leq 4\pi$ due to (a) of proposition. 3.1.

We can find a function $\mathcal{X} \in C^\infty$, such that $\mathcal{X}(\omega) = \begin{cases} 0 & |\omega| \geq 2\pi \\ 1 & |\omega| \leq \frac{4}{3}\pi \end{cases}$, then

$$e^{-ih\omega}\widehat{\varphi}(\omega) = e^{-ih\omega}\widehat{\varphi}(\omega)\mathcal{X}(\omega - c),$$

where $c = \frac{a+b}{2}$.

Let $m(\omega) = \sum_k e^{-ih(\omega+4k\pi)}\mathcal{X}_{(\omega+4k\pi-c)}$. We get

$$\widehat{\tau_h\varphi(\omega)} = e^{-ih\omega}\widehat{\varphi}(\omega) = m(\omega)\widehat{\varphi}(\omega), \tag{3.20}$$

Since $m(\omega)$ is a 4π–periodic function in C^∞, (3.20) implies that $\tau_h\varphi \in V_1$.

4. Three Bands Scaling Function and Open Problem

Let $\varphi(t)$ be the scaling function of MRA $\{V_m\}$ with supp$\widehat{\varphi}(\omega) = (a,b) \cup (c,d) \cup (e,f)$, we assume that $0 \in (a,b), d < a$ and $e > b$.

By (a) of proposition 2.2

$$(\frac{a}{2}, \frac{b}{2}) \cup (\frac{c}{2}, \frac{d}{2}) \cup (\frac{e}{2}, \frac{f}{2}) \subset (a,b) \cup (c,d) \cup (e,f).$$

We get

$$\frac{f}{2} \leq b \quad \text{and} \quad \frac{c}{2} \geq c. \tag{4.1}$$

Therefore

$$m_0((e,f) \cup (\frac{f}{2}, b) \cup (\frac{b}{2}, \frac{e}{2}) \cup (c,d) \cup (a, \frac{c}{2}) \cup (\frac{d}{2}, \frac{a}{2}) = 0,$$

due to $\widehat{\varphi}(2\omega) = m_0(\omega)\widehat{\varphi}(\omega)$.

The 2π–periodicity of $m_0(\omega)$ implies that

$$m_0((e,f) \cup (\frac{f}{2}, b) \cup (\frac{b}{2}, \frac{e}{2}) \cup (c,d) \cup (a, \frac{c}{2}) \cup (\frac{d}{2}, \frac{a}{2}) + 2k\pi) = 0$$

if $a + 2\pi \in (\frac{b}{2}, \frac{e}{2})$, then

$$\frac{c}{2} + 2\pi = \frac{e}{2} \quad \text{and} \quad \frac{d}{2} + 2\pi = \frac{f}{2}.$$

We have

$$f - d = e - c = 4\pi.$$

Since $b - a \geq 2\pi$, and $\frac{1}{2}(b-a) + f - e \leq 2\pi$, we have $f - e \leq \pi$, that is to say

$$f - c \leq 5\pi.$$

Otherwise, if $a + 2\pi > \frac{f}{2}$ and $b - 2\pi < \frac{c}{2}$, then

$$b - a \leq 4\pi + \frac{c-f}{2}.$$

But (4.1) tells us

$$b - a \geq \frac{f-c}{2},$$

therefore $f - c \leq 4\pi$.

Similarly from the proof of proposition 3.1. we also can get $b - a \leq \frac{8\pi}{3}$. This is the following theorem.

Theorem 4.1. *Let $\varphi(t)$ be a scaling function with supp$\widehat{\varphi}(\omega) = (a,b) \cup (c,d) \cup (e,f), 0 \in (a,b)$ and $d < a$, $e > b$. Then $b - a \leq \frac{8\pi}{3}$ and $f - c \leq 5\pi$.*

Remark 4.2. In general case we can discuss the scaling function with finite intervals band or infinite intervals, but it is not so easy, by now there are no general results. The results in Theorem 4.1 is not sharp. We are studying the sharp conditions. In fact, we can prove that $f - c < \frac{32}{7}\pi$ which will be proposed in another paper concerning more general cases. Similarly we can prove the three interval bands case is weakly translation invariant as proposition 3.8, and therefore has the characterization as proposition 3.2.

Open problem 4.3. Let $\varphi(t)$ be a scaling function with supp$\widehat{\varphi}(\omega) = \Omega$ being bounded open set. Can we show $|\Omega| \leq \frac{8\pi}{3}$? As we know it is correct when Ω is an interval.

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