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SUPERAPPROXIMATION PROPERTIES OF THE INTERPOLATION OPERATOR OF PROJECTION TYPE AND APPLICATIONS*[1)]

Tie Zhang
(*Department of Mathematics, Northeastern University, Shenyang 110006, China*)

Yan-ping Lin   R.J. Tait
(*Department of Mathematical Sciences, University of Alberta, Canada, T6G 2G1*)

Abstract

Some superapproximation and ultra-approximation properties in function, gradient and two-order derivative approximations are shown for the interpolation operator of projection type on two-dimensional domain. Then, we consider the *Ritz* projection and *Ritz-Volterra* projection on finite element spaces, and by means of the superapproximation elementary estimates and *Green* function methods, derive the superconvergence and ultraconvergence error estimates for both projections, which are also the finite element approximation solutions of the elliptic problems and the Sobolev equations, respectively.

Key words: Interpolation operator of projection type, Finite element, Superconvergence.

1. Introduction

Finite element superconvergence property has long attracted considerable attentions since its practical importance in enhancing the accuracy of finite element calculation and in adaptive computing via a posteriori error estimate. In this field affluent research results have been achieved. Some recent developments include the patch recovery technique by Zienkiewicz and Zhu [1], the computer-based approach by Babuska, Strouboulis, et al. [2], superconvergence via local symmetry by Schatz, Sloan and Wahlbin [3], and the integral identity method by Lin and his colleagues [4-5], etc. For a more complete literature on superconvergence, the reader is referred to Wahlbin's book [6], Chen and Huang's book [7], and a recent conference proceeding edited by Krizek et al. [8].

In article [4,9], Lin Qun presented a new type of interpolation operator into the finite element spaces, ie. the interpolation operator of projection type, and remarked that it will approximate the finite element solutions much better than the usual *Lagrange* interpolation. Thus, the interpolation operator of projection type provides a new powerful means in the research of finite element superconvergence. In this paper, we first investigate the interpolation operator of projection type in two-dimensional rectangular domain case and find out many delicate superapproximation and ultra-approximation properties in function, gradient and two-order derivative approximations, some of them are unknown for the *Lagrange* interpolation operator, where by ultra-approximation we mean that the approximate order is two-order higher than the global optimal approximate order. Then, we consider the *Ritz* projection on the tensor-product

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finite element spaces associated with the general partial differential operator of second order

$$A = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} a_{ij} \frac{\partial}{\partial x_i} + \sum_{i=1}^{2} a_i \frac{\partial}{\partial x_i} + a_0 I$$

Utilizing the superapproximation properties of interpolation operator of projection type and the *Green* function methods, we prove that the *Lobatto*, *Gauss* and *quasi-Lobatto* points on each subdivision element are superconvergence points of *Ritz* projection in function, gradient and two-order derivative approximations, successively. Furthermore, in the case of $A = -\Delta + a_0 I$, the ultraconvergence results are obtained successively for the function approximation at mesh nodal points, gradient approximation under arithmetic mean of two *Gauss* points and two-order mixed derivative approximation at *Gauss* points. Generally speaking, the finite element solution itself does not share the global superconvergence, but, in this paper, we obtain superconvergence on whole domain for the two-order mixed derivative approximation, this is a somewhat remarkable result. Finally, we study the *Ritz-Volterra* projection on finite element spaces which has been widely used in the finite element analysis for various evolution equations [11−13], and prove that those superconvergence and ultraconvergence properties shared by *Ritz* projection also hold for the *Ritz-Volterra* projection. A direct application of this projection here is the analysis of semidiscret finite element approximation to the *Sobolev* equations.

We would like to indicate that some superconvergence results concerning the *Ritz* projection in our paper may be not new[6,7,10], but they are derived by a different approach and for more general case (see operator A).

In Section 2 we introduce the interpolation operator of projection type and investigate its various superapproximation properties. Section 3 and 4 are devoted to the superconvergence of *Ritz* projection and *Ritz-Volterra* projection, respectively, and finally the semidiscret finite element approximation to the initial-boundary value problems of *Sobolev* equations is discussed.

In this article we shall use letter C to denote a generic constant which may not be the same in each occurrence and also use the standard notions for the *Sobolev* spaces and norms.

2. Interpolation Operator of Projection Type and Its Superapproximation Properties

Let element $e = e_1 \times e_2 = (x_e - h_e, x_e + h_e) \times (y_e - k_e, y_e + k_e)$, $\{L_j(x)\}_{j=0}^{\infty}$ and $\{\widetilde{L}_j(y)\}_{j=0}^{\infty}$ be the normalized orthogonal *Legendre* polynomial systems on $L_2(e_1)$ and $L_2(e_2)$, respectively. Set

$$\omega_0(x) = \widetilde{\omega}_0(y) = 1, \quad \omega_{j+1}(x) = \int_{x_e-h_e}^{x} L_j(x)\,dx, \ \widetilde{\omega}_{j+1}(y) = \int_{y_e-k_e}^{y} L_j(y)\,dy, \ j \geq 0$$

It is well known that polynomials $\omega_{k+1}(x), L_k(x)$ and $L'_k(x)$ $(k \geq 1)$ have successively $k+1$, k and $k-1$ zeros on element $e_1 = (x_e - h_e, x_e + h_e)$, and these zeros are symmetrically distributed with respect to the middle point x_e. Denote the three kinds of zero point set successively as *Lobatto*, *Gauss* and *quasi-Lobatto* point set by $N_k^{(s)} = \{g_j^{(s)}\}$, $s = 0, 1, 2$. Moreover, these polynomials also possess the following symmetry and antisymmetry

$$\omega_{2j}(x_e + x) = \omega_{2j}(x_e - x), \quad \omega_{2j-1}(x_e + x) = -\omega_{2j-1}(x_e - x) \tag{1}$$

$$L_{2j}(x_e + x) = L_{2j}(x_e - x), \quad L_{2j-1}(x_e + x) = -L_{2j-1}(x_e - x) \tag{2}$$

$$L'_{2j}(x_e + x) = -L'_{2j}(x_e - x), \quad L'_{2j-1}(x_e + x) = L'_{2j-1}(x_e - x) \tag{3}$$

It should be indicated that according to the *Legendre* differential equation on the standard element $\hat{e} = (-1,1)$

$$k(k+1)L_k = -((1-t^2)L_k')', \ t \in (-1,1), \ k \geq 1$$

one can easily see that $x_e \pm h_e$ are *Lobatto* points of $\omega_{k+1}(x)$, and the *Lobatto* point set $N_k^{(0)}$ is identical with the *quasi-Lobatto* point set $N_k^{(2)}$ except points $x_e \pm h_e$

The completely parallel conclusions hold for the polynomials $\tilde{\omega}_{k+1}(y)$, $\tilde{L}_k(y)$ and $\tilde{L}'_k(y)$ on element $e_2 = (y_e - k_e, y_e + k_e)$.

Below we denote the *Lobatto*, *Gauss* and *quasi-Lobatto* point set on element $e = e_1 \times e_2$ by $N_k^{(s)} \times \tilde{N}_k^{(s)} = \{ G_{ij}^{(s)} = (g_i^{(s)}, \tilde{g}_j^{(s)}) \}$, $s = 0,1,2$, and also denote the *Lobatto*, *Gauss* and *quasi-Lobatto* lines by $G_{x,j}^{(s)} = \{ (x, \tilde{g}_j^{(s)}) ; x \in e_1, \tilde{g}_j^{(s)} \in \tilde{N}_k^{(s)} \}$ and $G_{i,y}^{(s)} = \{ (g_i^{(s)}, y) ; g_i^{(s)} \in N_k^{(s)}, y \in e_2 \}$, $s = 0,1,2$. Now let $u \in H^2(e)$, then we have *Fourier* expansion [4]

$$u(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \beta_{ij}\, \omega_i(x)\, \tilde{\omega}_j(y), \ (x,y) \in e \tag{4}$$

$$\beta_{00} = u(x_e - h_e, y_e - k_e), \quad \beta_{ij} = \int_e u_{xy}\, L_{i-1}(x)\, \tilde{L}_{j-1}(y)\, dxdy \tag{5}$$

$$\beta_{i0} = \int_{e_1} u_x(x, y_e - k_e)\, L_{i-1}(x)\, dx, \quad \beta_{0j} = \int_{e_2} u_y(x_e - h_e, y)\, \tilde{L}_{j-1}(y)\, dy, \ i,j \geq 1 \tag{6}$$

Introduce the k-order and bicomplete k-order polynomial spaces P_k and Q_k, ie.

$$p(x,y) = \sum_{i+j=0}^{k} a_{ij}\, x^i y^j, \ p \in P_k; \quad q(x,y) = \sum_{i=0}^{k} \sum_{j=0}^{k} a_{ij}\, x^i y^j, \ q \in Q_k$$

Denote the bicomplete k-order interpolation operator of projection type by $\pi_k : H^2(e) \to Q_k(e)$ such that

$$\pi_k u(x,y) = \sum_{i=0}^{k} \sum_{j=0}^{k} \beta_{ij}\, \omega_i(x)\, \tilde{\omega}_j(y), \ (x,y) \in e \tag{7}$$

Then π_k is uniquely solvable with respect to $Q_k(e)$ and for $k \geq 1$ possesses properties[4]

$$\pi_k u(x_e \pm h_e, y_e \pm k_e) = u(x_e \pm h_e, y_e \pm k_e) \tag{8}$$

$$|u - \pi_k u|_{m.p.e} \leq C h^{k+1-m} |u|_{k+1.p.e}, \ 1 \leq p \leq \infty, \ 0 \leq m \leq k+1 \tag{9}$$

here $h = \sqrt{h_e^2 + k_e^2}$. From (4) and (7) we have

$$u - \pi_k u = \Big(\sum_{i=0}^{k} \sum_{j=k+1}^{\infty} + \sum_{i=k+1}^{\infty} \sum_{j=0}^{k} + \sum_{i=k+1}^{\infty} \sum_{j=k+1}^{\infty} \Big) \beta_{ij}\, \omega_i(x)\, \tilde{\omega}_j(y) \tag{10}$$

Lemma 1. *Let $u \in Q_k(e) \bigcup \{x^{k+1}, y^{k+1}\}$. Then*

$$u - \pi_k u = \beta_{0,k+1}\, \tilde{\omega}_{k+1}(y) + \beta_{k+1,0}\, \omega_{k+1}(x), \qquad (u - \pi_k u)_{xy} = 0$$

$$(u - \pi_k u)_x = \beta_{k+1,0} L_k(x), \quad (u - \pi_k u)_y = \beta_{0,k+1}\, \tilde{L}_k(y)$$

$$(u - \pi_k u)_{xx} = \beta_{k+1,0} L_k'(x), \quad (u - \pi_k u)_{yy} = \beta_{0,k+1}\, \tilde{L}'_k(y)$$

Proof. Let $u \in Q_k(e)\bigcup\{x^{k+1}, y^{k+1}\}$ so that $u_{xy} \in Q_{k-1}(e)$, $u_x \in P_k(e_1)$ and $u_y \in P_k(e_2)$. Then, from (5) − (6) and the orthogonality of *Legendre* polynomial system, we have

$$\beta_{ij} = 0,\ i \geq k+1, j \geq 1 \text{ or } i \geq 1,\ j \geq k+1, \quad \beta_{i0} = \beta_{0j} = 0,\ i \geq k+2,\ j \geq k+2$$

taking this into (10) to obtain the representation of $u - \pi_k u$, the other representations follow from taking partial derivatives to the representation formula of $u - \pi_k u$, the proof is completed.

Theorem 1. *Denote* $D_1 = \frac{\partial}{\partial x}, D_2 = \frac{\partial}{\partial y}$, $D^l = D_1^{l_1} D_2^{l_2}$, $l = (l_1, l_2)$, $|l| = l_1 + l_2$. *Then, the interpolation operator* π_k *possesses the following superapproximation properties*

$$|D^l(u - \pi_k u)(G_{ij}^{(|l|)})| \leq Ch^{k+2-|l|}|u|_{k+2.\infty.e}, \quad |l| = 0, 1, 2 \tag{11}$$

here $k \geq 1$, *or* $k \geq 2$ *when* $l_1 = 2$ *or* $l_2 = 2$. *Moreover, for the mixed derivative approximation, the global superapproximation holds*

$$\|D_1 D_2(u - \pi_k u)\|_{0.\infty.e} \leq Ch^k|u|_{k+2.\infty.e}\ ,\ k \geq 1 \tag{12}$$

Proof. Introduce bilinear transformation $F : \hat{e} \rightarrow e$ by $(x, y) = F(\xi, \eta) = (x_e + \xi h_e, y_e + \eta k_e)$, $(\xi, \eta) \in \hat{e} = (-1, 1) \times (-1, 1)$. Denote $\hat{u}\,(\xi, \eta) = u(F(\xi, \eta))$, $\hat{D}^{\,l} = D_\xi^{l_1} D_\eta^{l_2}$. For fixed point $G_{ij}^{(|l|)} = F(\xi_i^{(|l|)}, \eta_j^{(|l|)})$, $|l| = 0, 1, 2$, and smooth function u, define the linear functional

$$E^{(l)}(\hat{u}) = \hat{D}^{\,l}(\hat{u} - \hat{\pi}_k\hat{u})(\xi_i^{(|l|)}, \eta_j^{(|l|)}) = h_e^{l_1} k_e^{l_2}\, D^l(u - \pi_k u)(G_{ij}^{(|l|)})\ ,\ |l| = 0, 1, 2 \tag{13}$$

From (9) we see that $E^{(l)}$ is a linear bounded functional on $W_\infty^{k+2}(\hat{e})$, and it follows from Lemma 1 that

$$E^{(l)}(\hat{u}) = 0,\ \forall\ \hat{u} \in\ P_{k+1}(\hat{e}),\ |l| = 0, 1, 2$$

Then, according to the *Bramble − Hilbert* Lemma [14], we have

$$|E^{(l)}(\hat{u})| \leq C|\ \hat{u}\ |_{k+2.\infty.\hat{e}} \leq Ch^{k+2}|u|_{k+2.\infty.e}\ ,\ |l| = 0, 1, 2$$

Combining this with (13), estimate (11) is proved. Similarly, by using the representation of $(u - \pi_k u)_{xy}$ in Lemma 1, the estimate (12) can be derived, the proof is completed.

Remark 1. From the proof of Theorem 1, it is easy to see that the superapproximation points $G_{ij}^{(|l|)}$ in (11) can be replaced by superapproximation lines $G_{x,j}^{(|l|)}$ (or $G_{i,y}^{(|l|)}$) when $l_1 = 1, 2$ (or $l_2 = 1, 2$).

Let $x_e \leq g_i^{(1)} \in N_k^{(1)}$, $y_e \leq \tilde{g}_j^{(1)} \in \tilde{N}_k^{(1)}$ and $\overline{g}_i^{(1)}$ ($\overline{\tilde{g}}_j^{(1)}$) be the symmetric *Gauss* point of $g_i^{(1)}$ ($\tilde{g}_j^{(1)}$) with respect to x_e (y_e). Denote the arithmetic mean operators

$$\overline{D}_1 u(\, g_i^{(1)}, y) = \frac{1}{2}[\, D_1 u(\, g_i^{(1)}, y) + D_1 u(\, \overline{g}_i^{(1)}, y)\,],\ \ y \in e_2$$

$$\overline{D}_2 u(\, x, \tilde{g}_j^{(1)}) = \frac{1}{2}[\, D_2 u(\, x, \tilde{g}_j^{(1)}) + D_2 u(\, x, \overline{\tilde{g}}_j^{(1)})\,],\ \ x \in e_1$$

Theorem 2. *Let Gauss points* $(\, g_i^{(1)}, \tilde{g}_j^{(1)}) \in N_k^{(1)} \times \tilde{N}_k^{(1)}$ *(or* $(\, g_i^{(1)}, \tilde{g}_j^{(1)}) \in N_{k+1}^{(1)} \times \tilde{N}_{k+1}^{(1)}$ *when* k *is odd), Lobatto points* $(\, g_i^{(0)}, \tilde{g}_j^{(0)}) \in N_k^{(0)} \times \tilde{N}_k^{(0)}$. *Then, interpolation operator* π_k *possesses the ultra-approximation properties*

$$|\overline{D}_1(u - \pi_k u)(\, g_i^{(1)}, \tilde{g}_j^{(0)})| \leq C\, h^{k+2}|\, u\,|_{k+3.\infty.e}\ ,\ \ k \geq 1 \tag{14}$$

$$|\overline{D}_2(u - \pi_k u)(g_i^{(0)}, \tilde{g}_j^{(1)})| \leq C\, h^{k+2}|\, u\,|_{k+3.\infty.e}\ ,\ \ k \geq 1 \tag{15}$$

Moreover, on Gauss point set $G_{ij}^{(1)} = (g_i^{(1)}, \widetilde{g}_j^{(1)}) \in N_k^{(1)} \times \widetilde{N}_k^{(1)}$, *the following ultra-approximation property also holds*

$$|D_1 D_2(u - \pi_k u)(G_{ij}^{(1)})| \le Ch^{k+1}|u|_{k+3.\infty.e}, \ k \ge 1 \tag{16}$$

Proof. First consider (14). From (10),(5)-(6) and the orthogonality of *Legendre* polynomial system, we have when $u \in P_{k+2}(e)$ that

$$\begin{aligned} D_1(u - \pi_k u)(x,y) &= \sum_{i=1}^{k} \beta_{i,k+1} L_{i-1}(x)\, \widetilde{\omega}_{k+1}(y) + \sum_{j=1}^{k} \beta_{k+1,j}\, \widetilde{\omega}_j(y) L_k(x) \\ &+ \beta_{k+1,0} L_k(x) + \beta_{k+2,0} L_{k+1}(x) + \beta_{k+1,k+1} L_k(x)\, \widetilde{\omega}_{k+1}(y) \end{aligned} \tag{17}$$

taking $y = \widetilde{g}_j^{(0)} \in \widetilde{N}_k^{(0)}$, and then using the symmetry and antisymmetry of $L_k(x)$ with respect to x_e (see (2)), it implies that

$$\begin{aligned} & D_1(u - \pi_k u)(x_e + \tau, \widetilde{g}_j^{(0)}) + D_1(u - \pi_k u)(x_e - \tau, \widetilde{g}_j^{(0)}) \\ &= \begin{cases} 2(\sum_{m=1}^{k} \beta_{k+1,m}\, \widetilde{\omega}_m(\widetilde{g}_j^{(0)}) + \beta_{k+1,0}) L_k(x_e + \tau), & k \text{ is even} \\ 2\beta_{k+2,0} L_{k+1}(x_e + \tau), \quad k \text{ is odd}, & \forall u \in P_{k+2}(e) \end{cases} \end{aligned}$$

Letting $\tau = g_i^{(1)} - x_e = x_e - \overline{g}_i^{(1)}$, $g_i^{(1)} \in N_k^{(1)}$ (or $g_i^{(1)} \in N_{k+1}^{(1)}$ when k is odd), we obtain

$$\overline{D}_1(u - \pi_k u)(g_i^{(1)}, \widetilde{g}_j^{(0)}) = 0, \ \forall u \in P_{k+2}(e) \tag{18}$$

From (18) and *Bramble* − *Hilbert* Lemma, the estimate (14) is proved by similar argument as in Theorem 1. The proof of (15) is similar. For estimate (16), differentiating (17) with respect to y and then taking $(x,y) = (g_i^{(1)}, \widetilde{g}_j^{(1)})$ to obtain

$$D_1 D_2(u - \pi_k u)(g_i^{(1)}, \widetilde{g}_j^{(1)}) = 0, \forall u \in P_{k+2}(e)$$

Thus, the estimate (16) is proved by *Bramble* − *Hilbert* Lemma.

3. Superconvergence of Ritz Projection

Let $\Omega \subset R^2$ be a rectangular domain, $J_h = \{e\}$ be a sequence of subdivisions of $\overline{\Omega}$ parameterized by mesh size h so that $\overline{\Omega} = \bigcup_{e \in J_h} \overline{e}$. We assume that the partition is regular and all elements $\{e\}$ are rectangles with sides parallel to the coordinate axes, respectively. Denote the finite element space to be

$$S_h = \{ v \in C(\overline{\Omega}) \bigcap H_0^1(\Omega);\ v|_e \in Q_k(e), \forall e \in J_h, k \ge 1\}$$

On each element $e \in J_h$, we define the bicomplete k-order interpolation operator of projection type π_k as in Section 2 so that π_k is determined on J_h. From article [4, Section 1.4.3], we know for $k \ge 2$ that

$$\int_e (u - \pi_k u)\, q\, dxdy = 0, \ \forall q \in Q_{k-2}(e), e \in J_h \tag{19}$$

$$\int_l (u - \pi_k u)\, p\, ds = 0, \ \forall p \in P_{k-2}(l), \text{ line segment } l \in \partial e \tag{20}$$

Thus, properties (8) and (20) make $\pi_k : H^2(\Omega) \to S_h$, $k \geq 1$. Define the bilinear form

$$A(u,v) = \sum_{i,j=1}^{2} (a_{ij}\, D_i u, D_j v) + \sum_{i=1}^{2} (a_i\, D_i u, v) + (a_0\, u, v) \tag{21}$$

where (,) represents the inner product on $L_2(\Omega)$, $a_{ij}(x,y)$, $a_i(x,y)$ and $a_0(x,y)$ are proper smooth functions. It is well known that elementary estimate of interpolation operator (also called as interpolation week estimate) play an important role in the research of superconvergence. Many detail discussions have been given for the interpolation operator of *Lagrange* type [10]. For the interpolation operator of projection type, by properties (19) and (20) and using the Bilinear Lemma [14], we can prove that (a detail proof can be found in article [15,Theorem 2])

Theorem 3. [15] *Let bilinear form $A(u,v)$ be defined by (21) (not necessary positive definition and symmetry), $u \in H_0^1(\Omega) \bigcap W_p^{k+2}(\Omega)$. Then, the interpolation operator π_k satisfies the following superapproximation elementary estimates*

$$|A(u - \pi_k u, v_h)| \leq C h^{k+1} \|u\|_{k+2.p} \|v_h\|_{1.q}\,,\ k \geq 1 \tag{22}$$

$$|A(u - \pi_k u, \pi_k v)| \leq C h^{k+2} \|u\|_{k+2.p} \|v\|_{2.q}\,,\ k \geq 2 \tag{23}$$

where $v_h \in S_h$, $v \in H_0^1(\Omega) \bigcap H^2(\Omega)$, $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

For bilinear form $A(u,v)$ in simple case, we may have elementary estimates better.

Theorem 4. *Let bilinear form $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$, $u \in H_0^1(\Omega) \bigcap W_p^{k+3}(\Omega)$. Then, the interpolation operator π_k satisfies the following ultra-approximation elementary estimates*

$$|A(u - \pi_k u, v_h)| \leq C h^{k+2} \|u\|_{k+3.p} \|v_h\|_{1.q}\,,\ k \geq 2 \tag{24}$$

$$|A(u - \pi_k u, \pi_k v)| \leq C h^{k+3} \|u\|_{k+3.p} \|v\|_{2.q}\,,\ k \geq 3 \tag{25}$$

where $v_h \in S_h$, $v \in H_0^1(\Omega) \bigcap H^2(\Omega)$, $2 \leq p \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

The estimate of term $(\nabla(u - \pi_k u), \nabla w_h)$ with $w_h = v_h$ or $\pi_k v$, has been given in article [4, Section 2.12], we only need to estimate the function term. Let $q_m(w)$ be the piecewise m-order interpolation polynomial approximation of function w on J_h. From (19) and approximation property we have

$$(a_0(u - \pi_k u), v_h) = (u - \pi_k u, a_0 v_h - q_0(a_0 v_h)) \leq C h^{k+2} \|u\|_{k+1.p} \|v_h\|_{1.q}\,,\ k \geq 2$$

the estimate (24) is obtained. Again using (19) and approximation property, for $k \geq 3$, we have

$$(a_0(u - \pi_k u), \pi_k v) = (a_0(u - \pi_k u), \pi_k v - v) + (u - \pi_k u, a_0 v - q_1(a_0 v)) \leq C h^{k+3} \|u\|_{k+1.p} \|v\|_{2.q}$$

then (25) is derived, the proof of Theorem 4 is completed.

Let bilinear form $A(u,v)$ be defined by (21), and further assume that $A(u,v)$ is a continuous and uniformly elliptic on $H_0^1(\Omega) \times H_0^1(\Omega)$ (not necessary symmetry). For function $u \in H_0^1(\Omega)$, define its *Ritz* projection on finite element space S_h by $R_h u \in S_h$ such that

$$A(u - R_h u, v_h) = 0,\ \forall\, v_h \in S_h \tag{26}$$

Below we will assume that the partition J_h is quasi-regular, ie. the inverse properties hold on S_h. First we need to introduce the smooth *Green* functions and give some properties related with the functions. For detail discussion, the reader is referred to Chapter 3 in article [10]. Introduce the adjoint form of $A(u,v)$ as follows

$$A^*(u,v) = \sum_{i,j=1}^{2} (a_{ij}\, D_j u, D_i v) + \sum_{i=1}^{2} (a_i\, u, D_i v) + (a_0\, u, v)$$

then $A^*(v,u) = A(u,v)$. for any fixed point $z \in \Omega$, define the smooth *Green* functions in function and derivative types, respectively, by G^z, $\partial_z G^z \in H_0^1(\Omega) \bigcap H^2(\Omega)$ such that

$$A^*(G^z, v) = P_h v(z)\,,\ \forall\, v \in H_0^1(\Omega) \tag{27}$$

$$A^*(\partial_z G^z, v) = \partial_z P_h v(z)\,,\ \forall\, v \in H_0^1(\Omega) \tag{28}$$

Where $P_h : L_2(\Omega) \to S_h$ is the L_2 projection operator, ∂_z represents the derivative with respect to z along any indicated direction. Define their finite element approximations by G_h^z, $\partial_z G_h^z \in S_h$ such that

$$A^*(G^z - G_h^z, v_h) = 0,\ \forall\, v_h \in S_h \tag{29}$$

$$A^*(\partial_z G^z - \partial_z G_h^z, v_h) = 0,\ \forall\, v_h \in S_h \tag{30}$$

According to the research conclusions in article [10], we have

$$\|\partial_z G_h^z\|_{1.1} + \|G^z\|_{2.1} + h^{-1}\|G^z - G_h^z\|_{1.1} \le C|\,ln\, h| \tag{31}$$

Lemma 2. *Let u and $R_h u$ satisfy equation (26). Then, the following superapproximation estimates hold*

$$\|\pi_k u - R_h u\|_{1.\infty} \le Ch^{k+1}|\,ln\, h|\,\|u\|_{k+2.\infty},\ k \ge 1 \tag{32}$$

$$\|\pi_k u - R_h u\|_{0.\infty} \le Ch^{k+2}|\,ln\, h|\,\|u\|_{k+2.\infty},\ k \ge 2 \tag{33}$$

For the special case $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$, the following ultra-approximation estimates hold

$$\|\pi_k u - R_h u\|_{1.\infty} \le Ch^{k+2}|\,ln\, h|\,\|u\|_{k+3.\infty},\ k \ge 2 \tag{34}$$

$$\|\pi_k u - R_h u\|_{0.\infty} \le Ch^{k+3}|\,ln\, h|\,\|u\|_{k+3.\infty},\ k \ge 3 \tag{35}$$

Proof. Denote $e_h = \pi_k u - R_h u$. By means of equations (30),(28),(26), and estimates (22) and (31), we have

$$\partial_z e_h(z) = A^*(\partial_z G_h^z, e_h) = A(e_h, \partial_z G_h^z) = A(\pi_k u - u, \partial_z G_h^z) \le Ch^{k+1}|\,ln\, h|\,\|u\|_{k+2.\infty}$$

consequently, estimate (32) is derived. Again using equations (29),(27),(26), estimates (22)-(23), we obtain

$$\begin{aligned} e_h(z) = A^*(G_h^z, e_h) = A(e_h, G_h^z) = A(\pi_k u - u, G_h^z - \pi_k G^z) + \\ + A(\pi_k u - u, \pi_k G^z) \le Ch^{k+1}\|u\|_{k+2.\infty}\|G_h^z - \pi_k G^z\|_{1.1} + Ch^{k+2}\|u\|_{k+2.\infty}\|G^z\|_{2.1} \end{aligned}$$

Then estimate (33) follows from (31) and the approximation property. For the special case, estimates (34) and (35) can be proved by the same argument as above, and at this time elementary estimates (24) and (25) will be used.

Note that, at vertices of elements, $u = \pi_k u$, then from Lemma 2 we immediately obtain

Corollary. *Let (x_i, y_j) be vertices of elements. Then, we have the superconvergence estimate*

$$|(u - R_h u)(x_i, y_j)| \le Ch^{k+2}|\,ln\, h|\,\|u\|_{k+2.\infty},\ k \ge 2$$

For the special case $A(u,v) = (\nabla u, \nabla v) + (a_0 u, v)$, we have the ultraconvergence estimate

$$|(u - R_h u)(x_i, y_j)| \le Ch^{k+3}|\,ln\, h|\,\|u\|_{k+3.\infty},\ k \ge 3$$

Theorem 5. *Let u and $R_h u$ satisfy equation (26). Then, on the Lobatto, Gauss and quasi-Lobatto point set $N_k^{(s)} \times N_k^{(s)} = \{G_{ij}^{(s)}\}$, $s = 0, 1, 2$, hold the superconvergence estimates*

$$|D^l(u - R_h u)(G_{ij}^{(|l|)})| \le Ch^{k+2-|l|} |\ln h| \, \|u\|_{k+2.\infty}, \quad |l| = 0, 1, 2$$

here $k \ge 1$, or $k \ge 2$ when $|l| = 0$ or $l_1 = 2$ or $l_2 = 2$. Furthermore, for mixed derivative approximation, hold the global superconvergence estimate

$$\max_{e \in J_h} \|D_1 D_2(u - R_h u)\|_{0.\infty.e} \le Ch^k |\ln h| \, \|u\|_{k+2.\infty}\,, \; k \ge 1$$

Proof. Combining Theorem 1 with Lemma 2 and utilizing triangle inequality, the proof is completed immediately. Note that when $|l| = 2$, we need to apply the finite element inverse inequality to those results in Lemma 2 to derive the estimates involving two-order derivatives.

Theorem 6. *Let u and $R_h u$ satisfy equation (26) with bilinear form $A(u, v) = (\nabla u, \nabla v) + (a_0 u, v)$, arithmetic mean operators $\overline{D}_1$ and $\overline{D}_2$ be defined as in Section 2, Gauss points $(g_i^{(1)}, \tilde{g}_j^{(1)}) \in N_k^{(1)} \times \tilde{N}_k^{(1)}$ (or $(g_i^{(1)}, \tilde{g}_j^{(1)}) \in N_{k+1}^{(1)} \times \tilde{N}_{k+1}^{(1)}$ when k is odd), Lobatto points $(g_i^{(0)}, \tilde{g}_j^{(0)}) \in N_k^{(0)} \times \tilde{N}_k^{(0)}$. Then, the ultraconvergence estimates hold*

$$|\overline{D}_1(u - R_h u)(g_i^{(1)}, \tilde{g}_j^{(0)})| \le C\, h^{k+2} |\ln h| \, \|u\|_{k+3.\infty}\,, \quad k \ge 2 \tag{36}$$

$$|\overline{D}_2(u - R_h u)(g_i^{(0)}, \tilde{g}_j^{(1)})| \le C\, h^{k+2} |\ln h| \, \|u\|_{k+3.\infty}\,, \quad k \ge 2 \tag{37}$$

Moreover, on Gauss point set $N_k^{(1)} \times \tilde{N}_k^{(1)} = \{G_{ij}^{(1)}\}$, also hold the following ultraconvergence estimate

$$|D_1 D_2(u - R_h u)(G_{ij}^{(1)})| \le Ch^{k+1} |\ln h| \, \|u\|_{k+3.\infty}\,, \; k \ge 2 \tag{38}$$

Proof. Combining Theorem 2 with Lemma 2 and utilizing triangle inequality, the proof is completed immediately. When concerning two-order mixed derivative estimate, the inverse inequality should be used.

4. Superconvergence of Ritz-Volterra Projection and Application

In this section, we will prove that those superconvergence and ultraconvergence properties shared by *Ritz* projection in Section 3 also hold for the *Ritz-Volterra* projection[11−13]. As a direct application, we shall also discuss the semidiscrete finite element approximation to initial boundary value problems of *Sobolev* equations, and verify the same type superconvergence and ultraconvergence results as those of *Ritz-Volterra* projection.

Let $J = (0, T]$, $T > 0$ both $A(t)$ and $B(t, \tau)$, $t, \tau \in \overline{J}$, be linear partial differential operators of second order with general forms

$$A(t) \quad = \quad -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} a_{ij}(t, x) \frac{\partial}{\partial x_i} + a_0(t, x) \tag{39}$$

$$B(t, \tau, x) \quad = \quad -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j} b_{ij}(t, \tau, x) \frac{\partial}{\partial x_i} + \sum_{i=1}^{2} b_i(t, \tau, x) \frac{\partial}{\partial x_i} + b_0(t, \tau, x) \tag{40}$$

Hereafter we assume that the coefficient functions appearing in operators $A(t)$ and $B(t, \tau)$ possess the smoothness and boundedness required in our demonstrations for variables $t, \tau \in \overline{J}$ and $x \in \overline{\Omega}$. Further assume $A(t)$ be symmetric and uniformly positive definite. Let X be a *Banach* space, we use the notation $L_p(J; X)$, $1 \le p \le \infty$, to represent the usual X-value integrable function space with element $u(t) : J \to X$ and $\|u(t)\|_{L_p(J;X)} \leq \infty$. For simplicity, sometimes we shall use notation

$$|||u(t)|||_{m.p} = \|u(t)\|_{m.p} + \int_0^t \|u(\tau)\|_{m.p}\, d\tau,\ \ t \in\ J,\ 1 \leq p \leq \infty,\ m = 0, 1, \cdots$$

Introduce the bilinear form

$$V(t; u(t), v(t)) = A(t; u(t), v(t)) + \int_0^t B(t, \tau; u(\tau), v(t))\, d\tau \tag{41}$$

where both $A(t; u, v)$ and $B(t, \tau; u, v)$ are bilinear forms on $H_0^1(\Omega) \times H_0^1(\Omega)$ associated with operators $A(t)$ and $B(t, \tau)$, respectively. Now, for $u(t) : J \to H_0^1(\Omega)$, define its *Ritz-Volterra* projection on finite element space S_h by $V_h u(t) = (V_h u)(t) : J \to S_h$ such that

$$V(t; u(t) - V_h u(t), \chi) = 0,\ \ \forall\, \chi\, \in\, S_h,\ t \in J \tag{42}$$

Obviously, the *Ritz-Volterra* projection is a natural generalization of the *Ritz* projection, when $B(t, \tau) \equiv 0$, $V_h = R_h$ holds. On the unique existence, stability and approximation properties of this projection, the reader is refereed to [11-13,16,19] and references therein. Now, utilizing the elementary estimates (22)-(25) and the time dependence type *Green* functions[12], similar to the demonstration in article [12-13, 19-20], we can prove the following results.

Lemma 3. *Let $u(t)$ and $V_h u(t)$ satisfy equation (42). Then, the following superconvergence estimates hold*

$$\|\pi_k u(t) - V_h u(t)\|_{1.\infty} \leq Ch^{k+1}|ln\, h|\, |||u(t)|||_{k+2.\infty},\ \ k \geq 1\,,\ t \in\ J \tag{43}$$
$$\|\pi_k u(t) - V_h u(t)\|_{0.\infty} \leq Ch^{k+2}|ln\, h|\, |||u(t)|||_{k+2.\infty},\ \ k \geq 2\,,\ t \in\ J \tag{44}$$

Further, if $A(t; u, v) = (\nabla u, \nabla v) + (a_0 u, v)$ and $B(t, \tau; u, v) = (\nabla u, \nabla v) + (b_0 u, v)$, then

$$\|\pi_k u(t) - V_h u(t)\|_{1.\infty} \leq Ch^{k+2}|ln\, h|\, |||u(t)|||_{k+3.\infty},\ \ k \geq 2\,,\ t \in\ J \tag{45}$$
$$\|\pi_k u(t) - V_h u(t)\|_{0.\infty} \leq Ch^{k+3}|ln\, h|\, |||u(t)|||_{k+3.\infty},\ \ k \geq 3\,,\ t \in\ J \tag{46}$$

Theorem 7. *Let $u(t)$ and $V_h u(t)$ satisfy equation (42). Then, on the Lobatto, Gauss and quasi-Lobatto point set of element $N_k^{(s)} \times N_k^{(s)} = \{G_{ij}^{(s)}\}$, $s = 0, 1, 2$, hold the superconvergence estimates*

$$|D^l(u - V_h u)(t, G_{ij}^{(|l|)})| \leq Ch^{k+2-|l|}|\, ln\, h|\, |||u(t)|||_{k+2.\infty},\ \ |l| = 0, 1, 2,\ \ t \in J$$

here $k \geq 1$, or $k \geq 2$ when $|l| = 0$ or $l_1 = 2$ or $l_2 = 2$. Furthermore, for mixed derivative approximation, hold the global superconvergence estimate

$$\max_{e\, \in J_h} \|D_1 D_2(u - V_h u)(t)\|_{0.\infty.e} \leq Ch^k|\, ln\, h|\, |||u(t)|||_{k+2.\infty}\,,\ k \geq 1,\ \ t \in J$$

Proof. Combining Theorem 1 with $(43) - (44)$ in Lemma 3 and utilizing triangle inequality, the proof is completed immediately. Note that when $|l| = 2$, we need to apply the finite element inverse inequality to those results in Lemma 3 to derive the estimates involving two-order derivatives.

Theorem 8. *Let $u(t)$ and $V_h u(t)$ satisfy equation (42) with bilinear forms $A(t; u, v) = (\nabla u, \nabla v) + (a_0 u, v)$, $B(t, \tau; u, v) = (\nabla u, \nabla v) + (b_0 u, v)$, and arithmetic mean operators $\overline{D}_1$ and $\overline{D}_2$ be defined as in Section 2, Gauss points $(\, g_i^{(1)}, \widetilde{g}_j^{(1)}) \in N_k^{(1)} \times \widetilde{N}_k^{(1)}$ (or $(\, g_i^{(1)}, \widetilde{g}_j^{(1)})$*

$\in N^{(1)}_{k+1}\times \tilde{N}^{(1)}_{k+1}$ *when k is odd), Lobatto points* $(g^{(0)}_i, \tilde{g}^{(0)}_j) \in N^{(0)}_k \times \tilde{N}^{(0)}_k$. *Then, the ultraconvergence estimates hold*

$$|\overline{D}_1(u - V_h u)(t, g^{(1)}_i, \tilde{g}^{(0)}_j)| \leq C\, h^{k+2}|\ln h|\, |||u(t)|||_{k+3.\infty}\,, \quad k \geq 2,\ t \in J \tag{47}$$

$$|\overline{D}_2(u - V_h u)(t, g^{(0)}_i, \tilde{g}^{(1)}_j)| \leq C\, h^{k+2}|\ln h|\, |||u(t)|||_{k+3.\infty}\,, \quad k \geq 2,\ t \in J \tag{48}$$

Moreover, on Gauss point set $N^{(1)}_k \times \tilde{N}^{(1)}_k = \{G^{(1)}_{ij}\}$, *also hold the following ultraconvergence estimate*

$$|D_1 D_2(u - V_h u)(t, G^{(1)}_{ij})| \;\leq\; Ch^{k+1}|\ln h|\, |||u(t)|||_{k+3.\infty}\,,\ k \geq 2,\ t \in J \tag{49}$$

Proof. Combining Theorem 2 with $(45) - (46)$ in Lemma 3 and utilizing triangle inequality, the proof is completed immediately. When concerning two-order mixed derivative estimate, the inverse inequality should be used.

As applications of *Ritz-Volterra* projection, let us consider the initial boundary value problems of *Sobolev* equations

$$\begin{array}{rcll} A(t)u_t + B(t)u &=& f(t), & in\ \Omega,\ t \in J \\ u(t,x) &=& 0\ , & on\ \partial\Omega,\ t \in J \\ u(0,x) &=& u_0(x), & in\ \Omega \end{array} \tag{50}$$

Where f and u_0 are known functions, $A(t)$ and $B(t)$ are second order partial differential operators (unconcerning with τ at this time) with the forms as (39)-(40), and $A(t)$ is symmetric and uniformly positive definite. Problems (50) are employed to model a variety of physical processes, and researches on the unique existence and smoothness of the solution as well as various numerical methods for this problems have been carried successively [20−25].

Now, the semidiscrete finite element approximation of problem (50) is defined as $u_h(t)$: $J \to S_h$ such that

$$\begin{array}{rcll} A(t; u_{h,t}, \chi) + B(t; u_h, \chi) &=& (f, \chi), & \chi \in S_h,\ t \in J \\ A(0; u_0 - u_h(0), \chi) &=& 0, & \chi \in S_h \end{array} \tag{51}$$

It is easy to show that problem (51) admits a unique solution. Below we do the error analysis. From equations (50) and (51), we have

$$A(t; (u - u_h)_t(t), \chi) + B(t; u(t) - u_h(t), \chi) = 0,\ \chi \in S_h,\ t \in J$$

Integrate with respect to variable t and using (51) to obtain

$$A(t; u(t) - u_h(t), \chi) + \int_0^t \overline{B}(\tau; u(\tau) - u_h(\tau), \chi)\, d\tau = 0,\ \chi \in S_h,\ t \in J \tag{52}$$

Where $\overline{B}(t; u, v) = B(t; u, v) - A_t(t; u, v)$ is the bilinear form associated with operator $B(t) - A_t(t)$, and $A_t(t)$ is the operator derived from $A(t)$ by differentiating the coefficients with respect to t. Now comparing (52) with the definition (42), we find immediately that $u_h(t)$ turns to be the *Ritz-Volterra* projection of the exact solution $u(t)$ with $B(t,\tau) = B(t) - A_t(t)$, ie. $u_h(t) = V_h u(t)$. Thus, by means of the analysis of *Ritz-Volterra* projection mentioned above, we directly obtain conclusion: the superconvergence and ultraconvergence results in Theorem 7 and Theorem 8 maintain to hold for the error function $u(t) - u_h(t)$, where $u(t)$ and $u_h(t)$ are solutions of problems (50) and (51), respectively.

Remark 2. As indicated in Remark 1, the superconvergence points $G_{ij}^{(|l|)}$ in Theorem 5 and Theorem 7 can be replaced by superconvergence lines $G_{x,j}^{(|l|)}$ ($G_{i,y}^{(|l|)}$) when $l_1 = 1, 2$ ($l_2 = 1, 2$).

Remark 3. By means of the superconvergence estimates in Lemma 2-3 and the postprocessing techniques based on finite element interpolation methods[4−5], one can easily derive the posteriori error estimates for error functions $u - R_h u$ and $u(t) - V_h u(t)$, which are very useful in the practical computation when adaptive procedures are used.

Remark 4. In this article we assume that Ω is a rectangle domain and require that the solution u belongs to $W_\infty^{k+2}(\Omega)$ or $W_\infty^{k+3}(\Omega)$. But this is seldom the case in practice. In order to concentrate on superconvergence, we have not discuss the treatments of general domain and corner singularities. However, by the local analysis methods [4,6,10], our results should maintain to hold on a subdomain of Ω on which the solution u is smooth enough.

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