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STABILITY ANALYSIS OF RUNGE-KUTTA METHODS FOR NONLINEAR SYSTEMS OF PANTOGRAPH EQUATIONS *1)

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Abstract

This paper is concerned with numerical stability of nonlinear systems of pantograph equations. Numerical methods based on $(k,l)-$algebraically stable Runge-Kutta methods are suggested. Global and asymptotic stability conditions for the presented methods are derived.

Mathematics subject classification: 65L06, 65L20.
Key words: Nonlinear pantograph equations, Runge-Kutta methods, Numerical stability, Asymptotic stability.

1. Introduction

Consider the following systems of the pantograph equations

$$\begin{cases} y'(t) = f(t, y(t), y(pt)), & t > 0, \\ y(0) = \eta, & \eta \in C^N, \end{cases} \tag{1.1}$$

where $f : [0, +\infty) \times C^N \times C^N \to C^N$ is a given function and $p \in (0, 1)$ is a real constant. For applications of the systems(1.1), we refer to Iserles[1].

In order to investigate the stability of numerical methods for the pantograph equations, the scalar linear pantograph equations

$$y'(t) = \lambda y(t) + \mu y(pt),$$

where $\lambda, \mu \in C$ and $p \in (0, 1)$ are constants, have been used as the test problem and many significant results have been derived(cf.[2-10, 16, 17]). However, little attention has been paid to the nonlinear case of the form (1.1). In 2002, Zhang and Sun[11] considered nonlinear stability of one-leg $\theta-$methods for (1.1) and obtained some results of global and asymptotic stability. On the basis of their works, the present paper further deal with numerical stability of (k,l)-algebraically stable Runge-Kutta methods with variable stepsize (introduced by Liu[9]) for the nonlinear systems (1.1). Global and asymptotic stability conditions for the presented methods are derived.

2. Runge-Kutta Methods with Variable Stepsize

In this section, we consider the adaptation of Runge-Kutta methods for solving (1.1). Let (A, b, c) denotes a given Runge-Kutta method with matrix $A = (a_{ij}) \in R^{s\times s}$ and vectors $b = (b_1, b_2, \ldots, b_s)^T \in R^s$, $c = (c_1, c_2, \ldots, c_s)^T \in R^s$. In this paper, we always assume that $c_i \in [0, 1]$, $i = 1, 2, \ldots, s$. The application of the Runge-Kutta method (A, b, c) to (1.1) yields

$$\begin{cases} Y_i^{(n)} = y_n + h_{n+1} \sum\limits_{j=1}^{s} a_{ij} f(t_n + c_j h, Y_j^{(n)}, \widetilde{Y}_j^{(n)}), & i = 1, 2, \ldots, s, \\ y_{n+1} = y_n + h_{n+1} \sum\limits_{i=1}^{s} b_i f(t_n + c_i h, Y_i^{(n)}, \widetilde{Y}_i^{(n)}), & n = 0, 1, 2, \ldots, \end{cases} \tag{2.1}$$

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where $h_{n+1}=t_{n+1}-t_n$, y_n, $Y_i^{(n)}$ and $\widetilde{Y}_i^{(n)}(n\geq 0, i=1,2,\ldots,s)$ are approximations to $y(t_n)$, $y(t_n+c_ih_{n+1})$ and $y(p(t_n+c_ih_{n+1}))$ respectively.

Since a serious storage problem is created when the computation for (1.1) with constant stepsize is run on any computer, we consider a variable stepsize strategy introduced by Liu[9] and Bellen et al.[2] to resolve the storage problem. The grid points are selected as follows(cf. [11]).

First, divide $[0,+\infty)$ into a set of infinite bounded intervals, that is

$$[0,+\infty)=\bigcup_{l=0}^{\infty} D_l,$$

where $D_0=[0,\gamma]$ with a given positive number γ and $D_l=(T_{l-1},T_l](l\geq 1)$ with $T_l=p^{-l}\gamma$. Then, partition every primary interval $D_l(l\geq 1)$ into equal m subintervals. Thus the grid points on $[0,+\infty)/D_0$ are determined by

$$t_n=T_{\lfloor (n-1)/m\rfloor}+(n-\lfloor (n-1)/m\rfloor m)h_n, \qquad n\geq 1,$$

where $\lfloor x\rfloor$ denotes the maximal integer which not exceeds x. On D_0, choose $t_0=\gamma$, $t_{-(m+1)}=0$, $t_{-i}=pt_{m-i}$, $i=m,m-1,\ldots,1$, as grid points. The corresponding numerical solutions y_0, y_{-i} and $Y_j^{(-i)}(i=m+1,m,\ldots,1,\ j=1,2,\ldots,s)$ are assumed to exist. So the function $\varphi(t):=pt$ has these properties:

$$\begin{array}{lll} [S1] & \varphi(t_n)=t_{n-m}, & n\geq 0,\\ [S2] & \varphi(D_{n+1})=D_n, & n\geq 1,\\ [S3] & \varphi(h_n)=h_{n-m}, & n\geq 1, \end{array}$$

and the stepsize sequence $\{h_n\}$ is determined by

$$h_n=\begin{cases} p\gamma, & n=-m,\\ \frac{(1-p)\gamma}{m}, & n=-m+1,-m+2,\ldots,-1,0,\\ \frac{(1-p)\gamma}{mp^{\lfloor (n-1)/m\rfloor+1}}, & n=1,2,3,\ldots. \end{cases} \tag{2.2}$$

Properties [S1]-[S3] imply that the choice of grid points has removed the computational storage problem for (1.1) and the method (2.1) can be written as

$$\begin{cases} Y_i^{(n)}=y_n+h_{n+1}\sum\limits_{j=1}^{s}a_{ij}f(t_n+c_jh,Y_j^{(n)},Y_j^{(n-m)}), & i=1,2,\ldots,s,\\ y_{n+1}=y_n+h_{n+1}\sum\limits_{i=1}^{s}b_if(t_n+c_ih,Y_i^{(n)},Y_i^{(n-m)}), & n=0,1,2,\ldots, \end{cases} \tag{2.3}$$

3. Stability Analysis of the Methods

In order to study the stability of the methods (2.3), consider the perturbed systems of (1.1)

$$\begin{cases} z'(t)=f(t,z(t),z(pt)), & t>0,\\ z(0)=\varsigma, & \varsigma\in C^N, \end{cases} \tag{3.1}$$

Similarly, applying method (2.3) to the systems (3.1) yields

$$\begin{cases} Z_i^{(n)}=z_n+h_{n+1}\sum\limits_{j=1}^{s}a_{ij}f(t_n+c_jh,Z_j^{(n)},Z_j^{(n-m)}), & i=1,2,\ldots,s,\\ z_{n+1}=z_n+h_{n+1}\sum\limits_{i=1}^{s}b_if(t_n+c_ih,Z_i^{(n)},Z_i^{(n-m)}), & n=0,1,2,\ldots, \end{cases} \tag{3.2}$$

where z_n and $Z_i^{(n)}$ are approximations to $z(t_n)$ and $z(t_n+c_ih_{n+1})$ respectively.

Both (1.1) and (3.1), we assume that the function f satisfies

$$\begin{cases} Re\langle u_1-u_2,f(t,u_1,v)-f(t,u_2,v)\rangle\leq\alpha\|u_1-u_2\|^2, & t>0, \quad u_1,u_2,v\in C^N,\\ \|f(t,u,v_1)-f(t,u,v_2)\|\leq\beta\|v_1-v_2\|, & t>0, \quad u,v_1,v_2\in C^N, \end{cases} \tag{3.3}$$

where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote a given inner product and the corresponding norm in complex $N-$dimensional space C^N respectively. In the following, all systems (1.1) with (3.3) will be called class $D_p(\alpha,\beta)$(cf.[11]). For systems (1.1) and (3.1) of class $D_p(\alpha,\beta)$, it follows from the arguments given by Zennaro[12] that the solutions $y(t)$ and $z(t)$ satisfy

$$\|y(t)-z(t)\| \le \|\eta-\varsigma\|, \qquad t>0,$$

and

$$\lim_{t\to+\infty}\|y(t)-z(t)\|=0,$$

whenever $0<\beta\le -p\alpha$.

Definition 3.1[13]**.** *Let k,l be real constants. A Runge-Kutta method (A,b,c) is called to be $(k,l)-$algebraically stable if there exists a diagonal nonnegative matrix $D=diag(d_1,d_2,\ldots,d_s)$ such that $M=[m_{ij}]\in R^{(s+1)\times(s+1)}$ is nonnegative definite, where*

$$M=\begin{pmatrix} k-1-2le^TDe & e^TD-b^T-2le^TDA \\ De-b-2lA^TDe & DA+A^TD-bb^T-2lA^TDA \end{pmatrix}$$

with $e=[1,1,\ldots,1]^T\in R^s$. In particular, the $(1,0)-$algebraically stable Runge-Kutta method is called algebraically stable.

Now, we focus on the stability analysis of the methods(2.3).

Let

$$w_n=y_n-z_n, \qquad W_i^{(n)}=Y_i^{(n)}-Z_i^{(n)},$$

$$Q_i^{(n)}=f(t_n+c_ih,Y_i^{(n)},Y_i^{(n-m)})-f(t_n+c_ih,Z_i^{(n)},Z_i^{(n-m)}), \quad i=1,2,\ldots,s.$$

Then, we have

$$W_i^{(n)}=w_n+h_{n+1}\sum_{j=1}^{s}a_{ij}Q_j^{(n)}, \qquad i=1,2,\ldots,s,$$

$$w_{n+1}=w_n+h_{n+1}\sum_{i=1}^{s}b_iQ_i^{(n)}, \qquad n=0,1,2,\ldots.$$

Theorem 3.1. *Assume that the Runge-Kutta method (A,b,c) is $(k,l)-$algebraically stable with $k\le 1$, then the numerical solutions y_n and z_n, produced by the corresponding method (2.3) applying to the systems (1.1) and (3.1) of the class $D_p(\alpha,\beta)$ with $0<\beta\le-p\alpha$ and $(\alpha+\frac{\beta}{p})h_1\le l$ respectively, satisfy*

$$\|y_n-z_n\|\le C(\max_{-m\le j\le 0}\|y_j-z_j\|+\max_{-m\le j\le -1}\max_{1\le i\le s}\|Y_i^{(j)}-Z_i^{(j)}\|), \qquad n>0, \tag{3.4}$$

where C depends only on the method, α,β,p and γ, and the inequality(3.4) characterize the global stability of the method (2.3).

Proof. As in Burrage and Butcher [13], we can obtain

$$\|w_{n+1}\|^2-k\|w_n\|^2-2\sum_{i=1}^{s}d_iRe\langle W_i^{(n)},h_{n+1}Q_i^{(n)}-lW_i^{(n)}\rangle=-\sum_{i=1}^{s+1}\sum_{j=1}^{s+1}m_{ij}\langle\gamma_i,\gamma_j\rangle, \tag{3.5}$$

where $\gamma_1=w_n,\gamma_j=h_{n+1}Q_{j-1}^{(n)},j=2,3,\ldots,s+1$.

By means of $(k,l)-$algebraic stability of the method and $k\le 1$, we have

$$\|w_{n+1}\|^2\le\|w_n\|^2+2\sum_{i=1}^{s}d_iRe\langle W_i^{(n)},h_{n+1}Q_i^{(n)}-lW_i^{(n)}\rangle. \tag{3.6}$$

It follows from (3.3) that

$$\begin{aligned} 2Re\langle W_i^{(n)},Q_i^{(n)}\rangle &\le 2\alpha\|W_i^{(n)}\|^2+2\beta\|W_i^{(n)}\|\|W_i^{(n-m)}\| \\ &\le (2\alpha+\beta)\|W_i^{(n)}\|^2+\beta\|W_i^{(n-m)}\|^2. \end{aligned} \tag{3.7}$$

Inserting (3.7) into (3.6) gives

$$\|w_{n+1}\|^2\le\|w_n\|^2+\sum_{i=1}^{s}d_i(h_{n+1}((2\alpha+\beta)\|W_i^{(n)}\|^2+\beta\|W_i^{(n-m)}\|^2)-2l\|W_i^{(n)}\|^2). \tag{3.8}$$

Further, an induction yields

$$\|w_{n+1}\|^2 \leq \|w_0\|^2 + \sum_{i=1}^{s} d_i \sum_{j=0}^{n} (h_{j+1}(2\alpha+\beta) - 2l)\|W_i^{(j)}\|^2 + \beta \sum_{i=1}^{s} d_i \sum_{j=0}^{n} h_{j+1} \|W_i^{(j-m)}\|^2. \quad (3.9)$$

Moreover, (2.2) leads to

$$\begin{aligned}\sum_{j=0}^{n} h_{j+1}\|W_i^{(j-m)}\|^2 &= \sum_{j=-m}^{n-m} h_{m+j+1}\|W_i^{(j)}\|^2 \\ &= \frac{1}{p}\sum_{j=-m}^{n-m} h_{j+1}\|W_i^{(j)}\|^2 \\ &= \frac{1}{p}(\sum_{j=0}^{n-m} h_{j+1}\|W_i^{(j)}\|^2 + \sum_{j=-m}^{-1} h_{j+1}\|W_i^{(j)}\|^2) \\ &\leq \frac{1}{p}(\sum_{j=0}^{n-m} h_{j+1}\|W_i^{(j)}\|^2 + (1-p)\gamma \max_{-m\leq j\leq -1}\|W_i^{(j)}\|^2). \quad (3.10)\end{aligned}$$

Substituting (3.10) into (3.9) yields that

$$\|w_{n+1}\|^2 \leq \|w_0\|^2 + \sum_{i=1}^{s} d_i \sum_{j=0}^{n} (h_{j+1}(2\alpha+\beta+\frac{\beta}{p}) - 2l)\|W_i^{(j)}\|^2 + \frac{\beta(1-p)\gamma}{p}\sum_{i=1}^{s} d_i \max_{-m\leq j\leq -1}\max_{1\leq i\leq s}\|W_i^{(j)}\|^2. \quad (3.11)$$

Using condition $\beta \leq -p\alpha$ and $(\alpha + \frac{\beta}{p})h_1 \leq l$, we have

$$\begin{aligned} h_{j+1}(2\alpha+\beta+\frac{\beta}{p}) - 2l &\leq 2h_{j+1}(\alpha+\frac{\beta}{p}) - 2l \\ &\leq 2h_1(\alpha+\frac{\beta}{p}) - 2l \leq 0.\end{aligned}$$

Then (3.11) leads to

$$\|w_{n+1}\|^2 \leq \|w_0\|^2 + \frac{\beta(1-p)\gamma}{p}\sum_{i=1}^{s} d_i \max_{-m\leq j\leq -1}\max_{1\leq i\leq s}\|W_i^{(j)}\|^2, \qquad n \geq 0,$$

which shows that the inequality (3.4) is satisfied and the proof is completed.

Theorem 3.2. *Assume that the Runge-Kutta method (A, b, c) is $(k, l)-$algebraically stable with $k < 1$, then the numerical solutions y_n and z_n, produced by the corresponding method (2.3) applying to the systems (1.1) and (3.1) of the class $D_p(\alpha, \beta)$ with $0 < \beta \leq -p\alpha$ and $(\alpha + \frac{\beta}{p})h_1 \leq l$ respectively, satisfy*

$$\lim_{n\to+\infty}\|y_n - z_n\| = 0,$$

which characterize the asymptotic stability of the method (2.3).

Proof. Let

$$\overline{k} = max\{k,\ (\frac{2p\alpha+\beta}{(2\alpha+\beta)p})^{1/m}\},$$

when $0 < \beta \leq -p\alpha$ and $p \in (0, 1)$, we have $k \leq \overline{k} < 1$.

In view of $(k, l)-$algebraic stability of the Runge-Kutta method, it follows from (3.5) that

$$\|w_{n+1}\|^2 \leq \overline{k}\|w_n\|^2 + 2\sum_{i=1}^{s} d_i Re\langle W_i^{(n)}, h_{n+1}Q_i^{(n)} - lW_i^{(n)}\rangle$$

$$\leq \overline{k}\|w_n\|^2 + \sum_{i=1}^{s} d_i(h_{n+1}((2\alpha+\beta)\|W_i^{(n)}\|^2 + \beta\|W_i^{(n-m)}\|^2) - 2l\|W_i^{(n)}\|^2).$$

By induction, we have

$$\|w_{n+1}\|^2 \leq \overline{k}^{n+1}\|w_0\|^2 + \sum_{i=1}^{s} d_i \sum_{j=0}^{n} \overline{k}^{n-j}(h_{j+1}((2\alpha+\beta)\|W_i^{(j)}\|^2 + \beta\|W_i^{(j-m)}\|^2) - 2l\|W_i^{(j)}\|^2). \quad (3.12)$$

Moreover, (2.2) leads to

$$\begin{aligned}\sum_{j=0}^{n} h_{j+1}\overline{k}^{n-j}\|W_i^{(j-m)}\|^2 &= \sum_{j=-m}^{n-m} h_{m+j+1}\overline{k}^{n-j-m}\|W_i^{(j)}\|^2 \\ &= \frac{1}{p}\sum_{j=-m}^{n-m} h_{j+1}\overline{k}^{n-j-m}\|W_i^{(j)}\|^2 \\ &= \frac{1}{p}(\sum_{j=0}^{n-m} h_{j+1}\overline{k}^{n-j-m}\|W_i^{(j)}\|^2 + \sum_{j=-m}^{-1} h_{j+1}\overline{k}^{n-j-m}\|W_i^{(j)}\|^2) \\ &\leq \frac{1}{p}(\sum_{j=0}^{n-m} h_{j+1}\overline{k}^{n-j-m}\|W_i^{(j)}\|^2 + \overline{k}^{n-m}(1-p)\gamma \max_{-m\leq j\leq -1}\|W_i^{(j)}\|^2). \quad (3.13)\end{aligned}$$

Substituting (3.13) into (3.12) yields that

$$\begin{aligned}\|w_{n+1}\|^2 \leq\ & \overline{k}^{n+1}\|w_0\|^2 + \sum_{i=1}^{s} d_i \sum_{j=0}^{n}(h_{j+1}(\overline{k}^{n-j}(2\alpha+\beta) + \overline{k}^{n-j-m}\frac{\beta}{p}) - \overline{k}^{n-j}2l)\|W_i^{(j)}\|^2 \\ & + \overline{k}^{n-m}\frac{\beta(1-p)\gamma}{p}\sum_{i=1}^{s} d_i \max_{-m\leq j\leq -1}\max_{1\leq i\leq s}\|W_i^{(j)}\|^2 \\ =\ & \overline{k}^{n+1}\|w_0\|^2 + \sum_{i=1}^{s} d_i \sum_{j=0}^{n}\overline{k}^{n-j-m}((\overline{k}^{m}(2\alpha+\beta) + \frac{\beta}{p})h_{j+1} - \overline{k}^{m}2l)\|W_i^{(j)}\|^2 \\ & + \overline{k}^{n-m}\frac{\beta(1-p)\gamma}{p}\sum_{i=1}^{s} d_i \max_{-m\leq j\leq -1}\max_{1\leq i\leq s}\|W_i^{(j)}\|^2. \quad (3.14)\end{aligned}$$

On the other hand, $(k,l)-$algebraically stable with $k<1$ leads to $l<0$. Then, using $0<\overline{k}<1$, $0<\beta\leq -p\alpha$ and $(\alpha+\frac{\beta}{p})h_1 \leq l$, we have

$$-2\overline{k}^{m}l \leq -2l,$$

and

$$\begin{aligned}(\overline{k}^{m}(2\alpha+\beta) + \frac{\beta}{p})h_{j+1} - \overline{k}^{m}2l &\leq (\frac{2p\alpha+\beta}{p} + \frac{\beta}{p})h_{j+1} - 2l \\ &\leq \frac{2(p\alpha+\beta)}{p}h_1 - 2l \leq 0.\end{aligned}$$

Noting that $d_i \geq 0$, $i=1,2,\ldots,s$, and $0<\overline{k}<1$, (3.14) leads to

$$\|w_{n+1}\|^2 \leq \overline{k}^{n+1}\|w_0\|^2 + \overline{k}^{n-m}\frac{\beta(1-p)\gamma}{p}\sum_{i=1}^{s} d_i \max_{-m\leq j\leq -1}\max_{1\leq i\leq s}\|W_i^{(j)}\|^2. \quad (3.15)$$

It is easily obtained from (3.15) that

$$\lim_{n\to+\infty}\|w_{n+1}\|^2 = 0,$$

i.e.

$$\lim_{n\to+\infty}\|y_n - z_n\| = 0.$$

Hence, the proof is completed.

Corollary 3.1. *Assume that for every $l < 0$ there exists $k < 1$ such that the Runge-Kutta method (A, b, c) is $(k, l)-$algebraically stable, then the corresponding method (2.3) is asymptotically stable for the class $D_p(\alpha, \beta)$ whenever $0 < \beta \leq -p\alpha$.*

To examine the conditions of the above corollary, we have the following theorem which introduced by Huang et al. [15]. Here the meaning that a method is irreducible refers to [14].

Theorem 3.3. *Assume that an algebraically stable irreducible Runge-Kutta method (A, b, c) satisfies $\det A \neq 0$. Then for every $l < 0$ there exists $k < 1$ such that the method is $(k, l)-$algebraically stable if and only if $|1 - b^T A^{-1} e| < 1$.*

It is well known that the formulae Radau IA, Radau IIA and Lobatto IIIC (for ODEs) satisfy the conditions of theorem 3.3. Therefore, in terms of corollary 3.1, the methods induced by them are asymptotically stable for the class $D_p(\alpha, \beta)$ whenever $0 < \beta \leq -p\alpha$.

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