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THE EIGENVALUE PERTURBATION BOUND FOR ARBITRARY MATRICES *1)

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Abstract

In this paper we present some new absolute and relative perturbation bounds for the eigenvalue for arbitrary matrices, which improves some recent results. The eigenvalue inclusion region is also discussed.

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1. Introduction

Let A be an $n \times n$ matrix and $\widetilde{A} = A + E$ whose spectrum are $\{\lambda_1, ..., \lambda_n\}$ and $\{\mu_1, ..., \mu_n\}$, respectively. Let $\|\cdot\|_F$ and $\|\cdot\|_2$ denote the Frobenius norm and the spectral norm, respectively. For a positive integer n, let $\langle n\rangle = \{1, 2, ..., n\}$.

Classical absolute type perturbation bounds were established by the well-known Hoffman-Wielandt theorem [1]. When A and $\widetilde{A}$ are normal matrices, there exists a permutation τ of $\langle n\rangle$ such that

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \|E\|_F. \tag{1.1}$$

In the case that A is normal but $\widetilde{A}$ is arbitrary, Sun proved [2,3] that there exists a permutation τ of $\langle n\rangle$ such that

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n}\|E\|_F. \tag{1.2}$$

The factor $\sqrt{n}$ in (1.2) is optimal in some sense [2]. Furthermore, Song [4] studied the more general case. For two arbitrary matrices, he obtained

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n}(1 + \sqrt{n-p}) \max\left\{\|Q^{-1}EQ\|_F,\ \|Q^{-1}EQ\|_F^{1/m}\right\} \tag{1.3}$$

and

$$|\mu_{\tau(i)} - \lambda_i| \leq \sqrt{n}(1 + \sqrt{n-p}) \max\left\{\|\sqrt{n}Q^{-1}EQ\|_2,\ \|\sqrt{n}Q^{-1}EQ\|_F^{1/m}\right\} \tag{1.4}$$

where $Q^{-1}AQ = diag(J_1, ..., J_p)$ defines the Jordan form of A and m is the order of the largest Jordan block.

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As well known for any matrix $\widetilde{A} \in \mathbf{C}^{n\times n}$, there is a unitary matrix U such that $U^*\widetilde{A}U = diag(\widetilde{A}_1, ..., \widetilde{A}_s)$, where $\widetilde{A}_i$ is an upper triangular matrix, $i = 1, ..., s$. In [5] the authors showed that if A is normal then for any matrix $\widetilde{A}$ there exists a permutation τ such that

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n-s+1}||E||_F. \tag{1.5}$$

It is noted that s in (1.5) is not unique. In fact, it need not to decompose $\widetilde{A}$ so that $\widetilde{A}_i$ is upper triangular. Now let $\widetilde{A} \in \mathbf{C}^{n\times n}$, we denote by $s(\widetilde{A})$

$$s(\widetilde{A}) = \max_{U \text{ is unitary}} \{q : U^*\widetilde{A}U = diag(\widetilde{A}_1, ..., \widetilde{A}_q),\ \widetilde{A}_i \text{ is square},\ i = 1, ..., q\}. \tag{1.6}$$

This means that $s(\widetilde{A})$ is the most diagonal block numbers for which $\widetilde{A}$ is unitarily similar to a block diagonal matrix. Hence $s(\widetilde{A})$ exists and is unique for any matrix, and for any unitary matrix Q, $s(Q^*\widetilde{A}Q) = s(\widetilde{A}) \geq 1$. Notice if $\widetilde{A}$ is normal, then $s(\widetilde{A}) = n$.

By (1.5) it is easy to prove the following result.

Theorem 1.1. *Let $s(\widetilde{A})$ be given by (1.6), and let A be normal. Then for any matrix $\widetilde{A}$ there exists a permutation τ such that*

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n-s(\widetilde{A})+1}||E||_F. \tag{1.7}$$

In this paper, we shall improve the bounds in (1.3) and (1.4) for arbitrary matrices $\widetilde{A}$ and A based on Theorem 1.1. The relative bound and the eigenvalue inclusion region are also considered.

2. The Absolute Bound

First we write A into its Jordan canonical form

$$Q^{-1}AQ = J = diag(J_1, ..., J_p), \tag{2.1}$$

where J_i be an $m_i \times m_i$ Jordan block matrix with the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix},\ i = 1, ..., p.$$

For $\varepsilon \neq 0$, let

$$T = diag(T_1, ..., T_p),\ T_i = diag(1, \varepsilon, \varepsilon^2, ..., \varepsilon^{m_i-1}), i = 1, ..., p. \tag{2.2}$$

Then from (2.1) it is easy to check that

$$T^{-1}Q^{-1}AQT = \Lambda' + \Delta'_\varepsilon,$$

where $\Lambda' = diag(\lambda_1 I_{m_1}, ..., \lambda_p I_{m_p})$, $\Delta'_\varepsilon = diag(\Delta'_1, ..., \Delta'_p)$ and

$$\Delta'_i = \begin{bmatrix} 0 & \varepsilon & & 0 \\ & 0 & \ddots & \\ & & \ddots & \varepsilon \\ 0 & & & 0 \end{bmatrix}_{m_i\times m_i}, \quad i = 1, ..., p.$$

For two general matrices A and $\widetilde{A}$, we have the following bound.

Theorem 2.1. *Let* $A\in \mathbf{C}^{n\times n}$ *with (2.1) and* $\widetilde{A}=A+E$. *Then there exists a permutation* τ *of* $\langle n\rangle$ *such that*

$$\sum_{i=1}^{n}|\mu_{\tau(i)}-\lambda_i|^2 \leq \begin{cases} (n-s_1+1)\left[(n-p+1)+2\sqrt{n-p}\|Q^{-1}EQ\|_F\right]\|Q^{-1}EQ\|_F^{\frac{2}{m}},\ if\ \|Q^{-1}EQ\|_F<1 \\ (n-s_2+1)\left[\|Q^{-1}EQ\|_F+(n-p+2\sqrt{n-p})\right]\|Q^{-1}EQ\|_F,\ if\ \|Q^{-1}EQ\|_F\geq 1 \end{cases} \tag{2.3}$$

where $m=\max_i m_i$, $s_1=s(T^{-1}Q^{-1}\widetilde{A}QT)$, $s_2=s(Q^{-1}\widetilde{A}Q)$ and T is as in (2.2) with $\varepsilon=(\|Q^{-1}EQ\|_F)^{\frac{1}{m}}$

Proof. Let T be as in (2.2). Clearly, we have

$$T^{-1}Q^{-1}EQT+\Delta'_\varepsilon=T^{-1}Q^{-1}\widetilde{A}QT-\Lambda'.$$

Hence

$$\|T^{-1}Q^{-1}\widetilde{A}QT-\Lambda'\|_F^2=\|T^{-1}Q^{-1}EQT+\Delta'_\varepsilon\|_F^2$$

It is easy to see that

$$\begin{aligned} &\|T^{-1}Q^{-1}EQT+\Delta'_\varepsilon\|_F^2 \\ =\ &\|T^{-1}Q^{-1}EQT\|_F^2+\|\Delta'_\varepsilon\|_F^2+2\,\mathrm{Re}\,tr(\Delta'^*_\varepsilon T^{-1}Q^{-1}EQT) \\ \leq\ &\|T^{-1}\|_2^2\|T\|_2^2\|Q^{-1}EQ\|_F^2+(n-p)\varepsilon^2+2\,\mathrm{Re}\,tr(\Delta'^*_\varepsilon T^{-1}Q^{-1}EQT) \end{aligned} \tag{2.4}$$

Let $\widetilde{E}=Q^{-1}EQ$. Now partition $\widetilde{E}$ into the block form $\widetilde{E}=(\widetilde{E}_{ij})_{p\times p}$ conformable with (2.2), then

$$\begin{aligned} \mathrm{Re}\,tr(\Delta'^*_\varepsilon T^{-1}Q^{-1}EQT) &= \mathrm{Re}\,tr(\Delta'^*_\varepsilon T^{-1}\widetilde{E}T)=\mathrm{Re}\sum_{i=1}^{p}tr(\Delta'^*_i T_i^{-1}\widetilde{E}_{ii}T_i) \\ &= \mathrm{Re}\sum_{i=1}^{p}\sum_{k=2}^{m_i}\varepsilon(T_i^{-1}\widetilde{E}_{ii}T_i)_{k-1,k}. \end{aligned}$$

It is easy to see

$$(T_i^{-1}\widetilde{E}_{ii}T_i)_{k-1,k}=\varepsilon(\widetilde{E}_{ii})_{k-1,k}.$$

Thus

$$\begin{aligned} \mathrm{Re}\,tr(\Delta'^*_\varepsilon T^{-1}Q^{-1}EQT) &= \mathrm{Re}\sum_{i=1}^{p}\sum_{k=2}^{m_i}\varepsilon^2(\widetilde{E}_{ii})_{k-1,k} \\ &\leq \varepsilon^2\sum_{i=1}^{p}\sum_{k=2}^{m_i}|(\widetilde{E}_{ii})_{k-1,k}| \\ &\leq \varepsilon^2\sqrt{n-p}\sqrt{\sum_{i=1}^{p}\sum_{k=2}^{m_i}|(\widetilde{E}_{ii})_{k-1,k}|^2} \\ &\leq \varepsilon^2\sqrt{n-p}\sqrt{\sum_{i=1}^{p}\|\widetilde{E}_{ii}\|_F^2}, \end{aligned} \tag{2.5}$$

which together with (2.4) gives

$$\begin{aligned} &\|T^{-1}Q^{-1}EQT-\Delta'_\varepsilon\|_F^2 \\ \leq\ &\max\{\varepsilon^{2(m-1)},\varepsilon^{2(1-m)}\}\|Q^{-1}EQ\|_F^2+(n-p)\varepsilon^2 \\ &+2\varepsilon^2\sqrt{n-p}\|Q^{-1}EQ\|_F. \end{aligned} \tag{2.6}$$

If $||Q^{-1}EQ||_F < 1$, take $\varepsilon = (||Q^{-1}EQ||_F)^{\frac{1}{m}}$. By (2.6) we have

$$\|T^{-1}Q^{-1}EQT - \Delta'_\varepsilon\|_F^2 \le \varepsilon^2(n-p+1) + 2\sqrt{n-p}\varepsilon^{m+2}. \tag{2.7}$$

Applying the inequality (1.5) to $T^{-1}Q^{-1}\widetilde{A}QT$ and Λ', one may deduce

$$\sum_{i=1}^n |\mu_{\tau(i)} - \lambda_i|^2 \le (n-s_1+1)||T^{-1}Q^{-1}\widetilde{A}QT - \Lambda'||_F^2,$$

which together with (2.7) gives the first inequality in (2.3).

If $||Q^{-1}EQ||_F \ge 1$, take $\varepsilon = 1$. Hence by (2.6) we have

$$\begin{aligned} & \|T^{-1}Q^{-1}EQT - \Delta'_\varepsilon\|_F^2 \\ \le\ & ||Q^{-1}EQ\|_F^2 + (n-p) + 2\sqrt{n-p}||Q^{-1}EQ||_F \\ \le\ & ||Q^{-1}EQ\|_F^2 + (n-p+2\sqrt{n-p})||Q^{-1}EQ||_F. \end{aligned}$$

Again applying the inequality (1.5) to $T^{-1}Q^{-1}\widetilde{A}QT$ and Λ', one may deduce the second inequality in (2.3).

The bound in (2.3) seems more complicated. But we can reduce it to some simple forms.

Corollary 2.2. *Let $A \in \mathbf{C}^{n\times n}$ with (2.1) and $\widetilde{A} = A + E$.*

(1) If A is diagonalizable, i.e., there exists a nonsingular matrix Q such that $A = Q\Lambda Q^{-1}$, then there exists a permutation τ of $\langle n\rangle$ such that

$$\sum_{i=1}^n |\mu_{\tau(i)} - \lambda_i|^2 \le (n-s+1)||Q^{-1}EQ\|_F^2 \le (n-s+1)\kappa^2(Q)\|E\|_F^2, \tag{2.8}$$

where $\kappa(Q) = \|Q^{-1}\|_2\|Q\|_2$ denotes the spectral condition number of Q and $s = s(Q^{-1}\widetilde{A}Q)$.

(2) If A is not diagonalizable, then

$$\sum_{i=1}^n |\mu_{\tau(i)} - \lambda_i|^2 \le (n-\widehat{s}+1)f(||Q^{-1}EQ\|_F)\max\{||Q^{-1}EQ\|_F^{\frac{2}{m}}, ||Q^{-1}EQ||_F\}. \tag{2.9}$$

where $f(x) = x + (n-p+2\sqrt{n-p})$, $\widehat{s} = \min\{s(T^{-1}Q^{-1}\widetilde{A}QT),\ s(Q^{-1}\widetilde{A}Q)\}$

Proof. (1) Since A is diagonalizable, from (2.1) and (2.2) we have $p = n$, $m_i = 1,\ i = 1, ..., n$ and $T = I$. This implies that $s_1 = s$. Then the first inequality in (2.8) follows from Theorem 2.1. The second in (2.8) follows from the fact that $||Q^{-1}EQ\|_F \le ||Q||_2||Q^{-1}||_2\|E\|_F = \kappa(Q)\|E\|_F$.

(2) Since A does not diagonalizable, we have $p < n$, and thus $2\sqrt{n-p} > 1$. Notice that if $||Q^{-1}EQ\|_F < 1$, then $(n-p+1) + 2\sqrt{n-p}||Q^{-1}EQ\|_F \le ||Q^{-1}EQ\|_F + (n-p+2\sqrt{n-p})$, from which and (2.3) one may deduce (2.9).

Remark 2.1. If $||Q^{-1}EQ\|_F < 1$, then $\max\{||Q^{-1}EQ\|_F^{\frac{2}{m}}, ||Q^{-1}EQ||_F\} = ||Q^{-1}EQ\|_F^{\frac{2}{m}}$ and $f(||Q^{-1}EQ\|_F) < (\sqrt{n-p}+1)^2$. If $||Q^{-1}EQ\|_F \ge 1$, then $\max\{||Q^{-1}EQ\|_F^{\frac{2}{m}}, ||Q^{-1}EQ||_F\} = ||Q^{-1}EQ\|_F$ and $f(||Q^{-1}EQ\|_F) \le (\sqrt{n-p}+1)^2||Q^{-1}EQ\|_F$. From this argument we have the bound below

$$\sqrt{\sum_{i=1}^n |\mu_{\tau(i)} - \lambda_i|^2} \le \sqrt{n-\widehat{s}+1}(1+\sqrt{n-p})\max\left\{\|Q^{-1}EQ\|_F,\ \|Q^{-1}EQ\|_F^{1/m}\right\}. \tag{2.10}$$

From (2.10) it is easy to see that our bounds (2.8) and (2.9) improve the bound (1.3). It is also noted that if A is normal, then (1.7) can be deduced from (2.8).

Remark 2.2. In practice the perturbation E may be small enough, so we may assume that $||E\|_F < \frac{1}{\kappa(Q)}$. Hence from Theorem 2.1 we may obtain a simple bound below.

$$\sqrt{\sum_{i=1}^n |\mu_{\tau(i)} - \lambda_i|^2} \le \sqrt{n-s_1+1}(1+\sqrt{n-p})||Q^{-1}EQ\|_F^{1/m}.$$

Applying our recent bound (1.5) we can deduce the following result.

We write $\widetilde{A}$ into Jordan canonical form

$$P^{-1}\widetilde{A}P = \widetilde{J} = diag(\widetilde{J}_1, ..., \widetilde{J}_{\widetilde{p}}), \tag{2.11}$$

where $\widetilde{J}_j$ be an $\widetilde{m}_j \times \widetilde{m}_j$ Jordan block matrix with the form

$$\widetilde{J}_i = \begin{bmatrix} \mu_j & 1 & & 0 \\ & \mu_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \mu_j \end{bmatrix}, \ j = 1, ..., \widetilde{p}.$$

Corollary 2.3. *Let $\widetilde{A} \in \mathbf{C}^{n\times n}$ with (2.11). If $P^{-1}AP$ is a normal matrix. Then there exists a permutation τ of $\langle n\rangle$ such that*

$$\sqrt{\sum_{i=1}^{n} |\mu_{\tau(i)} - \lambda_i|^2} \leq \sqrt{n - \widetilde{p} + 1}\kappa(P)\|E\|_F. \tag{2.12}$$

Proof. Applying (1.7) to the matrices $P^{-1}\widetilde{A}P$ and $P^{-1}AP$, it is easy to deduce the inequality (2.12).

From the bound (2.10) and the fact that $\|E\|_F \leq \sqrt{n}\|E\|_2$, we have the spectral norm bound as follows.

Corollary 2.4. *Let $A \in \mathbf{C}^{n\times n}$ with (2.1) and $\widetilde{A} = A + E$. Then there exists a permutation τ of $\langle n\rangle$ such that*

$$|\mu_{\tau(i)} - \lambda_i|^2 \leq \sqrt{n - s + 1}(1 + \sqrt{n-p}) \max\left\{\|\sqrt{n}Q^{-1}EQ\|_2, \|\sqrt{n}Q^{-1}EQ\|_2^{1/m}\right\}. \tag{2.13}$$

3. The Relative Bound

Relative perturbation bounds of eigenvalues were studied by Eisenstat and Ipsen [5] for diagonalizable matrices. It was proved in [6] that when $A = X\Lambda X^{-1}$, $\widetilde{A} = Y\Lambda' Y^{-1}$ and A is nonsingular, there exists a permutation τ of $\langle n\rangle$ such that

$$\sqrt{\sum_{i=1}^{n} \left|\frac{\mu_{\tau(i)} - \lambda_i}{\lambda_i}\right|^2} \leq \kappa(X)\kappa(Y)\|A^{-1}E\|_F\,. \tag{3.1}$$

(noting that a slight modification form of (3.1) can be found in [7]). In [5] the authors considered that case that A is normal and $\widetilde{A}$ is arbitrary, the new relative perturbation bound was obtained below.

Let $A \in \mathbf{C}^{n\times n}$ be a normal matrix with

$$A = V\Lambda V^*, \tag{3.2}$$

where V is a unitary matrix, and let $\widetilde{A} = A + E \in \mathbf{C}^{n\times n}$. Then there exists a permutation τ of $\langle n\rangle$ such that

$$\sqrt{\sum_{i=1}^{n} \left|\frac{\mu_{\tau(i)} - \lambda_i}{\lambda_i}\right|^2} \leq \sqrt{n - s(\widetilde{A}) + 1}||A^{-1}||_2||E||_F. \tag{3.3}$$

Here we consider the case that A is diagonalizable, i.e., there is a nonsingular matrix X such that

$$A = X\Lambda X^{-1} \tag{3.4}$$

and $\widetilde{A}$ is arbitrary with Schur upper triangular decomposition

$$\widetilde{A} = U(\Lambda' + \Delta)U^*, \tag{3.5}$$

where U is unitary. We have the following bound.

Theorem 3.1. *Let $\widetilde{A} = A + E \in \mathbf{C}^{n\times n}$, and let A be a nonsingular diagonalizable matrix with (3.4). Then there exists a permutation τ of $\langle n\rangle$ such that*

$$\sqrt{\sum_{i=1}^{n}\left|\frac{\mu_{\tau(i)}-\lambda_i}{\lambda_i}\right|^2} \le \min\left\{\rho(A^{-1})\sqrt{n - s(X^{-1}\widetilde{A}X)+1},\ ||A^{-1}||_2\sqrt{n}\right\}\kappa(X)||E||_F. \tag{3.6}$$

Proof. Since

$$A^{-1}\widetilde{A} - I = A^{-1}E\,,$$

by (3.4) and (3.5) we have

$$\Lambda^{-1}X^{-1}U\Lambda' - X^{-1}U = X^{-1}A^{-1}EU - X^{-1}A^{-1}U\Delta \tag{3.7}$$

where Δ is defined in (3.5). Let $T = X^{-1}U$, $\Lambda_1 = \Lambda^{-1}$, $\Lambda_2 = \Lambda'$ and $\Lambda_3 = \Lambda_4 = I$. Then by Theorem 3.2 of [8] we obtain

$$\begin{aligned}
\sigma_n(X^{-1}U)\sqrt{\sum_{i=1}^{n}|\frac{\mu_{\tau(i)}}{\lambda_i}-1|^2} &\le ||\Lambda^{-1}X^{-1}U\Lambda' - X^{-1}U||_F \\
&= ||X^{-1}A^{-1}EU - X^{-1}A^{-1}U\Delta||_F \\
&\le ||X^{-1}||_2||A^{-1}||_2||E - U\Delta U^*||_F \\
&= ||X^{-1}||_2||A^{-1}||_2||U^*EU - \Delta||_F
\end{aligned} \tag{3.8}$$

By the proof of Theorem 4.9 of [3] (see (4.41) of [3]) it is easy to see $\|U^*EU - \Delta\|_F^2 \le n\|U^*EU\|_F^2 = n\|E\|_F^2$, which together with (3.8) gives

$$\sqrt{\sum_{i=1}^{n}\left|\frac{\mu_{\tau(i)}-\lambda_i}{\lambda_i}\right|^2} \le \sqrt{n}\kappa(X)||A^{-1}||_2||E||_F. \tag{3.9}$$

On the other hand by (3.4) we have

$$X^{-1}\widetilde{A}X - \Lambda = X^{-1}EX. \tag{3.10}$$

Since Λ is normal, from (3.3) and (3.10) we have

$$\begin{aligned}
\sqrt{\sum_{i=1}^{n}\left|\frac{\mu_{\tau(i)}-\lambda_i}{\lambda_i}\right|^2} &\le \sqrt{n - s(X^{-1}\widetilde{A}X)+1}||\Lambda^{-1}||_2||X^{-1}EX||_F \\
&\le \sqrt{n - s(X^{-1}\widetilde{A}X)+1}\kappa(X)\rho(A^{-1})||E||_F,
\end{aligned}$$

which together with (3.9) gives (3.5).

By Theorem 3.1 it is easy to prove the following spectral norm relative bound.

Corollary 3.2. *Under the same assumption as Theorem 3.1, there exists a permutation τ of $\langle n\rangle$ such that*

$$\max\left|\frac{\mu_{\tau(i)}-\lambda_i}{\lambda_i}\right| \le \min\{\sqrt{n - s(X^{-1}\widetilde{A}X)+1}\rho(A^{-1}), \sqrt{n}||A^{-1}||_2\}\sqrt{n}\kappa(X)||E||_2 \tag{3.11}$$

Remark 3.1. The bound (3.6) reduces to (3.3) provided A is normal. In fact, if A is normal, then X in (3.4) is unitary, and thus $s(X^{-1}\widetilde{A}X) = s(\widetilde{A})$ and $\kappa(X) = 1$.

Remark 3.2. Although $\widetilde{A}$ is diagonalizable, in the following example it is known that the bound in (3.6) is better than one in (3.1). Let

$$A = \begin{pmatrix} -0.03 & 0.04 \\ 0.04 & 0.03 \end{pmatrix}, \ \widetilde{A} = \begin{pmatrix} -0.02 & 0.05 \\ 0.01 & 0.02 \end{pmatrix}.$$

Then A is Hermitian and $\widetilde{A}$ is diagonalizable, in fact,

$$\widetilde{A} = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0.03 & 0 \\ 0 & -0.03 \end{pmatrix} \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Let $Y = \begin{pmatrix} 1 & -5 \\ 1 & 1 \end{pmatrix}$. Then $\kappa(Y) = 4.4415$. A simple calculation gives $\kappa(Y)||A^{-1}(\widetilde{A}-A)||_F = 3.0772$ and $\sqrt{2}||A^{-1}||_2||\widetilde{A}-A||_F = 0.9798$, which show that our relative bound in (3.6) is sharper than those in (3.1).

4. The Eigenvalue Inclusion Region

As an application of the bound (1.5), in this section we consider the eigenvalue inclusion region. Let A be unitarily similar to a block diagonal matrix and $s(A)$ be given by (1.6).

A classical region is the Gersgorin discs, i.e., the all eigenvalues of $A = (a_{ij})$ are located in the union of n discs

$$\bigcup_{i=1}^{n} \{z \in \mathbf{C} \mid |z - a_{ii}| \leq \sum_{(i \neq)j=1}^{n} |a_{ij}|\}. \tag{4.1}$$

Let $\lambda = (\lambda_1, ..., \lambda_n)^T$ be an eigenvalue vector of A. Let τ be a permutation of $\langle n \rangle$. By λ_τ we denote $\lambda_\tau = (\lambda_{\tau(1)}, ..., \lambda_{\tau(n)})^T$. Let $D_A = diag(a_{11}, ..., a_{nn})$, $a = (a_{11}, ..., a_{nn})$ and $A' = A - D_A$. By the similar idea to [9], we can obtain a new result below.

Theorem 4.1. *Let $A = (a_{ij}) \in \mathbf{C}^{n \times n}$. Then there exists a permutation τ of $\langle n \rangle$ such that λ_τ is located in the following n-dimension sphere.*

$$\{z \in \mathbf{C}^n \mid ||z - a||_F^2 \leq (n - s(A) + 1)||A'||_F^2\}. \tag{4.2}$$

Proof. Clearly we have $A - D_A = A'$. Notice that D_A is normal. It follows from (1.5) that $\sum_{i=1}^{n} |\lambda_{\tau(i)} - a_{ii}|^2 \leq (n - s(A) + 1)||A'||_F^2$, which proves the result.

For some special matrices, e.g., a normal matrix, the eigenvalue inclusion region (4.2) can be written as follows.

Corollary 4.2. *Let $A = (a_{ij}) \in \mathbf{C}^{n \times n}$ be a normal matrix. Then each disc*

$$\{z \in \mathbf{C} \mid |z - a_{ii}| \leq (\sum_{i \neq j}^{n} |a_{ij}|^2)^{\frac{1}{2}}\} \tag{4.3}$$

contains at least one *eigenvalue of A. If there is an eigenvalue of A located on the boundary of (4.3), then the other $n-1$ eigenvalues of A are a_{jj}, $j \neq i$*

Proof. The first part of this corollary can be found in [10]. It needs only to prove the second part. Since A is normal, A is unitarily similar to a diagonal matrix, and hence $s(A) = n$. By Theorem 4.1 we have

$$||\lambda_\sigma - a||_F \leq ||A'||_F. \tag{4.4}$$

Let λ_k be an eigenvalue of A located on the boundary of (4.3), i.e., $|\lambda_k - a_{ii}| = (\sum_{i \neq j}^{n} |a_{ij}|^2)^{\frac{1}{2}} = ||A'||_F$. By (4.4) we have $|\lambda_{\sigma(j)} - a_{jj}| = 0$, $\sigma(j) \neq k$. Hence $\lambda_{\sigma(j)} = a_{jj}$ for $\sigma(j) \neq k$ and $j \neq i$. This proves the corollary.

Remark 4.1. It is difficult to compute $s(A)$ for general matrices. However, since $s(A) \geq 1$, by (4.2) we have

$$\lambda_\tau \in \{z \in \mathbf{C}^n \mid ||z - a||_F \leq \sqrt{n}||A'||_F\},$$

which is a result in [9].

Remark 4.2. If A is normal, then by (4.3) we provide a tighter region in some sense. For example,

$$A = \begin{pmatrix} 1 & -3 & -5 & -3 \\ -3 & 1 & 0 & 0 \\ -5 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}.$$

Then by Gersgorin discs (4.1) all eigenvalues of A are located in $\{z \in \mathbf{C} \mid |z-1| \leq 11\}$. But this matrix is Hermitian, and hence normal, then by (4.3) all eigenvalues of A are located in $\{z \in \mathbf{C} \mid |z-1| \leq \sqrt{86}\}$. This implies the region in (4.3) is tighter than one in (4.1). In fact, the eigenvalues of A are $1+\sqrt{43}, 1-\sqrt{43}, 1, 1$. This example also illustrates that the region in (4.2) is sharp.

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