Journal of Computational Mathematics, Vol.24, No.1, 2006, 45–58.

EXPANSIONS OF STEP-TRANSITION OPERATORS OF MULTI-STEP METHODS AND ORDER BARRIERS FOR DAHLQUIST PAIRS *1)

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Abstract

Using least parameters, we expand the step-transition operator of any linear multi-step method (LMSM) up to $O(\tau^{s+5})$ with order $s = 1$ and rewrite the expansion of the step-transition operator for $s = 2$ (obtained by the second author in a former paper). We prove that in the conjugate relation $G_3^{\lambda\tau} \circ G_1^{\tau} = G_2^{\tau} \circ G_3^{\lambda\tau}$ with G_1 being an LMSM, (1) the order of G_2 can not be higher than that of G_1; (2) if G_3 is also an LMSM and G_2 is a symplectic B-series, then the orders of G_1, G_2 and G_3 must be 2, 2 and 1 respectively.

Mathematics subject classification: 65L06
Key words: Linear Multi-Step Method; Step-Transition Operator; B-series; Dahlquist (Conjugate) pair; Symplecticity

1. Introduction

For an ordinarily differential equation (ODE)

$$\frac{d}{dt}Z = f(Z), \quad Z \in \mathbb{R}^p, \tag{1}$$

any compatible linear m-step difference scheme (DS)

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k f(Z_k) \quad \left(\sum_{k=0}^{m} \beta_k \neq 0\right) \tag{2}$$

can be characterized by a step-transition operator (STO) (also called underlying one-step method) G (also denoted by G^τ): $\mathbb{R}^p \to \mathbb{R}^p$ satisfying

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k f \circ G^k, \tag{3}$$

where G^k stands for k-time composition of G: $G \circ G \cdots \circ G$ (refer to [2,3,5,6,7]). This operator G^τ can be represented as a power series in τ with first term equal to the identity I. More

*Received September 27, 2004
1) This research is supported by the Informatization Construction of Knowledge Innovation Projects of the Chinese Academy of Sciences "Supercomputing Environment Construction and Application" (INF105-SCE), and by a grant (No. 10471145) from National Natural Science Foundation of China.

precisely, one can expand[9] the STO $G^\tau(Z)$ of any linear multi-step method (LMSM)[2] of form (2) with order $s \geq 2$ up to $O(\tau^{s+5})$:

$$G^\tau(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^{s+1} A(Z) + \tau^{s+2} B(Z) + \tau^{s+3} C(Z) + \tau^{s+4} D(Z) + O(\tau^{s+5}) \tag{4}$$

(where $Z^{[0]} = Z$, $Z^{[1]} = f(Z)$, $Z^{[k+1]} = \frac{\partial Z^{[k]}}{\partial Z} Z^{[1]} = Z_z^{[k]} Z^{[1]}$ for $k = 1, 2, \cdots$) with complete formulae for calculation of $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$.

Thus, the STO G^τ satisfying equation (3) completely characterizes the LMSM (2) as: $Z_1 = G^\tau(Z_0), \cdots, Z_m = G^\tau(Z_{m-1}) = [G^\tau]^m(Z_0), \cdots$.

When equation (1) is a hamiltonian system, i.e., $p = 2n$ and $f(Z) = J\nabla H(Z)$, where $J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$, ∇ stands for the gradient operator, and $H : \mathbb{R}^{2n} \to \mathbb{R}^1$ is a smooth function, (1), (2) and (3) become

$$\frac{dZ}{dt} = J\nabla H(Z), \quad Z \in \mathbb{R}^{2n}, \tag{5}$$

$$\sum_{k=0}^{m} \alpha_k Z_k = \tau \sum_{k=0}^{m} \beta_k J\nabla H(Z_k) \qquad \left(\sum_{k=0}^{m} \beta_k \neq 0\right), \tag{6}$$

$$\sum_{k=0}^{m} \alpha_k G^k = \tau \sum_{k=0}^{m} \beta_k J\nabla H \circ G^k, \tag{7}$$

and we can rewrite

$$\begin{aligned}
Z^{[0]} =& Z, \\
Z^{[1]} =& J\nabla H, \\
Z^{[2]} =& JH_{zz}J\nabla H = Z_z^{[1]} Z^{[1]}, \\
Z^{[3]} =& Z_{z^2}^{[1]} \left(Z^{[1]}\right)^2 + Z_z^{[1]} Z^{[2]}, \\
Z^{[4]} =& Z_{z^3}^{[1]} \left(Z^{[1]}\right)^3 + 3Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} + Z_z^{[1]} Z^{[3]}, \\
Z^{[5]} =& Z_{z^4}^{[1]} \left(Z^{[1]}\right)^4 + 6Z_{z^3}^{[1]} \left(Z^{[1]}\right)^2 Z^{[2]} + 3Z_{z^2}^{[1]} \left(Z^{[2]}\right)^2 \\
& + 4Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} + Z_z^{[1]} Z^{[4]},
\end{aligned} \tag{8}$$

and generally,

$$Z^{[r+1]} = \sum_{j=1}^{r} \sum_{i_1+i_2+\cdots+i_j=r; i_u \geq 1} \frac{r!}{j!} \frac{\Omega(i_1, i_2, \cdots, i_j)}{i_1! i_2! \cdots i_j!} J(\nabla H)_{z^j} Z^{[i_1]} Z^{[i_2]} \cdots Z^{[i_j]}$$

where $i_1 \leq i_2 \leq \cdots \leq i_j$, $\Omega(i_1, i_2, \cdots, i_j)$ is the number of all different permutations of $\{i_1, i_2, \cdots, i_j\}$, and $(\nabla H)_{z^j} Z^{[i_1]} Z^{[i_2]} \cdots Z^{[i_j]}$ stands for the multi-linear form

$$\sum_{1 \leq t_1, \cdots, t_j \leq 2n} \frac{\partial^j (\nabla H)}{\partial Z_{(t_1)} \cdots \partial Z_{(t_j)}} Z_{(t_1)}^{[i_1]} \cdots Z_{(t_j)}^{[i_j]},$$

$Z_{(t_u)}^{[i_u]}$ stands for the t_u-th component of the 2n-dim vector $Z^{[i_u]}$.

The expansion of STO (4) has been used to study the symplecticity of LMSM (refer to [3], [7]), and also the symplecticity of Dahlquist pair (refer to [8]).

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2) More generally, one can use an STO to characterize any DS compatible with ODE (1), and obviously the STO can be written in form (4).

Definition 1. (due to Feng and Tang[2],[7]) *An LMSM is said to be symplectic for Hamiltonian system (5) iff its STO G^τ defined by (7) is symplectic, i.e.,*

$$\left[\frac{\partial G^\tau(Z)}{\partial Z}\right]^\top J \left[\frac{\partial G^\tau(Z)}{\partial Z}\right] = J \tag{9}$$

for any hamiltonian function H and any sufficiently small step-size τ.

Definition 2. *If three B-serieses[3] G_1^τ, G_2^τ and G_3^τ in form (4) compatible with equation (1) satisfy*

$$G_3^{\lambda\tau} \circ G_1^\tau = G_2^\tau \circ G_3^{\lambda\tau} \tag{10}$$

for some real number λ and for any smooth function f and any sufficiently small step-size τ, then G_1^τ and G_2^τ are said to be a Dahlquist[4] pair or a conjugate pair via G_3^τ, and we call equation (10) a conjugate relation. A Dahlquist pair G_1^τ and G_2^τ is said to be symplectic if G_1^τ or G_2^τ is symplectic. In this case when one of G_1^τ and G_2^τ is symplectic, we also call the other conjugate-symplectic.

In the present paper, for any linear multi-step method (LMSM) with order $s = 1$, using 6 parameters we obtain the expansion of its step-transition operator in form (4) up to $O(\tau^6)$; and using 5 parameters we rewrite the expansion of the step-transition operator for $s = 2$ (obtained by Tang in a former paper [9] where 9 parameters are used) (in Section 2). We prove that in conjugate relation (10) with G_1 being an LMSM, (1) the order of G_2 can not be higher than that of G_1 (that means, conjugation will not improve the order of any LMSM); (2) if G_3 is also an LMSM and G_2 is a symplectic B-series, then the order of both G_1 and G_2 must be 2 (in Section 3).

2. Expansion of Step-Transition Operator

Theorem 1. *If scheme (2) is of order $s = 1$, then the corresponding step-transition operator defined by (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^2 C(Z) + \tau^3 D(Z) + \tau^4 E(Z) + \tau^5 F(Z) + O(\tau^6), \tag{11}$$

where $C(Z), D(Z), E(Z), F(Z)$ can be determined by 6 parameters ω, ρ, δ, σ, η, ν:

$$C = \omega Z^{[2]}, \quad \omega = \frac{\sum_{k=0}^m \left(k\beta_k - \frac{k^2}{2}\alpha_k\right)}{\sum_{k=0}^m k\alpha_k}; \tag{12.1}$$

$$D = \left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) Z^{[3]} + \omega(\omega + \frac{1}{2}) Z_z^{[1]} Z^{[2]}, \tag{12.2}$$

$$\rho = \frac{\sum_{k=0}^m \left[k^2\beta_k - \frac{k^3}{3}\alpha_k\right]}{\sum_{k=0}^m k\alpha_k}, \quad \delta = \frac{\sum_{k=0}^m k^2\alpha_k}{\sum_{k=0}^m k\alpha_k};$$

$$\begin{aligned} E = &\left(\frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6} - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4}\right) Z^{[4]} \\ &+ \left(\rho\omega - \frac{3\delta\omega^2}{4} + \frac{3\omega^2}{4} + \frac{\omega}{3}\right) Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &+ \left(-\delta\omega^2 + \frac{\omega^2}{2} + \rho\omega - \frac{\omega\delta}{4} + \frac{\omega}{6} + \frac{\rho}{4}\right) Z_z^{[1]} Z^{[3]} \end{aligned} \tag{12.3}$$

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3) For the details about B-series, one can refer to [4]. We would like to thank Ernst Hairer for the suggestion that the case when G_3^τ is a B-series should be considered in the conjugate relation.

4) It was G. Dahlquist[1] who first found that the trapezoid rule and the mid-point rule are a conjugate pair via the Euler-forward scheme.

$$+\left(\omega^3+\frac{3\omega^2}{4}-\frac{\omega^2\delta}{4}+\frac{\omega\rho}{2}+\frac{\omega}{6}\right)Z_z^{[1]}Z_z^{[1]}Z^{[2]},$$

$$\sigma=\frac{\sum_{k=0}^m\left(\frac{k^3}{2}\beta_k-\frac{k^4}{8}\alpha_k\right)}{\sum_{k=0}^m k\alpha_k}\omega,\quad \eta=\frac{\sum_{k=0}^m k^3\alpha_k}{\sum_{k=0}^m k\alpha_k};$$

$$\begin{aligned}F=&\nu Z^{[5]} \qquad (12.4)\\
&+\left\{-\frac{3\omega^2\delta}{4}+\frac{5\omega^2\delta^2}{8}-\frac{5\eta\omega^2}{12}+\frac{13\omega^2}{24}-\frac{3\omega\rho\delta}{4}+\frac{3\omega\rho}{4}+\sigma\omega+\frac{\omega}{8}\right\}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[2]}\\
&+\left\{\frac{\omega^3}{2}-\frac{\delta\omega^3}{2}+\frac{5\delta^2\omega^2}{8}-\frac{3\delta\omega^2}{4}+\frac{\rho\omega^2}{2}-\frac{5\eta\omega^2}{12}+\frac{17\omega^2}{24}-\frac{3\delta\rho\omega}{4}+\frac{3\omega\rho}{4}+\sigma\omega+\frac{\omega}{8}\right\}\\
&\quad Z_{z^2}^{[1]}\left(Z^{[2]}\right)^2\\
&+\left\{\frac{9\delta^2\omega^2}{8}-\frac{5\delta\omega^2}{4}-\frac{5\eta\omega^2}{12}+\frac{13\omega^2}{24}-\frac{7\delta\rho\omega}{4}+\frac{5\rho\omega}{4}+\sigma\omega-\frac{\omega\delta}{6}+\frac{\omega}{8}+\frac{\rho^2}{2}+\frac{\rho}{6}\right\}\\
&\quad Z_{z^2}^{[1]}Z^{[1]}Z^{[3]}\\
&+\left\{-\delta\omega^3+\omega^3+\frac{5\omega^2\delta^2}{8}-\frac{3\omega^2\delta}{4}+\rho\omega^2-\frac{5\eta\omega^2}{12}+\frac{7\omega^2}{8}-\frac{3\delta\rho\omega}{4}+\frac{3\rho\omega}{4}+\sigma\omega+\frac{\omega}{8}\right\}\\
&\quad Z_{z^2}^{[1]}Z^{[1]}Z_z^{[1]}Z^{[2]}\\
&+\left\{-\frac{\eta\omega^2}{3}+\frac{3\omega^2\delta^2}{4}-\frac{\omega^2\delta}{2}+\frac{\omega^2}{6}+\frac{\delta^2\omega}{8}-\delta\rho\omega+\frac{2\sigma\omega}{3}+\frac{\rho\omega}{2}-\frac{\omega\delta}{12}-\frac{\eta\omega}{12}+\frac{\omega}{24}\right.\\
&\quad\left.+\frac{\rho^2}{4}+\frac{\rho}{12}+\frac{\sigma}{6}-\frac{\delta\rho}{8}\right\}Z_z^{[1]}Z^{[4]}\\
&+\left\{-\frac{7\delta\omega^3}{4}+\omega^3-\frac{3\delta\omega^2}{4}+\frac{3\delta^2\omega^2}{8}+\frac{7\rho\omega^2}{4}-\frac{\eta\omega^2}{3}+\frac{7\omega^2}{12}-\frac{\delta\rho\omega}{2}+\frac{2\sigma\omega}{3}+\rho\omega+\frac{\omega}{12}\right\}\\
&\quad Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[2]}\\
&+\left\{-\frac{3\delta\omega^3}{2}+\frac{\omega^3}{2}+\frac{3\delta^2\omega^2}{8}-\delta\omega^2+\frac{3\rho\omega^2}{2}-\frac{\omega^2\eta}{12}+\frac{3\omega^2}{8}-\frac{3\delta\rho\omega}{4}+\frac{\sigma\omega}{3}+\rho\omega+\frac{\omega}{24}\right.\\
&\quad\left.\frac{\rho^2}{4}+\frac{\rho}{12}-\frac{\omega\delta}{12}\right\}Z_z^{[1]}Z_z^{[1]}Z^{[3]}\\
&+\left\{\omega^4-\frac{5\delta\omega^3}{4}+\frac{3\omega^3}{2}+\frac{\delta^2\omega^2}{4}-\frac{\eta\omega^2}{4}+\frac{5\rho\omega^2}{4}-\frac{\delta\omega^2}{2}+\frac{5\omega^2}{8}-\frac{\delta\rho\omega}{4}+\frac{\rho\omega}{2}+\frac{\delta\omega}{8}\right.\\
&\quad\left.-\frac{\eta\omega}{12}+\frac{\sigma\omega}{3}\right\}Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[2]},\end{aligned}$$

$$\begin{aligned}\nu=\frac{1}{\sum\limits_{k=0}^m k\alpha_k}\sum_{k=0}^m&\left\{\frac{k^4}{4!}\beta_k-\left[\frac{k^5}{5!}+\frac{k^4-2k^3+k^2}{24}\omega+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.\right.\\
&\left.\left.+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right]\alpha_k\right\}.\end{aligned}$$

Here we use the notation for example,

$$Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[2]}=\sum_{i,j,k=1}^p\frac{\partial^3Z^{[1]}}{\partial z_i\partial z_j\partial z_k}\left[Z^{[1]}\right]_{(i)}\left[Z^{[1]}\right]_{(j)}\left[Z^{[2]}\right]_{(k)}$$

where z_i is the i-th component of p-dim vector Z, and $\left[Z^{[r]}\right]_{(j)}$ stands for the j-th component of p-dim vector $Z^{[r]}$.

The proof of Theorem 1 is tedious but straightforward calculation, and similar to that for $s \geq 2$ given in [9]. A difference is that we here try to use least parameters in expressing $C(Z)$, $D(Z)$, $E(Z)$ and $F(Z)$. We give the complete proof of Theorem 1 later in Appendix 1.

Similar result for $s \geq 2$ is already given in [9], where 9 parameters λ, μ, ν, ρ, ξ, σ, χ, η and ζ are used for expressing $A(Z)$, $B(Z)$, $C(Z)$ and $D(Z)$ in (4). Using 5 parameters ω_2, ρ_2, δ_2, σ_2 and ν_2, we rewrite the result for $s = 2$ as follows:

Theorem 2. *If scheme (2) is of order $s = 2$, then the step-transition operator decided by equation (3) has the following expansion:*

$$G(Z) = \sum_{i=0}^{+\infty} \frac{\tau^i}{i!} Z^{[i]} + \tau^3 C(Z) + \tau^4 D(Z) + \tau^5 E(Z) + \tau^6 F(Z) + O(\tau^7), \tag{13}$$

where $C(Z), D(Z), E(Z), F(Z)$ can be expressed by 5 parameters ω_2, ρ_2, δ_2, σ_2 and ν_2:

$$C = \omega_2 Z^{[3]}, \quad \omega_2 = \frac{\sum_{k=0}^{m} \left\{\frac{k^2}{2}\beta_k - \frac{k^3}{6}\alpha_k\right\}}{\sum_{k=0}^{m} k\alpha_k}; \tag{14.1}$$

$$D = \left(\rho_2 - \frac{1}{2}\delta_2\omega_2 + \frac{\omega_2}{2}\right) Z^{[4]} + \frac{\omega_2}{2} Z_z^{[1]} Z^{[3]}, \tag{14.2}$$

$$\rho_2 = \frac{\sum_{k=0}^{m} \left[\frac{k^3}{6}\beta_k - \frac{k^4}{24}\alpha_k\right]}{\sum_{k=0}^{m} k\alpha_k}, \quad \delta_2 = \frac{\sum_{k=0}^{m} k^2\alpha_k}{\sum_{k=0}^{m} k\alpha_k};$$

$$\begin{aligned} E =& \sigma_2 Z^{[5]} + \left(\omega_2^2 + \frac{\omega_2}{6}\right) Z_z^{[1]} Z_z^{[1]} Z^{[3]} \\ &+ \left(\omega_2^2 - \frac{1}{4}\delta_2\omega_2 + \frac{\omega_2}{6} + \frac{\rho_2}{2}\right) Z_z^{[1]} Z^{[4]} + \left(2\omega_2^2 + \frac{\omega_2}{3}\right) Z_{z^2}^{[1]} Z^{[1]} Z^{[3]}, \end{aligned} \tag{14.3}$$

$$\sigma_2 = \frac{\sum_{k=0}^{m} \left[\frac{k^4}{24}\beta_k - \frac{k^5}{120}\alpha_k - \left\{\frac{2k^3 - 3k^2 + k}{12}\omega_2 + \frac{k^2 - k}{2}\left(\rho_2 - \frac{1}{2}\delta_2\omega_2 + \frac{\omega_2}{2}\right)\right\}\alpha_k\right]}{\sum_{k=0}^{m} k\alpha_k};$$

$$\begin{aligned} F =& \nu_2 Z^{[6]} \\ &+ \left\{-\delta_2\omega_2^2 + \frac{\omega_2^2}{2} + 2\rho_2\omega_2 + \frac{1}{24}\delta_2\omega_2 - \frac{\omega_2}{24} - \frac{\rho_2}{12} + \frac{\sigma_2}{2}\right\} Z_z^{[1]} Z^{[5]} \\ &+ \left\{-\delta_2\omega_2^2 + \omega_2^2 + 2\rho_2\omega_2 - \frac{1}{12}\delta_2\omega_2 + \frac{\omega_2}{24} + \frac{\rho_2}{6}\right\} Z_z^{[1]} Z_z^{[1]} Z^{[4]} \\ &+ \left\{-\frac{5}{2}\delta_2\omega_2^2 + \frac{3}{2}\omega_2^2 + 5\rho_2\omega_2 - \frac{1}{6}\delta_2\omega_2 + \frac{\omega_2}{8} + \frac{\rho_2}{3}\right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[4]} \\ &+ \left\{-\frac{1}{2}\delta_2\omega_2^2 + \omega_2^2 + \rho_2\omega_2 + \frac{\omega_2}{24}\right\} Z_z^{[1]} Z_z^{[1]} Z_z^{[1]} Z^{[3]} \\ &+ \left\{-\delta_2\omega_2^2 + 2\omega_2^2 + 2\rho_2\omega_2 + \frac{\omega_2}{12}\right\} Z_z^{[1]} Z_{z^2}^{[1]} Z^{[1]} Z^{[3]} \\ &+ \left\{-\frac{3}{2}\delta_2\omega_2^2 + \frac{3}{2}\omega_2^2 + 3\rho_2\omega_2 + \frac{\omega_2}{8}\right\} Z_{z^2}^{[1]} Z^{[1]} \left(Z_z^{[1]} Z^{[3]}\right) \\ &+ \left\{-\frac{3}{2}\delta_2\omega_2^2 + \frac{3}{2}\omega_2^2 + 3\rho_2\omega_2 + \frac{\omega_2}{8}\right\} Z_{z^2}^{[1]} Z^{[2]} Z^{[3]} \\ &+ \left\{-\frac{3}{2}\delta_2\omega_2^2 + \frac{3}{2}\omega_2^2 + 3\rho_2\omega_2 + \frac{\omega_2}{8}\right\} Z_{z^3}^{[1]} \left(Z^{[1]}\right)^2 Z^{[3]}, \end{aligned} \tag{14.4}$$

$$\nu_2 = \frac{\sum_{k=0}^{m}\left\{\frac{k^5}{5!}\beta_k - \left[\frac{k^6}{6!} + \frac{k^4-2k^3+k^2}{24}\omega_2 + \frac{2k^3-3k^2+k}{12}\left(\rho_2 - \frac{\delta_2\omega_2}{2} + \frac{\omega_2}{2}\right) + \frac{k^2-k}{2}\sigma_2\right]\alpha_k\right\}}{\sum_{k=0}^{m} k\alpha_k}.$$

3. Order Barriers for STOs in Conjugate Relation

Theorem 3. *In conjugate relation (10), if B-series G_1 stands for an LMSM, then the order of G_2 can not be higher than that of G_1.*

Proof. Supposing the orders of G_1^τ, G_1^τ and G_3^τ are u, v and $w-1$ respectively, we write their expansions as follows:

$$G_1^\tau(Z) = \sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]} + \tau^{u+1}A(Z) + O(\tau^{u+2}) \tag{15}$$

$$G_2^\tau(Z) = \sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]} + \tau^{v+1}M(Z) + O(\tau^{v+2}) \tag{16}$$

$$G_3^\tau(Z) = \sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]} + \tau^{w}B(Z) + O(\tau^{w+1}) \tag{17}$$

where $A(Z) \neq 0$, $B(Z) \neq 0$ and $M(Z) \neq 0$.

Provided $v > u$, there are three cases:

Case 1. $w > u$, expanding both sides of (10) and comparing the terms in τ^{u+1} we have

$$A(Z) = 0. \tag{18}$$

Case 2. $w = u$, expanding both sides of (10) and comparing the terms in τ^{u+1} we have

$$\lambda^w B_z Z^{[1]} + A(Z) = \lambda^w Z_z^{[1]} B. \tag{19}$$

Case 3. $w < u$, expanding both sides of (10) and comparing the terms in τ^{w+1} we have

$$\lambda^w B_z Z^{[1]} = \lambda^w Z_z^{[1]} B. \tag{20}$$

From Theorem 1, Theorem 2 in Section 2 above, and Lemma 1 in [7], we know that in fact $A(Z) = aZ^{[u+1]}$ for some $a \neq 0$. Since B-series G_3^τ is compatible with (3), $w \geq 2$. When $\lambda \neq 0$, it's easy to check that any of the cases (18), (19) and (20) is impossible; when $\lambda = 0$, equation (10) becomes into $G_1^\tau(Z) = G_2^\tau(Z)$ which contradicts $v > u$.

So the only possible case should be $v \leq u$.

Theorem 4. *In conjugate relation (10), if both G_1 and G_3 stand for LMSMs, and G_2 is a symplectic B-series, then the orders of G_1^τ, G_2^τ and G_3^τ are 2, 2 and 1 respectively.*

The result in Theorem 4 is a little different from that in Theorem 1 in [8], and the proof of the former will also based on the latter.

Proof of Theorem 4. Supposing the order of G_1^τ, G_1^τ and G_3^τ are u, v and $w-1$ respectively. Since they are compatible with (3), $u \geq 1$, $v \geq 1$ and $w \geq 2$. We write their expansions as (15-17). According to Theorem 1 in [8], if G_2 is symplectic, then the order of G_1^τ can not be greater than 2. So $1 \leq u \leq 2$. And according to Theorem 3 above, we know $v \leq u$. Let's discuss all the cases as follows:

Case 1. If $u = 1$, then $v = 1$. Expanding both sides of (10) and comparing the terms in τ^2 we have

$$A(Z) = M(Z). \tag{21}$$

Case 2. If $u = 2$, $v = 1$. Expanding both sides of (10) and comparing the terms in τ^2 we have

$$0 = M(Z). \tag{22}$$

Case 3. If $u=2$, $v=2$ and $w=2$. Expanding both sides of (10) and comparing the terms in τ^3 we have

$$\lambda^w B_z Z^{[1]} + A(Z) = M(Z) + \lambda^w Z_z^{[1]} B. \tag{23}$$

Case 4. If $u=2$, $v=2$ and $w>2$. Expanding both sides of (10) and comparing the terms in τ^3 we have

$$A(Z) = M(Z). \tag{24}$$

Since $A(Z) = aZ^{[u+1]}$ for some $a \neq 0$, and G_2 is a symplectic B-series with order v, cases (21), (22) and (24) are impossible. So the only possible case is (23), i.e., $u=v=w=2$.

Appendix 1. Proof of Theorem 1

When we set

$$G^k(Z) = \sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} Z^{[i]} + \tau^2 C_k(Z) + \tau^3 D_k(Z) + \tau^4 E_k(Z) + \tau^5 F_k(Z) + O(\tau^6), \tag{25}$$

then

$$\begin{aligned}
&\sum_{i=0}^{+\infty} \frac{(k+1)^i \tau^i}{i!} Z^{[i]} + \tau^2 C_{k+1}(Z) + \tau^3 D_{k+1}(Z) \\
&\qquad + \tau^4 E_{k+1}(Z) + \tau^5 F_{k+1}(Z) + O(\tau^6) \\
=&G^{k+1}(Z) = G^k\left[G(Z)\right] \\
=&\sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[G(Z)\right]^{[i]} + \tau^2 C_k\left[G(Z)\right] + \tau^3 D_k\left[G(Z)\right] \\
&\quad + \tau^4 E_k\left[G(Z)\right] + \tau^5 F_k\left[G(Z)\right] + O(\tau^6) \\
\equiv&\widetilde{I} + \widetilde{II} + \widetilde{III} + \widetilde{IV} + \widetilde{V} + O(\tau^6),
\end{aligned} \tag{26}$$

and

$$\begin{aligned}
\widetilde{I} =&\sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \Bigg[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]} \\
&\qquad +\tau^2 C_1(Z) + \tau^3 D_1(Z) + \tau^4 E_1(Z) + \tau^5 F_1(Z) + O(\tau^6)\Bigg]^{[i]} \\
=&\sum_{i=0}^{+\infty} \frac{k^i \tau^i}{i!} \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right]^{[i]} + \tau^2 C_1 + \tau^3 D_1 + \tau^4 E_1 + \tau^5 F_1 \\
&+ \frac{k\tau}{1!} \left\{ Z_z^{[1]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1 + \tau^3 D_1 + \tau^4 E_1\right) + \frac{1}{2!} Z_{z^2}^{[1]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1\right)^2 \right\} \\
&+ \frac{k^2\tau^2}{2!} \left\{ Z_z^{[2]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1 + \tau^3 D_1\right) \right\} \\
&+ \frac{k^3\tau^3}{3!} \left\{ Z_z^{[3]} \circ \left[\sum_{j=0}^{+\infty} \frac{\tau^j}{j!} Z^{[j]}\right] * \left(\tau^2 C_1\right) \right\} + O(\tau^6) \\
=&\sum_{l=0}^{+\infty} \frac{(k+1)^l \tau^l}{l!} Z^{[l]} + \tau^2 C_1 + \tau^3 \left\{ D_1 + k Z_z^{[1]} C_1 \right\} \\
&+ \tau^4 \left\{ E_1 + k Z_z^{[1]} D_1 + k Z_{z^2}^{[1]} Z^{[1]} C_1 + \frac{k^2}{2} Z_z^{[2]} C_1 \right\}
\end{aligned} \tag{26.1}$$

$$+\tau^5\left\{F_1+kZ_z^{[1]}E_1+kZ_{z^2}^{[1]}Z^{[1]}D_1+\frac{k}{2}Z_{z^2}^{[1]}Z^{[2]}C_1+\frac{k}{2}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2C_1\right.$$
$$\left.+\frac{k}{2}Z_{z^2}^{[1]}\left(C_1\right)^2+\frac{k^2}{2}Z_z^{[2]}D_1+\frac{k^2}{2}Z_{z^2}^{[2]}Z^{[1]}C_1+\frac{k^3}{6}Z_z^{[3]}C_1\right\}+O(\tau^6);$$

$$\begin{aligned}\widetilde{II}=&\tau^2C_k\circ\left(Z+\tau Z^{[1]}+\frac{\tau^2}{2}Z^{[2]}+\frac{\tau^3}{6}Z^{[3]}+\tau^2C_1+\tau^3D_1\right)+O(\tau^6)\\=&\tau^2C_k+\tau^3\left(C_k\right)_zZ^{[1]}+\tau^4\left\{\frac{1}{2}\left(C_k\right)_zZ^{[2]}+\left(C_k\right)_zC_1+\frac{1}{2}\left(C_k\right)_{z^2}\left[Z^{[1]}\right]^2\right\}\\&+\tau^5\left\{\frac{1}{6}\left(C_k\right)_zZ^{[3]}+\left(C_k\right)_zD_1+\left(C_k\right)_{z^2}Z^{[1]}C_1+\frac{1}{2}\left(C_k\right)_{z^2}\left[Z^{[1]}Z^{[2]}\right]\right.\\&\left.+\frac{1}{6}\left(C_k\right)_{z^3}\left[Z^{[1]}\right]^3\right\}+O(\tau^6);\end{aligned}\tag{26.2}$$

$$\begin{aligned}\widetilde{III}=&\tau^3D_k\circ\left(Z+\tau Z^{[1]}+\frac{\tau^2}{2}Z^{[2]}+\tau^2C_1\right)+O(\tau^6)\\=&\tau^3D_k+\tau^4\left(D_k\right)_zZ^{[1]}\\&+\tau^5\left\{\frac{1}{2}\left(D_k\right)_zZ^{[2]}+\left(D_k\right)_zC_z+\frac{1}{2}\left(D_k\right)_{z^2}\left[Z^{[1]}\right]^2\right\}+O(\tau^6);\end{aligned}\tag{26.3}$$

$$\begin{aligned}\widetilde{IV}=&\tau^4E_k\circ\left(Z+\tau Z^{[1]}\right)+O(\tau^6)\\=&\,\tau^4E_k+\tau^5\left(E_k\right)_zZ^{[1]}+O(\tau^6);\end{aligned}\tag{26.4}$$

$$\widetilde{V}=\tau^5F_k+O(\tau^6).\tag{26.5}$$

From (26), (26.1)–(26.5), we obtain

$$C_{k+1}=C_1+C_k;\tag{27.1}$$
$$D_{k+1}=D_1+kZ_z^{[1]}C_1+\left(C_k\right)_zZ^{[1]}+D_k;\tag{27.2}$$
$$\begin{aligned}E_{k+1}=&E_1+kZ_z^{[1]}D_1+kZ_{z^2}^{[1]}\left[Z^{[1]}C_1\right]+\frac{k^2}{2}Z_z^{[2]}C_1+\frac{1}{2}\left(C_k\right)_zZ^{[2]}\\&+\left(C_k\right)_zC_1+\frac{1}{2}\left(C_k\right)_{z^2}\left[Z^{[1]}\right]^2+\left(D_k\right)_zZ^{[1]}+E_k;\end{aligned}\tag{27.3}$$
$$\begin{aligned}F_{k+1}=&F_1+kZ_z^{[1]}E_1+kZ_{z^2}^{[1]}\left[Z^{[1]}D_1\right]+\frac{k}{2}Z_{z^2}^{[1]}\left[Z^{[2]}C_1\right]\\&+\frac{k}{2}Z_{z^3}^{[1]}\left[\left(Z^{[1]}\right)^2C_1\right]+\frac{k}{2}Z_{z^2}^{[1]}\left(C_1\right)^2+\frac{k^2}{2}Z_z^{[2]}D_1\\&+\frac{k^2}{2}Z_{z^2}^{[2]}\left[Z^{[1]}C_1\right]+\frac{k^3}{6}Z_z^{[3]}C_1+\frac{1}{6}\left(C_k\right)_zZ^{[3]}+\left(C_k\right)_zD_1\\&+\frac{1}{2}\left(C_k\right)_{z^2}\left[Z^{[1]}Z^{[2]}\right]+\left(C_k\right)_{z^2}Z^{[1]}C_1+\frac{1}{6}\left(C_k\right)_{z^3}\left[Z^{[1]}\right]^3\\&+\frac{1}{2}\left(D_k\right)_zZ^{[2]}+\left(D_k\right)_zC_1+\frac{1}{2}\left(D_k\right)_{z^2}\left[Z^{[1]}\right]^2+\left(E_k\right)_zZ^{[1]}\\&+F_k.\end{aligned}\tag{27.4}$$

From (3), we have

$$\begin{aligned}&\sum_{k=0}^{m}\alpha_k\left[\sum_{i=0}^{+\infty}\frac{k^i\tau^i}{i!}Z^{[i]}+\tau^2C_k(Z)+\tau^3D_k(Z)+\tau^4E_k(Z)+\tau^5F_k(Z)+O(\tau^6)\right]\\
=&\tau\sum_{k=0}^{m}\beta_k f\left(\sum_{i=0}^{+\infty}\frac{k^i\tau^i}{i!}Z^{[i]}+\tau^2C_k(Z)+\tau^3D_k(Z)+\tau^4E_k(Z)+O(\tau^5)\right)\\
=&\tau\sum_{k=0}^{m}\beta_k\left\{f\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]+f_z\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]*\left(\tau^2C_k+\tau^3D_k+\tau^4E_k\right)\right.\\
&\left.+\frac{1}{2}f_{z^2}\circ\left[\sum_{i=0}^{+\infty}\frac{\tau^i}{i!}Z^{[i]}\right]*\left(\tau^2C_k\right)^2\right\}+O(\tau^6) \qquad (28)\\
=&\sum_{l=0}^{+\infty}\sum_{k=0}^{m}\beta_k\frac{k^l\tau^{l+1}}{l!}Z^{[l+1]}+\tau^3\sum_{k=0}^{m}\beta_kZ_z^{[1]}C_k+\tau^4\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}D_k+kZ_{z^2}^{[1]}Z^{[1]}C_k\right\}\\
&+\tau^5\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}E_k+kZ_{z^2}^{[1]}Z^{[1]}D_k+\frac{k^2}{2}Z_{z^2}^{[1]}Z^{[2]}C_k+\frac{k^2}{2}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2C_k\right.\\
&\left.+\frac{1}{2}Z_{z^2}^{[1]}\left(C_k\right)^2\right\}+O(\tau^6),\end{aligned}$$

comparing the coefficients of τ^2, τ^3, τ^4 and τ^5 respectively on both sides of (28) we obtain

$$\sum_{k=0}^{m}\alpha_kC_k=\sum_{k=0}^{m}\left\{k\beta_k-\frac{k^2}{2!}\alpha_k\right\}Z^{[2]}; \qquad (29.1)$$

$$\sum_{k=0}^{m}\alpha_kD_k=\sum_{k=0}^{m}\left\{\frac{k^2}{2!}\beta_k-\frac{k^3}{3!}\alpha_k\right\}Z^{[3]}+\sum_{k=0}^{m}\beta_kZ_z^{[1]}C_k; \qquad (29.2)$$

$$\begin{aligned}\sum_{k=0}^{m}\alpha_kE_k=&\sum_{k=0}^{m}\left\{\frac{k^3}{3!}\beta_k-\frac{k^4}{4!}\alpha_k\right\}Z^{[4]} \qquad (29.3)\\
&+\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}D_k+kZ_{z^2}^{[1]}Z^{[1]}C_k\right\};\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{m}\alpha_kF_k=&\sum_{k=0}^{m}\left\{\frac{k^4}{4!}\beta_k-\frac{k^5}{5!}\alpha_k\right\}Z^{[5]}+\sum_{k=0}^{m}\beta_k\left\{Z_z^{[1]}E_k+kZ_{z^2}^{[1]}Z^{[1]}D_k\right. \qquad (29.4)\\
&\left.+\frac{k^2}{2}Z_{z^2}^{[1]}Z^{[2]}C_k+\frac{k^2}{2}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2C_k+\frac{1}{2}Z_{z^2}^{[1]}\left(C_k\right)^2\right\}.\end{aligned}$$

From relations (27.1) and (29.1) we deduce directly

$$C_k=kC_1\equiv kC, \qquad (30)$$

and (12.1). Substituting (30) into (27.2), we obtain

$$D_k=kD_1+\frac{k^2-k}{2}\omega\left(Z_z^{[1]}Z^{[s+1]}+Z^{[3]}\right), \qquad (31)$$

substituting (31) and (30) into (29.2), we obtain

$$\begin{aligned}\left(\sum_{k=0}^{m}k\alpha_k\right)D_1=&\sum_{k=0}^{m}\left\{\frac{k^2}{2!}\beta_k-\frac{k^3}{3!}\alpha_k-\frac{k^2-k}{2}\omega\alpha_k\right\}Z^{[3]}\\
&+\sum_{k=0}^{m}\left\{k\omega\beta_k-\frac{k^2-k}{2}\omega\alpha_k\right\}Z_z^{[1]}Z^{[2]}.\end{aligned}$$

Then we get (12.2), and

$$D_k = \left[\frac{k^2\omega}{2} + k\omega^2\right] Z_z^{[1]} Z^{[2]} + \left[\frac{k^2-k}{2}\omega + k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right] Z^{[3]}. \tag{32}$$

Substituting (30) and (32) into (27.3), we have

$$\begin{aligned} E_{k+1} =& E_1 + \frac{k^2\omega + 3k\omega^2 + k\omega}{2} Z_z^{[1]} Z_z^{[1]} Z^{[2]} \\ &+ \left[k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + k\omega^2 + \frac{k^2\omega}{2}\right] Z_z^{[1]} Z^{[3]} \\ &+ \frac{2k^2\omega + 3k\omega^2 + k\omega}{2} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &+ \left[k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{k^2\omega}{2}\right] Z^{[4]} + E_k, \end{aligned}$$

and then

$$\begin{aligned} E_k =& kE_1 + \left[\frac{k^2-k}{4}(3\omega^2 + \omega) + \frac{2k^3 - 3k^2 + k}{12}\omega\right] Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &+ \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{k^2-k}{2}\omega^2 + \frac{2k^3-3k^2+k}{12}\omega\right] Z_z^{[1]} Z^{[3]} \\ &+ \left[\frac{2k^3-3k^2+k}{6}\omega + \frac{k^2-k}{2}\left(\frac{3\omega^2}{2} + \omega\right)\right] Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &+ \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{2k^3-3k^2+k}{12}\omega\right] Z^{[4]} \end{aligned} \tag{33}$$

Substituting (33), (30) and (32) into (29.3), we obtain

$$\begin{aligned} &\left(\sum_{k=0}^{m} k\alpha_k\right) E_1 = \\ &\sum_{k=0}^{m}\left\{\frac{k^3}{3!}\beta_k - \frac{k^4}{4!}\alpha_k - \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{2k^3-3k^2+k}{12}\omega\right]\alpha_k\right\} Z^{[4]} \\ &+ \sum_{k=0}^{m}\left\{k^2\omega\beta_k - \frac{2k^3-3k^2+k}{6}\omega\alpha_k - \frac{k^2-k}{2}\left(\frac{3\omega^2}{2} + \omega\right)\alpha_k\right\} Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \\ &+ \sum_{k=0}^{m}\left\{\left[k\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + \frac{k^2-k}{2}\omega\right]\beta_k - \left[\frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right)\right.\right. \\ &\qquad \left.\left. + \frac{k^2-k}{2}\omega^2 + \frac{2k^3-3k^2+k}{12}\omega\right]\alpha_k\right\} Z_z^{[1]} Z^{[3]} \\ &+ \sum_{k=0}^{m}\left\{\left(k\omega^2 + \frac{k^2\omega}{2}\right)\beta_k - \left[\frac{k^2-k}{4}(3\omega^2 + \omega) + \frac{2k^3-3k^2+k}{12}\omega\right]\alpha_k\right\} Z_z^{[1]} Z_z^{[1]} Z^{[2]}, \end{aligned}$$

and we have (12.3), and

$$\begin{aligned} E_k =& \left[\frac{2k^3-3k^2+k}{12}\omega + \frac{k^2-k}{2}\left(\frac{\rho}{2} - \frac{\omega\delta}{2} + \frac{\omega}{2}\right) + k\left(\frac{\sigma}{3} - \frac{\eta\omega}{6} - \frac{\delta\omega}{4} + \frac{\omega}{6}\right.\right. \\ &\left.\left. - \frac{\delta\rho}{4} + \frac{\delta^2\omega}{4} + \frac{\rho}{4}\right)\right] Z^{[4]} \\ &+ \left[\frac{k^3\omega}{3} + \frac{3k^2\omega^2}{4} + k\rho\omega - \frac{3k\omega^2\delta}{4}\right] Z_{z^2}^{[1]} Z^{[1]} Z^{[2]} \end{aligned} \tag{34}$$

$$+\left[\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{2}\omega^2+\frac{2k^3-3k^2}{2}\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.$$
$$\left.+\frac{k\rho\omega}{2}-\frac{k\delta\omega^2}{2}\right]Z_z^{[1]}Z^{[3]}$$
$$+\left[\frac{k^3\omega}{6}+\frac{3k^2\omega^2}{4}+\frac{k\rho\omega}{2}+k\omega^3-\frac{k\omega^2\delta}{4}\right]Z_z^{[1]}Z_z^{[1]}Z^{[2]}.$$

Substituting (30), (32) and (34) into (27.4), we obtain

$$F_{k+1}=F_1+F_k$$
$$+\left\{\frac{k^2\omega^2}{2}+\frac{k^3}{6}\omega+\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right.$$
$$\left.+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k\omega\rho}{2}-\frac{k\delta\omega^2}{2}\right\}Z_z^{[1]}Z^{[4]}$$
$$+\left\{k\omega^3+\frac{3k^2}{4}\omega^2+\frac{2k^3-k}{12}\omega+2k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\rho\omega+\frac{k^2+k}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.$$
$$\left.-\frac{3k\omega^2\delta}{4}\right\}Z_z^{[1]}Z_z^{[1]}Z^{[3]}$$
$$+\left\{\frac{5k^2+2k}{4}\omega^2+\frac{2k^3+k^2}{4}\omega+2k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3}{2}k\rho\omega-\frac{5k\delta\omega^2}{4}\right.$$
$$\left.+\left(k^2+k\right)\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right\}Z_{z^2}^{[1]}Z^{[1]}Z^{[3]}$$
$$+\left\{3k\omega^3+\frac{6k^2+5k}{4}\omega^2+\frac{k^3+3k^2+k}{6}\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k\omega\rho}{2}-\frac{k\delta\omega^2}{4}\right\}$$
$$Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[2]}$$
$$+\left\{2k\omega^3+\frac{7k^2+5k}{4}\omega^2+\frac{2k^3+3k^2+2k}{6}\omega+\frac{3}{2}k\rho\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)-k\delta\omega^2\right\}$$
$$Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[2]}$$
$$+\left\{2k\omega^3+\frac{9k^2+6k}{4}\omega^2+\frac{2k^3+3k^2+2k}{4}\omega+2k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k\rho\omega}{2}-\frac{k\delta\omega^2}{4}\right\}$$
$$Z_{z^2}^{[1]}Z^{[1]}\left[Z_z^{[1]}Z^{[2]}\right]$$
$$+\left\{k\omega^3+\frac{7k^2+6k}{4}\omega^2+\frac{2k^3+3k^2+2k}{4}\omega+k\rho\omega+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)-\frac{3k\delta\omega^2}{4}\right\}$$
$$Z_{z^2}^{[1]}\left(Z^{[2]}\right)^2$$
$$+\left\{\frac{5k^2+4k}{4}\omega^2+\frac{2k^3+3k^2+2k}{4}\omega-\frac{3k\delta\omega^2}{4}+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\rho\omega\right\}$$
$$Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[2]}$$
$$+\left\{\frac{k^3}{6}\omega+\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right\}Z^{[5]},$$

and then

$$F_k=kF_1 \tag{35}$$
$$+\left\{\frac{2k^3-3k^2+k}{12}\omega^2+\frac{k^4-2k^3+k^2}{24}\omega+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.$$

$$
\begin{aligned}
&+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\\
&+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{4}\delta\omega^2\Bigg\}Z_z^{[1]}Z^{[4]}\\
+&\Bigg\{\frac{k^2-k}{2}\omega^3+\frac{2k^3-3k^2+k}{8}\omega^2+\frac{k^4-2k^3+k}{24}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\\
&+\frac{k^2-k}{2}\rho\omega+\frac{k^3-k}{6}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)-\frac{3\left(k^2-k\right)}{8}\delta\omega^2\Bigg\}Z_z^{[1]}Z_z^{[1]}Z^{[3]}\\
+&\Bigg\{\frac{10k^3-9k^2-k}{24}\omega^2+\frac{3k^4-4k^3+k}{24}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3\left(k^2-k\right)}{4}\rho\omega\\
&-\frac{5\left(k^2-k\right)}{8}\delta\omega^2+\frac{k^3-k}{3}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg\}Z_{z^2}^{[1]}Z^{[1]}Z^{[3]}\\
+&\Bigg\{\frac{3\left(k^2-k\right)}{2}\omega^3+\frac{4k^3-k^2-3k}{8}\omega^2+\frac{k^4+2k^3-3k^2}{24}\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\\
&+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{8}\delta\omega^2\Bigg\}Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[2]}\\
+&\Bigg\{\left(k^2-k\right)\omega^3+\frac{7k^3-3k^2-4k}{12}\omega^2+\frac{k^4-k}{12}\omega+\frac{3\left(k^2-k\right)}{4}\rho\omega-\frac{k^2-k}{2}\delta\omega^2\\
&+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg\}Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[2]}\\
+&\Bigg\{\left(k^2-k\right)\omega^3+\frac{6k^3-3k^2-3k}{8}\omega^2+\frac{k^4-k}{8}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\\
&+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{8}\delta\omega^2\Bigg\}Z_{z^2}^{[1]}Z^{[1]}\left[Z_z^{[1]}Z^{[2]}\right]\\
+&\Bigg\{\frac{k^2-k}{2}\omega^3+\frac{14k^3-3k^2-11k}{24}\omega^2+\frac{k^4-k}{8}\omega+\frac{k^2-k}{2}\rho\omega-\frac{3\left(k^2-k\right)}{8}\delta\omega^2\\
&+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg\}Z_{z^2}^{[1]}\left(Z^{[2]}\right)^2\\
+&\Bigg\{\frac{10k^3-3k^2-7k}{24}\omega^2+\frac{k^4-k}{8}\omega-\frac{3\left(k^2-k\right)}{8}\delta\omega^2+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\\
&+\frac{k^2-k}{2}\rho\omega\Bigg\}Z_{z^3}^{[1]}\left(Z^{[1]}\right)^2Z^{[2]}\\
+&\Bigg\{\frac{k^4-2k^3+k^2}{24}\omega+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\\
&+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg\}Z^{[5]}.
\end{aligned}
$$

Substituting (30), (32), (34) and (35) into (29.4) we have

$$\left(\sum_{k=0}^{m}k\alpha_k\right)F_1=$$

$$\sum_{k=0}^{m}\left\{\frac{k^4}{4!}\beta_k-\frac{k^5}{5!}\alpha_k-\frac{k^4-2k^3+k^2}{24}\alpha_k\omega-\frac{2k^3-3k^2+k}{12}\alpha_k\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.$$
$$\left.-\frac{k^2-k}{2}\alpha_k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right\}Z^{[5]}$$
$$+\sum_{k=0}^{m}\left\{\left[\frac{2k^3-3k^2+k}{12}\omega+\frac{k^2-k}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}\right.\right.\right.$$
$$\left.\left.+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)\right]\beta_k-\left[\frac{2k^3-3k^2+k}{12}\omega^2+\frac{2k^3-3k^2+k}{12}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.$$
$$+\frac{k^4-2k^3+k^2}{24}\omega+\frac{k^2-k}{2}\left(\frac{\sigma}{3}-\frac{\eta\omega}{6}-\frac{\delta\omega}{4}+\frac{\omega}{6}-\frac{\delta\rho}{4}+\frac{\delta^2\omega}{4}+\frac{\rho}{4}\right)$$
$$\left.\left.+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{4}\delta\omega^2\right]\alpha_k\right\}Z_z^{[1]}Z^{[4]}$$
$$+\sum_{k=0}^{m}\left\{\left[\frac{k^3\omega}{6}+\frac{3k^2\omega^2}{4}+\frac{k\rho\omega}{2}+k\omega^3-\frac{k\delta\omega^2}{4}\right]\beta_k-\left[\frac{3\left(k^2-k\right)}{2}\omega^3\right.\right.$$
$$+\frac{4k^3-k^2-3k}{8}\omega^2+\frac{k^4+2k^3-3k^2}{24}\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$
$$\left.\left.+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{8}\delta\omega^2\right]\alpha_k\right\}Z_z^{[1]}Z_z^{[1]}Z_z^{[1]}Z^{[2]}$$
$$+\sum_{k=0}^{m}\left\{\left[\frac{k^3\omega}{3}+\frac{3k^2}{4}\omega^2+k\rho\omega-\frac{3k}{4}\delta\omega^2\right]\beta_k-\left[\left(k^2-k\right)\omega^3+\frac{7k^3-3k^2-4k}{12}\omega^2\right.\right.$$
$$\left.\left.+\frac{k^4-k}{12}\omega+\frac{3\left(k^2-k\right)}{4}\rho\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)-\frac{k^2-k}{2}\delta\omega^2\right]\alpha_k\right\}$$
$$Z_z^{[1]}Z_{z^2}^{[1]}Z^{[1]}Z^{[2]}$$
$$+\sum_{k=0}^{m}\left\{\left[\frac{k^2-k}{2}\omega^2+\frac{2k^3-3k^2}{12}\omega+\frac{k^2}{2}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+k\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right.\right.$$
$$\left.+\frac{k\rho\omega}{2}-\frac{k\delta\omega^2}{2}\right]\beta_k-\left[\frac{k^2-k}{2}\omega^3+\frac{2k^3-3k^2+k}{8}\omega^2+\frac{k^4-2k^3+k}{24}\omega\right.$$
$$+\left(k^2-k\right)\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{2}\rho\omega+\frac{k^3-k}{6}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)$$
$$\left.\left.-\frac{3\left(k^2-k\right)}{8}\delta\omega^2\right]\alpha_k\right\}Z_z^{[1]}Z_z^{[1]}Z^{[3]}$$
$$+\sum_{k=0}^{m}\left\{\left[k^2\omega^2+\frac{k^2}{2}\omega+\frac{k^3-k^2}{2}\omega\right]\beta_k-\left[\left(k^2-k\right)\omega^3+\frac{6k^3-3k^2-3k}{8}\omega^2\right.\right.$$
$$\left.\left.+\frac{k^4-k}{8}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{4}\rho\omega-\frac{k^2-k}{8}\delta\omega^2\right]\alpha_k\right\}$$
$$Z_{z^2}^{[1]}Z^{[1]}\left(Z_z^{[1]}Z^{[2]}\right)$$
$$+\sum_{k=0}^{m}\left\{\left[\frac{k^3-k^2}{2}\omega+k^2\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\right]\beta_k-\left[\frac{10k^3-9k^2-k}{24}\omega^2\right.\right.$$

$$
\begin{aligned}
&+\frac{3k^4-4k^3+k}{24}\omega+\left(k^2-k\right)\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3\left(k^2-k\right)}{4}\rho\omega \\
&-\frac{5\left(k^2-k\right)}{8}\delta\omega^2+\frac{k^3-k}{3}\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)\Bigg]\alpha_k\Bigg\}Z^{[1]}_{z^2}Z^{[1]}Z^{[3]} \\
+\sum_{k=0}^{m}&\left\{\left[\frac{k^3}{2}\omega+\frac{k^2}{2}\omega^2\right]\beta_k-\left[\frac{k^2-k}{2}\omega^3+\frac{14k^3-3k^2-11k}{24}\omega^2+\frac{k^4-k}{8}\omega\right.\right. \\
&\left.\left.+\frac{k^2-k}{2}\rho\omega+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{3\left(k^2-k\right)}{8}\delta\omega^2\right]\alpha_k\right\}Z^{[1]}_{z^2}\left(Z^{[2]}\right)^2 \\
+\sum_{k=0}^{m}&\left\{\frac{k^3}{2}\beta_k\omega-\left[\frac{k^4-k}{8}\omega+\frac{10k^3-3k^2-7k}{24}\omega^2-\frac{3\left(k^2-k\right)}{8}\delta\omega^2\right.\right. \\
&\left.\left.+\frac{k^2-k}{2}\omega\left(\frac{\rho}{2}-\frac{\omega\delta}{2}+\frac{\omega}{2}\right)+\frac{k^2-k}{2}\rho\omega\right]\alpha_k\right\}Z^{[1]}_{z^3}\left(Z^{[1]}\right)^2Z^{[2]},
\end{aligned}
$$

and we obtain (12.4).

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