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ON THE OPTIMIZATION OF EXTRAPOLATION METHODS FOR SINGULAR LINEAR SYSTEMS*

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Abstract

We discuss semiconvergence of the extrapolated iterative methods for solving singular linear systems. We obtain the upper bounds and the optimum convergence factor of the extrapolation method as well as its associated optimum extrapolation parameter. Numerical examples are given to illustrate the theoretical results.

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1. Introduction

Consider a system of linear equations

$$Ax = b, \tag{1.1}$$

where $A \in \mathcal{C}^{n\times n}$ is singular, b, $x \in \mathcal{C}^n$ with b known and x unknown. We assume that the linear system (1.1) is solvable, i.e., it has at least one solution. In order to solve the linear system (1.1) with iterative methods, the coefficient matrix A is split into

$$A = M - N, \tag{1.2}$$

where M is nonsingular. Then a linear stationary iterative method for solving (1.1) can be described as follows.

$$x^{k+1} = Tx^k + M^{-1}b, \quad k = 0, 1, 2, \cdots, \tag{1.3}$$

where $T = M^{-1}N$ is the iteration matrix.

The iterative method (1.3) is called semiconvergent if for every x^0 the sequence defined by (1.3) converges to a solution of (1.1). It is well known that the iterative method (1.3) is semiconvergent if and only if the pseudo-spectral radius

$$\vartheta(T) \equiv \max\{|\mu|, \ \ \mu \in \sigma(T)\backslash\{1\}\}$$

is less than 1 and the elementary divisors associated with $\mu = 1 \in \sigma(T)$ are linear, i.e.,

$$\text{index}(I - T) = 1,$$

where $\sigma(T)$ denotes the spectrum of T and $\text{index}(B)$ denotes the index of the matrix B, i.e., the smallest nonnegative integer k such that $\text{rank}(B^{k+1}) = \text{rank}(B^k)$ ($\text{rank}(B)$ means the rank of

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B). In this case, the splitting (1.2) is also called semiconvergent and T is called a semiconvergent matrix. The associated convergence factor of T and the iterative method (1.3) is $\vartheta(T)$.

The semiconvergence of splitting (1.2) has been investigated by many papers (cf. [4, 11] and the references therein).

Moreover, some new results have been obtained by using matrix splittings and iterative methods to solve the linear complementarity problem (cf. [1, 2, 3, 17]).

For $\omega \in \mathcal{C}$ the extrapolation method of (1.3) can be defined by

$$x^{k+1} = T_\omega x^k + \omega M^{-1}b, \quad k = 0, 1, 2, \cdots, \tag{1.4}$$

where

$$T_\omega = (1-\omega)I + \omega T$$

is the iteration matrix and ω is called the extrapolation parameter (cf. [8]). Clearly, if $\omega = 0$ then $T_0 = I$, which leads to a trivial case. Thus, we assume that $\omega \neq 0$.

Now, we assume that

$$A = D - Q, \tag{1.5}$$

where $D = diag(a_{11}, \cdots, a_{nn})$ is nonsingular. Associated with the splitting (1.5), the Jacobi iteration matrix J can be expressed as

$$J = D^{-1}Q.$$

The extrapolated Jacobi method is also called JOR method (cf. [16]) with the iteration matrix J_ω, namely,

$$J_\omega = (1-\omega)I + \omega J.$$

The method (1.4) is consistent with (1.1) and is used to accelerate the convergence of the method (1.3). The extrapolation method for solving the singular systems has been discussed in many papers (cf. [7, 10, 12]).

Now, an interesting, important and also complicated problem is the determination of the optimum value ω_{opt} for ω, which minimizes $\vartheta(T_\omega)$. This problem has been discussed extensively by some researchers. It was treated by the geometrical method in [7].

In this paper, the determination of the sharp analytical upper bounds for $\min_\omega \vartheta(T_\omega)$ is achieved by an algebraic approach, which generalize the results in [15] to the singular case. On the other hand, these bounds are obtained for the good analytical values for the extrapolation parameter which coincide with the optimum ones under some additional conditions. In the theory presented no knowledge of the eigenvalues of T is required. Finally, some applications and numerical examples are given which support the theory developed. The paper is organized as follows. After establishing the bounds for $\min_\omega \vartheta(T_\omega)$ in Section 2, we extend the extrapolation theorem given in [6, 15] to the singular system and improve the corresponding results in [10, 12]. In Section 3, an application and the numerical results are given to illustrate the results presented in Sections 2.

2. Determination of Upper Bounds and Optimum Values

Lemma 2.1. ([10]) *For the singular linear system* (1.1) *the following results hold:*

(i) *for* $\lambda \in \sigma(T_\omega)$ *and* $\mu \in \sigma(T)$, *it holds that*

$$\lambda = 1 - \omega + \omega\mu; \tag{2.1}$$

(ii) $1 \in \sigma(T)$, $1 \in \sigma(T_\omega)$ *and*

$$\text{index}(I - T_\omega) = \text{index}(I - T);$$

(iii) *for* μ *and* λ *satisfying* (2.1) $\mu \in \sigma(T)\backslash\{1\}$ *if and only if* $\lambda \in \sigma(T_\omega)\backslash\{1\}$*;*
(iv) *if the extrapolated method* (1.4) *is semiconvergent, then* $\text{index}(I - T) = 1$.

Moreover, we have proved in [12] the following semiconvergence theorem.

Theorem 2.1. *The extrapolation method* (1.4) *is semiconvergent if and only if* $\text{index}(I-T) = 1$ *and one of the following conditions is satisfied.*
(i) $\text{Re}(\mu) < 1$, *for all* $\mu \in \sigma(T)\backslash\{1\}$, *and*

$$0 < \omega < \min_{\mu \in \sigma(T)\backslash\{1\}} \frac{2[1 - \text{Re}(\mu)]}{1 - 2\text{Re}(\mu) + |\mu|^2};$$

(ii) $\text{Re}(\mu) > 1$, *for all* $\mu \in \sigma(T)\backslash\{1\}$, *and*

$$0 > \omega > \max_{\mu \in \sigma(T)\backslash\{1\}} \frac{2[1 - \text{Re}(\mu)]}{1 - 2\text{Re}(\mu) + |\mu|^2}.$$

Now, we give some notations which will be used in the sequel. Let

$$S \equiv \sigma(T)\backslash\{1\} = \{\mu_j = x_j + iy_j,\ j = 1, \cdots, t\}$$

be a set consisting of the eigenvalues of T, excluding 1, and the real and the imaginary parts satisfy

$$x_m \le x_j \le x_M, \quad y_m \le |y_j| \le y_M, \quad j = 1, \cdots, t.$$

In addition, we denote

$$\begin{aligned} &\phi = (x_M - x_m)(1 - x_M), \quad \psi = 2y_M^2, \quad \widetilde{\phi} = (x_M - x_m)(x_m - 1), \\ &\omega_1 = \frac{1 - x_M}{(x_M - 1)^2 + y_M^2}, \quad \omega_0 = \frac{2}{2 - (x_m + x_M)}, \quad \widetilde{\omega}_1 = \frac{x_m - 1}{(x_m - 1)^2 + y_M^2}. \end{aligned} \tag{2.2}$$

Then, we have the following main result.

Theorem 2.2. *Assume that* $\text{index}(I - T) = 1$. *If* $x_M < 1$, *then for the extrapolated method* (1.4)*, it holds that*

$$\min_\omega \vartheta(T_\omega) \le \left\{ \begin{array}{ll} \dfrac{y_M}{[(x_M - 1)^2 + y_M^2]^{\frac{1}{2}}}, & \textit{if} \quad \phi \le \psi \\ \dfrac{[(x_M - x_m)^2 + 4y_M^2]^{\frac{1}{2}}}{2 - x_m - x_M}, & \textit{if} \quad \phi \ge \psi \end{array} \right\} < 1.$$

Moreover, if $x_m + iy_M$, $x_M + iy_M \in S$, *then we have*

$$\omega_{opt} = \left\{ \begin{array}{ll} \omega_1, & \textit{if} \quad \phi \le \psi, \\ \omega_0, & \textit{if} \quad \phi \ge \psi, \end{array} \right.$$

and

$$\min_{\omega} \vartheta(T_\omega) = \vartheta(T_{\omega_{opt}}) = \begin{cases} \dfrac{y_M}{[(x_M-1)^2+y_M^2]^{\frac{1}{2}}}, & \text{if} \quad \omega_{opt}=\omega_1, \\ \dfrac{[(x_M-x_m)^2+4y_M^2]^{\frac{1}{2}}}{2-x_m-x_M}, & \text{if} \quad \omega_{opt}=\omega_0, \end{cases}$$

which satisfies

$$\vartheta(T_{\omega_{opt}}) \leq \vartheta(T) = \max\{(x_m^2+y_M^2)^{\frac{1}{2}}, (x_M^2+y_M^2)^{\frac{1}{2}}\}$$

with equality holding if and only if one of the following conditions holds:
(i) $x_m + x_M = 0$ *and* $x_m^2 + y_M^2 - x_M \leq 0$, *when* $\omega_{opt} = \omega_0$,
(ii) $x_m + x_M \geq 0$ *and* $x_M^2 + y_M^2 - x_M = 0$, *when* $\omega_{opt} = \omega_1$.

Proof. According to Lemma 2.1, we have

$$\text{index}(I - T_\omega) = \text{index}(I - T) = 1$$

and

$$\sigma(T_\omega)\backslash\{1\} = \{1 - \omega + \omega\mu_j \mid \mu_j \in S, j = 1, \cdots, t\},$$

which implies that

$$\begin{aligned} \vartheta^2(T_\omega) &= \max_j\{\mid 1-\omega+\omega x_j \mid^2 + \omega^2 \mid y_j \mid^2\} \\ &\leq (\max_j \mid 1-\omega+\omega x_j \mid)^2 + \omega^2 y_M^2. \end{aligned} \tag{2.3}$$

Theorem 2.1 shows that the necessity for semiconvergence is $\omega > 0$.

It can be shown that

$$\max_j \mid 1-\omega+\omega x_j \mid = \begin{cases} \omega x_M + 1 - \omega, & \text{if} \quad 0 < \omega \leq \omega_0, \\ \omega - 1 - \omega x_m, & \text{if} \quad \omega \geq \omega_0, \end{cases}$$

where

$$\omega_0 = \frac{2}{2-(x_m+x_M)}.$$

Therefore, from (2.3) we have

$$\vartheta^2(T_\omega) \leq \begin{cases} (\omega x_M + 1 - \omega)^2 + \omega^2 y_M^2 = g_1(\omega), & \text{if} \quad 0 < \omega \leq \omega_0, \\ (\omega - 1 - \omega x_m)^2 + \omega^2 y_M^2 = g_2(\omega), & \text{if} \quad \omega \geq \omega_0. \end{cases} \tag{2.4}$$

As a result, it holds that

$$\min_\omega \vartheta^2(T_\omega) \leq \begin{cases} \min\limits_\omega g_1(\omega), & \text{if} \quad 0 < \omega \leq \omega_0, \\ \min\limits_\omega g_2(\omega), & \text{if} \quad \omega \geq \omega_0. \end{cases}$$

Noting that $g_1'(\omega) \geq 0$ if and only if $\omega \geq \omega_1$, $g_2'(\omega) \geq 0$ if and only if $\omega \geq \omega_2$, and $\omega_2 \leq \omega_0$, where

$$\begin{cases} \omega_1 = \dfrac{1-x_M}{(x_M-1)^2+y_M^2}, \\ \omega_2 = \dfrac{1-x_m}{(x_m-1)^2+y_M^2}, \end{cases}$$

we can conclude that

$$\begin{cases} \min\limits_{0<\omega<\omega_0} g_1(\omega) = g_1(\omega_1), & \text{if} \quad \omega_1 \le \omega_0, \\ \min\limits_{0<\omega<\omega_0} g_1(\omega) = g_1(\omega_0), & \text{if} \quad \omega_1 \ge \omega_0, \\ \min\limits_{\omega\ge\omega_0} g_2(\omega) = g_2(\omega_0) = g_1(\omega_0). \end{cases}$$

It is easily seen that $\omega_1 \le \omega_0$ is equivalent to $\phi \le \psi$, where ϕ and ψ are given in (2.2). Then we have

$$\min_{\omega} \vartheta^2(T_\omega) \le \begin{cases} g_1(\omega_1), & \text{if} \quad \phi \le \psi, \\ g_1(\omega_0), & \text{if} \quad \phi \ge \psi, \end{cases}$$

which is equivalent to

$$\min_{\omega} \vartheta(T_\omega) \le \begin{cases} \dfrac{y_M}{\left[(x_M-1)^2 + y_M^2\right]^{\frac{1}{2}}}, & \text{if} \quad \phi \le \psi, \\ \dfrac{\left[(x_M-x_m)^2 + 4y_M^2\right]^{\frac{1}{2}}}{2-x_m-x_M}, & \text{if} \quad \phi \ge \psi. \end{cases}$$

Under the conditions that $x_m + iy_M$ and $x_M + iy_M$ are eigenvalues of T, we can obtain by combining (2.4) and the above analysis that

$$\omega_{opt} = \begin{cases} \omega_1, & \text{if} \quad \phi \le \psi, \\ \omega_0, & \text{if} \quad \phi \ge \psi, \end{cases}$$

and

$$\min_{\omega} \vartheta(T_\omega) = \vartheta(T_{\omega_{opt}}) = \begin{cases} \dfrac{y_M}{\left[(x_M-1)^2 + y_M^2\right]^{\frac{1}{2}}}, & \text{if} \quad \omega_{opt} = \omega_1, \\ \dfrac{\left[(x_M-x_m)^2 + 4y_M^2\right]^{\frac{1}{2}}}{2-x_m-x_M}, & \text{if} \quad \omega_{opt} = \omega_0. \end{cases}$$

If $\phi \ge \psi$, we have $2y_M^2 \le (x_M - x_m)(1 - x_M)$. Since

$$(x_M - x_m)(1 - x_M) < 2(1 - x_m)(1 - x_M),$$

we obtain

$$y_M^2 < (1 - x_m)(1 - x_M),$$

which implies

$$\frac{\left[(x_M-x_m)^2 + 4y_M^2\right]^{\frac{1}{2}}}{2-x_m-x_M} < 1,$$

and therefore $\vartheta(T_{\omega_{opt}}) < 1$. It can be shown that

$$\omega_0 = 1, \quad \phi \ge \psi; \qquad \omega_1 = 1, \quad \phi \le \psi,$$

are equivalent to

$$\begin{aligned} x_m + x_M = 0, \quad & x_m^2 + y_M^2 - x_M \le 0, \quad \omega_{opt} = \omega_0, \\ x_m + x_M \ge 0, \quad & x_M^2 + y_M^2 - x_M = 0, \quad \omega_{opt} = \omega_1, \end{aligned}$$

respectively. The proof is then complete. $\square$

Similar to Theorem 2.2, we have the following result.

Theorem 2.3. *Assume that* $\mathrm{index}(I-T)=1$. *If* $x_m>1$, *then for the extrapolated method* (1.4) *it holds that*

$$\min_{\omega}\vartheta(T_\omega)\le\left\{\begin{array}{ll}\dfrac{y_M}{[(1-x_m)^2+y_M^2]^{\frac{1}{2}}}, & \text{if}\quad \widetilde{\phi}\le\psi\\ \dfrac{[(x_M-x_m)^2+4y_M^2]^{\frac{1}{2}}}{x_M+x_m-2}, & \text{if}\quad \widetilde{\phi}\ge\psi\end{array}\right\}<1.$$

Moreover, if x_m+iy_M, $x_M+iy_M\in S$, *then we have*

$$\omega_{opt}=\left\{\begin{array}{ll}-\widetilde{\omega}_1, & \text{if}\quad \widetilde{\phi}\le\psi,\\ \omega_0, & \text{if}\quad \widetilde{\phi}\ge\psi\end{array}\right.$$

and

$$\min_{\omega}\vartheta(T_\omega)=\vartheta(T_{\omega_{opt}})=\left\{\begin{array}{ll}\dfrac{y_M}{[(1-x_m)^2+y_M^2]^{\frac{1}{2}}}, & \text{if}\quad \omega_{opt}=-\widetilde{\omega}_1,\\ \dfrac{[(x_M-x_m)^2+4y_M^2]^{\frac{1}{2}}}{x_M+x_m-2}, & \text{if}\quad \omega_{opt}=\omega_0,\end{array}\right.$$

which satisfies

$$\vartheta(T_{\omega_{opt}})\le\vartheta(T)=\max\{[(2-x_M)^2+y_M^2]^{\frac{1}{2}},\ [(2-x_m)^2+y_M^2]^{\frac{1}{2}}\}$$

with equality holding if and only if one of the following conditions holds:
(i) $x_m+x_M=4$ *and* $(2-x_M)^2+y_M^2-2+x_m\le 0$, *when* $\omega_{opt}=\omega_0$,
(ii) $x_m+x_M<4$ *and* $(2-x_m)^2+y_M^2-2+x_m=0$, *when* $\omega_{opt}=-\widetilde{\omega}_1$.

Proof. According to Lemma 2.1, we have

$$\mathrm{index}(I-T_\omega)=\mathrm{index}(I-T)=1$$

and

$$\sigma(T_\omega)\backslash\{1\}=\{1-\omega+\omega\mu_j\mid \mu_j\in S, j=1,\cdots,t\},$$

which implies that

$$\begin{aligned}\vartheta^2(T_\omega)&=\max_j\{|\,1-\omega+\omega x_j\,|^2+\omega^2\,|\,y_j\,|^2\}\\&\le\max_j|\,1-\omega+\omega x_j\,|^2+\omega^2y_M^2.\end{aligned}$$

Theorem 2.1 shows that under the condition $x_m>1$ the necessity for the semiconvergence is $\omega<0$. Now, let $\widetilde{\omega}=-\omega$, $\widetilde{x}_j=2-x_j$ and $\widetilde{y}_j=-y_j$. Then

$$\begin{aligned}\vartheta^2(T_\omega)&=\max_j\{|\,1-\widetilde{\omega}+\widetilde{\omega}\widetilde{x}_j\,|^2+\widetilde{\omega}^2|\,\widetilde{y}_j\,|^2\}\\&\le\max_j|\,1-\widetilde{\omega}+\widetilde{\omega}\widetilde{x}_j\,|^2+\widetilde{\omega}^2y_M^2,\end{aligned}$$

where

$$\left\{\begin{array}{l}2-x_M\le\widetilde{x}_j\le 2-x_m<1,\\ \max\widetilde{y}_j=\max y_j=y_M,\\ \widetilde{\omega}>0.\end{array}\right.$$

From Theorem 2.2 and its proof, we get the results immediately. This completes the proof. □

From the above discussion, we know that if the real part of any eigenvalue of the iteration matrix is smaller than 1 or the real part of any eigenvalue of the iteration matrix is larger than 1, then the extrapolated method can be applied directly.

Under some conditions, we have a more general conclusion as follows.

Theorem 2.4. *Assume that* $\operatorname{index}(I-T)=1$, *and* $x_j \neq 1,\ j=1,\cdots,t$. *If*

$$y_M < \min_j \mid 1-x_j \mid,\ j=1,\cdots,t$$

or

$$y_m > \max_j \mid 1-x_j \mid,\ j=1,\cdots,t,$$

then the iteration

$$x^{(k+1)} = T(2I-T)x^{(k)} + (I-T)M^{-1}b, \quad k=0,1,2,\cdots \tag{2.5}$$

is consistent with (1.1), *and the extrapolated method can be applied directly.*

Proof. There exists a nonsingular matrix P such that

$$T = P^{-1}\begin{pmatrix} I & 0 \\ 0 & K \end{pmatrix} P,$$

where $I-K$ is nonsingular. Then we can prove that

$$(I-T)x = M^{-1}b \tag{2.6}$$

is equivalent to

$$(I-T)(I-T)x = (I-T)M^{-1}b. \tag{2.7}$$

In fact, if we denote

$$PM^{-1}b = \widetilde{b} = \begin{pmatrix} \widetilde{b}_1 \\ \widetilde{b}_2 \end{pmatrix}, \quad Px = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

then (2.6) and (2.7) are respectively equivalent to

$$\begin{pmatrix} 0 & 0 \\ 0 & I-K \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \widetilde{b}_1 \\ \widetilde{b}_2 \end{pmatrix} \tag{2.8}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & (I-K)^2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I-K \end{pmatrix}\begin{pmatrix} \widetilde{b}_1 \\ \widetilde{b}_2 \end{pmatrix}. \tag{2.9}$$

Since (1.1) is solvable, we have $\widetilde{b}_1 = 0$. Now, it is easy to see that (2.8) and (2.9) have the same set of solutions. On the other hand,

$$\operatorname{index}(I-T(2I-T)) = 1,$$

and the real part of any of $\sigma(T(2I-T))\backslash\{1\}$ is

$$x_j(2-x_j)+y_j^2, \quad j=1,\cdots,t,$$

which is smaller than 1 whenever

$$y_M < \min_j \mid 1-x_j \mid, \quad j=1,\cdots,t,$$

or larger than 1 whenever

$$y_m > \max_j \mid 1-x_j \mid, \quad j=1,\cdots,t.$$

According to Theorems 2.2 and 2.3, the proof is then complete. $\square$

If all the eigenvalues of T are real, i.e., $\sigma(T)\in\mathcal{R}$, then we have the following result.

Corollary 2.1. *Assume that* $\sigma(T)\in\mathcal{R}$, $\operatorname{index}(I-T)=1$.

(i) *If either* $x_M<1$ *or* $x_m>1$, *then for the extrapolation* (1.4), *it holds that*

$$\omega_{opt}=\frac{2}{2-x_M-x_m}, \quad \vartheta(T_{\omega_{opt}})=\frac{x_M-x_m}{|2-x_M-x_m|};$$

(ii) *If* $x_m<1$ *and* $x_M>1$, *then the extrapolated method can be applied to* (2.5) *directly and*

$$\omega_{opt}=\frac{2}{2-\bar{x}_M-\bar{x}_m}, \quad \vartheta(\bar{T}_{\omega_{opt}})=\frac{\bar{x}_M-\bar{x}_m}{|2-\bar{x}_M-\bar{x}_m|},$$

where

$$\begin{aligned}
&\bar{T}=T(2I-T),\\
&\bar{x}_m=\min\{2x_m-x_m^2,\, 2x_M-x_M^2\},\\
&\bar{x}_M=\max\{2x_j-x_j^2\}.
\end{aligned}$$

Proof. For the case (i), noting that

$$y_M=0<\min_j \mid 1-x_j \mid$$

and according to Theorems 2.1-2.4, the results are immediate.

For the case (ii), let $\bar{x}_j=2x_j-x_j^2$. Then $\bar{x}_j$, $j=1,\cdots,n$, are the eigenvalues of the matrix $T(2I-T)$ and $\bar{x}_j=1$ if and only if $x_j=1$. It is easy to check that $\bar{x}_j\le 1$ and

$$\bar{x}_m=\min\{2x_m-x_m^2, \quad 2x_M-x_M^2\}.$$

The rest of the proof is obvious. $\square$

3. An Application

Definition 3.1. ([16]) *A matrix* $A=(a_{ij})$ *of order* n *has Property A if there exist two disjoint subsets* S_1 *and* S_2 *of* W, *the set of the first* n *positive integers, such that* $S_1\cup S_2=W$ *and such that if* $i\neq j$ *and if either* $a_{ij}\neq 0$ *or* $a_{ji}\neq 0$, *then* $i\in S_1$ *and* $j\in S_2$ *or else* $i\in S_2$ *and* $j\in S_1$.

If a matrix has Property A with non-vanishing diagonal elements, by [16] it follows that the eigenvalues of Jacobi matrix occur in $\pm$ pairs. So we have the following theorem, which is an application of Theorem 2.2.

Theorem 3.1. *Assume A in* (1.1) *has non-vanishing diagonal elements, and let* (1.3) *be the Jacobi method. We assume that* $\mathrm{index}(I-J)=1$ *and let*

$$\widetilde{S} \equiv \{\mu_j = x_j + iy_j \in \sigma(J)\backslash\{1,-1\}, j = 1, \cdots, m-1\}.$$

Suppose that $\mu = \gamma + i\delta \in \widetilde{S}$, where $0 \le \gamma < 1, \delta \ge 0$, and all other eigenvalues of J lie in the rectangle

$$\mid x_j \mid \le \gamma, \quad \mid y_j \mid \le \delta.$$

If $\mid \mu \mid^2 \ge 1 - \delta^2$, then the optimum JOR method is always better than the Jacobi method, that is,

$$\vartheta(J_{\omega_{opt}}) < \min\{\vartheta(J), \ 1\},$$

where

$$\omega_{opt} = \frac{1-\gamma}{(\gamma-1)^2 + \delta^2} < 1.$$

Proof. Since the eigenvalues of J occur in plus minus pairs, we have $x_m = -1$, $x_M = \gamma$ and $y_M = \delta$. By Theorem 2.2, the result is true. $\square$

Example 3.1. *Consider the homogeneous system of equations*

$$(I-P)v = 0, \tag{3.1}$$

which arises from the stationary probability distribution of a Markov chain. Here v is the stationary distribution vector associated with the chain and P is a $n \times n$ transition matrix with

$$P = D_n \equiv \begin{pmatrix} 0 & 1 & & & & \\ q & 0 & p & & & \\ & q & 0 & p & & \\ & & \ddots & \ddots & \ddots & \\ & & & q & 0 & p \\ & & & & 1 & 0 \end{pmatrix}, \tag{3.2}$$

where $0 < p < 1$ and $q = 1 - p$. This chain is called a random walk and it can be used to model various physical situations.

Iterative methods for solving Markov chains are investigated by many authors. As usual, we split $I - P$ into

$$I - P = I - L - U,$$

where L and U are the strictly lower and strictly upper triangular parts of the matrix P respectively. Then, we obtain the Jacobi iteration matrix $J = P$ and the Gauss-Seidel iteration matrix $G = (I - L)^{-1}U$. The corresponding Jacobi and Gauss-Seidel extrapolated iterative methods for solving (3.1) are

$$v^{k+1} = [(1-\omega)I + \omega J]v^k, \quad k = 0, 1, 2, \cdots \tag{3.3}$$

and

$$v^{k+1} = [(1-\omega)I + \omega G]v^k, \quad k = 0, 1, 2, \cdots, \tag{3.4}$$

respectively.

For clarity, we first give the following results.

Lemma 3.1. ([5]) *Let $A \in Z^{n\times n}$, i.e., the off-diagonal elements of A are not positive. If A has all positive diagonal entries, then* $\text{index}(I - J) \le 1$ *if and only if* $\text{index}(I - G) \le 1$.

Lemma 3.2. *Let $n \times n$ matrix P be given by* (3.2). *Then the following statements are true:*

(i) *The eigenvalues of the matrix J in* (3.3) *are*

$$\lambda_1 = 1, \quad \lambda_s = 2\sqrt{pq}\cos(\frac{s\pi}{n-1}), \quad s = 1, \cdots, n-2, \quad \lambda_{n-1} = -1 \tag{3.5}$$

and $\text{index}(I - J) = 1$.

(ii) *If λ is an eigenvalue of the matrix G in* (3.4)*, then there exists an eigenvalue μ of the matrix J in* (3.3) *such that*

$$\lambda^2 = \lambda\mu^2 \tag{3.6}$$

and $\text{index}(I - G) = 1$.

Proof. For the matrix D_n defined by (3.2), by induction, we have

$$\det(P - \lambda I_n) = \lambda^2 \det(I - \lambda D_{n-2}) + \det(I - \lambda D_{n-3}) + pq\det(I - \lambda D_{n-4})$$

and

$$\det(I - \lambda D_{n-2}) = -\det(I - \lambda D_{n-3}) - pq\det(I - \lambda D_{n-4}).$$

Thus

$$\det(P - \lambda I_n) = (\lambda^2 - 1)\det(I - \lambda D_{n-2}).$$

Here $\det(\cdot)$ denotes the determination of the corresponding matrix.

As shown in [9, 13], the eigenvalues of the matrix D_{n-2} are

$$\lambda_s = 2\sqrt{pq}\cos(\frac{s\pi}{n-1}), \quad s = 1, 2, \cdots, n-2,$$

As a result, we obtain (3.5).

Note that $I - J$ is a tri-diagonal matrix with non-vanishing diagonal elements. By [16, Theorem 5-2.2], the relation (3.6) holds.

Since J is an irreducible nonnegative matrix and $Je = e$, where e is a column vector with unit entries, according to Perron-Frobenius Theorem in [14], $\rho(J) = 1$ is a simple eigenvalue of the matrix J. Consequently, $\text{index}(I - J) = 1$. Obviously, $I - J \in Z^{n\times n}$, by Lemma 3.1, we have $\text{index}(I - G) = 1$. This completes the proof of this lemma. $\square$

As a direct application of Theorem 2.1 and Corollary 2.1, we have the following theorem.

Theorem 3.2. *For the matrix P given in* (3.2), *we have the following results:*

(i) *The Jacobi extrapolated iterative method* (3.3) *is semiconvergent if and only if*

$$0 < \omega < 1.$$

Furthermore,

$$\omega_{opt} = \frac{2}{3 - 2\sqrt{pq}\cos(\frac{\pi}{n-1})}, \quad \vartheta(\omega_{opt}) = \frac{1 + 2\sqrt{pq}\cos(\frac{\pi}{n-1})}{3 - 2\sqrt{pq}\cos(\frac{\pi}{n-1})}.$$

(ii) *The Gauss-Seidel extrapolated iterative method* (3.4) *is semiconvergent if and only if*

$$0 < \omega < 2.$$

Furthermore,

$$\omega_{opt} = \frac{2}{2 - 4pq\cos^2(\frac{\pi}{n-1})}, \quad \vartheta(\omega_{opt}) = \frac{2pq\cos^2(\frac{\pi}{n-1})}{1 - 2pq\cos^2(\frac{\pi}{n-1})}.$$

Now, we give the numerical results for the iterative methods (3.3) and (3.4). In Table 3.1, the initial approximation v^0 is taken as $(2, 3, 4, 1, 1, \cdots, 1)^T$. It is easy to see that the solutions of (3.1) is $c(1, 1, \cdots, 1)^T$ where c is any constant. Here,

$$\frac{\|(I-P)v^k}{\|(I-P)v^0}\| \leq 10^{-6}$$

is used as the stopping criterion. The maximum number of iterations for all the numerical experiments is set to 10000. In addition, we choose $p = \frac{1}{10}$ and $q = \frac{9}{10}$.

All experiments were executed on a PC using the MATLAB programming package. MATLAB carries out calculations in double precision by default.

In Table 3.1, we report the pseudo-spectral radius of the corresponding iteration matrix (SPEC), the CPU time elapsed for the iteration (CPU) and the number of iterations (IT). It should be pointed out that the final approximation solution is affected by the initial approximation. In the table, c means that the approximation solution is $c(1, 1, \cdots, 1)^T$. Notations J_ω, G_ω, ω_J and ω_G represent Jacobi extrapolation matrix, Gauss-Seidel extrapolation matrix and the corresponding extrapolation parameters respectively. The data with underline are the optimal parameter in theory. N means that P is an $N \times N$ matrix.

During our experiments, we found that the approximation solution depends on the initial approximation v^0, e.g., Table 3.1 shows that for the given initial approximation, extrapolated Jacobi method is semiconvergent to $2.59671(1, 1, \cdots, 1)^T$, while the approximation solution of the extrapolated Gauss-Seidel method is $3.07407(1, 1, \cdots, 1)^T$. From the table, we can also see that when N is not large, the optimum values can be obtained at the optimal parameters. The larger the N is, the more serious the perturbation is. This is due to the singularity of the coefficient matrix $I - P$. In this case, the optimal parameter is referenced by the choice of the other parameters. Normally, the parameter for the practical optimum value is not very far from the theoretic optimal one.

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Table 3.1: Pseudo-spectral radius, CPU time and number of iterations (IT).

N	J_ω $(c = 2.59671)$				$G_\omega(c = 3.07407)$			
	ω_J	IT	CPU	SPEC	ω_G	IT	CPU	SPEC
10	0.5	58	0	0.781908	0.7	21	0	0.522522
	0.6	47	0	0.738289	0.9	15	0	0.386099
	0.7	39	0	0.694671	1.1	10	0	0.249677
	0.8	33	0	0.651052	$\underline{1.18898}$	10	0	0.188981
	$\underline{0.820956}$	33	0	0.641912	1.2	10	0	0.2
	0.9	61	0	0.8	1.7	46	0	0.7
20	0.5	103	0	0.795908	1	16	0	0.350247
	0.6	84	0	0.75509	1.1	14	0	0.285272
	0.7	70	0	0.714272	1.2	12	0	0.220297
	0.8	60	0	0.673453	$\underline{1.2123}$	11	0	0.212303
	$\underline{0.830502}$	57	0	0.661003	1.3	13	0	0.3
	0.9	63	0	0.8	1.5	23	0	0.5
100	0.3	617	0.078	0.879909	1	36	0	0.359633
	0.5	361	0.047	0.799849	1.1	32	0	0.295601
	0.7	251	0.031	0.719789	1.2	28	0	0.231565
	0.8	216	0.031	0.679758	$\underline{1.21924}$	27	0	0.219243
	$\underline{0.833228}$	206	0.016	0.666457	1.3	24	0	0.3
	0.9	189	0.016	0.8	1.7	46	0	0.7
1000	0.5	2822	103.703	0.799999	1	186	6.875	0.359996
	0.6	2339	86.844	0.759998	1.2	151	5.657	0.231996
	0.7	1994	74.422	0.719998	$\underline{1.21951}$	148	5.531	0.21951
	0.8	1734	63.891	0.679998	1.3	137	5.266	0.3
	$\underline{0.833332}$	1662	61.375	0.666665	1.8	88	3.297	0.8
	0.9	1532	56.421	0.8	1.9	167	6.234	0.9
3000	0.5	8042	2698.81	0.8	1	476	157.516	0.36
	0.6	6681	2415.95	0.76	1.2	389	128.907	0.232
	0.7	5708	1954.55	0.72	$\underline{1.21951}$	383	126.578	0.219512
	0.8	4977	1651.28	0.68	1.3	356	118.125	0.3
	$\underline{0.833333}$	4772	1579.53	0.666666	1.7	259	85.703	0.7
	0.9	4407	1460.59	0.8	1.9	224	74.469	0.9

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References

[1] Z.-Z. Bai, Experimental study of the asynchronous multisplitting relaxation methods for linear complementarity problems, *J. Comput. Math.*, **20** (2002), 561-574.

[2] Z.-Z. Bai and Y.-G. Huang, Relaxed asynchronous iterations for the linear complementarity problem, *J. Comput. Math.*, **20** (2002), 97-112.

[3] Z.-Z. Bai and Y.-G. Huang, A class of asynchronous parallel multisplitting relaxation methods for the large sparse linear complementarity problems, *J. Comput. Math.*, **21** (2003), 773-790.

[4] Z.-Z. Bai, L. Wang and J.-Y. Yuan, Weak-convergence theory of quasi-nonnegative splittings for singular matrices, *Appl. Numer. Math.*, **47** (2003), 75-89.

[5] A. Berman, R. J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Society for Industrial and Applied Mathematics, Philadelphia, 1994.

[6] Z. Cao, A convergence theorem on an extrapolated iterative method and its applications, *Appl. Numer. Math.*, **27** (1998), 203-209.

[7] A. Hadjidimos, On the optimization of classical iterative schemes for the solution of complex singular linear systems, *SIAM Journal on Algebraic and Discrete Methods*, **6** (1985), 555-566.

[8] A. Hadjidimos and A. Yeyios, The principle of extrapolation in connection with the accelerated overrelaxation method, *Linear Algebra Appl.*, **30** (1980), 115-128.

[9] J.W. Hu and H.M. Tang, Numerical Methods for Differential Equations (In Chinese), Academic Press, Peking, 1999.

[10] Y. Song, Semiconvergence of extrapolated iterative methods for singular linear systems, *J. Comput. Appl. Math.*, **106** (1999), 117-129.

[11] Y. Song, Semiconvergence of nonnegative splittings for singular matrices, *Numer. Math.*, **85** (2000), 109-127.

[12] Y. Song and L. Wang, On the semiconvergence of extrapolated iterative methods for singular linear systems, *Appl. Numer. Math.*, **44** (2003), 404-413.

[13] J.W. Thomas, Numerical Partial Differential Equations Finite Difference Methods, Springer-Verlag, New York, 1997

[14] R.S. Varga, Matrix Iterative Analysis, Springer Series in Computational Mathematics: 27, Springer-Verlag, Berlin, 2000.

[15] A. Yeyios, On the optimization of an extrapolation method, *Linear Algebra Appl.*, **57** (1984), 191-203.

[16] D.M. Young, Iterative Solution of Large Linear Systems, Academic Press, New York, 1971.

[17] D. Yuan and Y. Song, Modified AOR methods for linear complementarity problem, *Appl. Math. Comput.*, **140** (2003), 53-67.