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TWO-GRID DISCRETIZATION SCHEMES OF THE NONCONFORMING FEM FOR EIGENVALUE PROBLEMS *

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Abstract

This paper extends the two-grid discretization scheme of the conforming finite elements proposed by Xu and Zhou (Math. Comput., 70 (2001), pp.17-25) to the nonconforming finite elements for eigenvalue problems. In particular, two two-grid discretization schemes based on Rayleigh quotient technique are proposed. By using these new schemes, the solution of an eigenvalue problem on a fine mesh is reduced to that on a much coarser mesh together with the solution of a linear algebraic system on the fine mesh. The resulting solution still maintains an asymptotically optimal accuracy. Comparing with the two-grid discretization scheme of the conforming finite elements, the main advantages of our new schemes are twofold when the mesh size is small enough. First, the lower bounds of the exact eigenvalues in our two-grid discretization schemes can be obtained. Second, the first eigenvalue given by the new schemes has much better accuracy than that obtained by solving the eigenvalue problems on the fine mesh directly.

Mathematics subject classification: 65N25, 65N30.
Key words: Nonconforming finite elements, Rayleigh quotient, Two-grid schemes, The lower bounds of eigenvalue, High accuracy.

1. Introduction

Xu [10–12] first proposed two-grid discretization methods for nonsymmetric and nonlinear elliptic problems. Later, Xu and Zhou [13] proposed a two-grid discretization scheme of conforming finite elements for eigenvalue problems. In [14], Xu and Zhou proposed some local and parallel finite element algorithms based on [13]. Yang [15] extended the method in [13] to the Wilson nonconforming element and demonstrated by numerical experiments that the first eigenvalue given by the two-grid discretization scheme approximates the exact eigenvalue from below and has much better accuracy than that obtained by solving the eigenvalue problem on a fine mesh directly.

In this paper we will discuss two-grid discretization schemes of the nonconforming finite elements for any n-dimensional eigenvalue problems. We propose a new two-grid discretization scheme (see Scheme 1) and extend the scheme in [13, 15] (see Scheme 2). Using these two new schemes, the solution of an eigenvalue problem on a fine mesh is reduced to the solution of an eigenvalue problem on a much coarser mesh and the solution of a linear algebraic system on the fine mesh and the resulting solution still maintains an asymptotically optimal accuracy. Comparing with the two-grid discretization scheme of the conforming finite elements (see [13]), the main advantages of our new schemes are twofold when the mesh size is small enough. First, the lower bounds of the exact eigenvalues in our two two-grid discretization schemes can be

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obtained. Second, the first eigenvalue given by these new schemes has much better accuracy than that obtained by solving the eigenvalue problems on fine mesh directly.

Finding the lower bounds of eigenvalues by using the nonconforming elements has attracted many attentions in the past. In 1967, Zienkiewicz etc. [19] discovered that the nonconforming Morley element approximates eigenvalues from below. As for the vibration plate problems, Rannacher [6] provided numerical results in 1979, which indicated that the nonconforming Morley and Adini element can be used to obtain lower bounds of eigenvalues. Yang [16] proved that on a rectangular domain, the Adini element approximates the exact eigenvalues from below. For Laplace operator eigenvalue problems, Armentano and Duran [1] proved that piecewise linear nonconforming Crouzeix-Raviart element approximates the exact eigenvalues from below, Lin and Lin [5] proved that the nonconforming EQ_1^{rot} element approximates the exact eigenvalues from below, and Zhang et al. [18] proved that the nonconforming Wilson element approximates the exact eigenvalues from below. In this paper, we will show that the proposed two-grid discretization schemes maintain the above properties of approximation from below.

By the minimum-maximum principle we can conclude that the first eigenvalue given by the two-grid discretization scheme in [13] has much lower accuracy than that obtained by solving the eigenvalue problems on the fine mesh directly for conforming finite elements. However, to our surprise, it is exactly opposite for two-grid discretization scheme of most nonconforming finite elements. In particular, the two-grid discretization schemes of nonconforming finite elements are very efficient for eigenvalue problems.

The rest of the paper is organized as follows. In Section 2, we shall describe some notation and properties of the nonconforming finite element approximation for eigenvalue problems. In Section 3, we propose two two-grid discretization schemes of the nonconforming finite elements for eigenvalue problems and discuss approximation properties of the schemes. In Section 4, we apply the results in Section 3 to several representative nonconforming finite elements such as Wilson, Crouzeix-Raviart and Adini nonconforming elements.

2. Preliminaries

Let Ω be a bounded open connected subset of R^n with a Lipschitz-continuous boundary. Let V be a mth-order Sobolev space over Ω with inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$ (m=1, 2), and let W be a sth-order Sobolev space over Ω with inner product $(\cdot,\cdot)_W$ and norm $\|\cdot\|_W$ $(0 \leq s < m), V \subset W$ with a compact imbedding.

Suppose that $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric and continuous bilinear forms on $V \times V$ and $W \times W$, respectively, which satisfy

$$\begin{aligned}
&\mid a(u,v) \mid \leq M_1 \|u\|_V \|v\|_V, && \forall u,v \in V,\\
&a(u,u) \geq \alpha_1 \|u\|_V^2, && \forall u \in V,\\
&\mid b(u,v) \mid \leq M_2 \|u\|_W \|v\|_W, && \forall u,v \in W,\\
&b(u,u) \geq \alpha_2 \|u\|_W^2, && \forall u \in W.
\end{aligned}$$

Define $\|\cdot\|_b = b(\cdot,\cdot)^{\frac{1}{2}}$. Noting that $\|\cdot\|_b$ and $\|\cdot\|_W$ are two equivalent norms on W, we shall use $b(\cdot,\cdot)$ and $\|\cdot\|_b$ as the inner product and norm on W, respectively.

Consider the $2m$th-order elliptic differential operator eigenvalue problems: Find $(\lambda, u) \in R \times V, \|u\|_b = 1$ satisfying

$$a(u,v) = \lambda b(u,v), \quad \forall v \in V. \tag{2.1}$$

Let π_h be a mesh of Ω with the mesh size h, and V_h be a nonconforming finite element space on π_h, $V_h \subset W$ and $V_h \not\subset V$. The nonconforming finite element discretization of (2.1) is given by: Find $(\lambda_h, u_h) \in R \times V_h$, $\|u_h\|_b = 1$ satisfying

$$a_h(u_h, v) = \lambda_h b(u_h, v), \qquad \forall v \in V_h, \tag{2.2}$$

where $a_h(\cdot,\cdot)$ is an approximation to $a(\cdot,\cdot)$.

Define $\|\cdot\|_h = a_h(\cdot,\cdot)^{\frac{1}{2}}$. Assume that $a_h(\cdot,\cdot)$ is uniformly V_h-elliptic, symmetric and continuous, and $\|\cdot\|_h$ is a norm on $V + V_h$. Define $T : W \to V$ such that

$$a(Tf, v) = b(f, v), \quad \forall f \in W, \forall v \in V.$$

Moreover, define $T_h : W \to V_h$ such that

$$a_h(T_h f, v) = b(f, v), \quad \forall f \in W, \forall v \in V_h.$$

It is clear that (2.1) and (2.2) are equivalent to

$$\lambda T u = u \tag{2.3}$$

and

$$\lambda_h T_h u_h = u_h, \tag{2.4}$$

respectively.

Using the facts that the bilinear forms $a(\cdot,\cdot)$, $a_h(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are symmetric, continuous and $V \subset W$ with a compact imbedding, one can prove that T is self-adjoint completely continuous operator and T_h are self-adjoint operators of finite rank (see [2]). In fact,

$$b(Tf, g) = b(g, Tf) = a(Tg, Tf) = a(Tf, Tg) = b(f, Tg), \qquad \forall f, g \in W,$$

and

$$b(T_h f, g) = b(g, T_h f) = a_h(T_h g, T_h f) = a_h(T_h f, T_h g) = b(f, T_h g), \quad \forall f, g \in W.$$

It is also known that the spectrum of (2.1) consists of an infinite sequence of isolated real eigenvalues with finite algebraic multiplicity,

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \nearrow +\infty.$$

In the sequence $\{\lambda_k\}$, the λ_k are repeated according to algebraic multiplicity, and the corresponding eigenspaces are denoted by $M(\lambda_k)$ with $M(\lambda_k) = M(\lambda_l)$ in case $\lambda_k = \lambda_l$. Moreover, the spectrum of (2.2) consists exactly of $N_h = \dim V_h$ real eigenvalues,

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \lambda_{3,h} \leq \cdots \leq \lambda_{N_h,h}.$$

We make the following assumptions.

C1: $\{T_h\}$ is a family of operators satisfying

$$\|T_h - T\|_b \to 0 \ (h \searrow 0).$$

C2: For the nonconforming finite element approximation of the source problem corresponding to (2.1) with the right hand side $b(f, v)$, there exist three positive constants C, q_1 and q_2 ($q_1 \leq q_2$) independent of h and $f \in W$, such that the following error estimates hold:

$$\|Tf - T_h f\|_h \leq Ch^{q_1}\|f\|_b, \quad \|Tf - T_h f\|_b \leq Ch^{q_2}\|f\|_b.$$

The above conditions are satisfied for most of the nonconforming finite elements in practice.

Assume the algebraic multiplicity of λ_k is equal to q, i.e.,

$$\lambda_k = \lambda_{k+1} = \cdots = \lambda_{k+q-1}.$$

Let $M_h(\lambda_k)$ be the direct sum of the eigenspaces of (2.2) corresponding to eigenvalues $\lambda_{j,h}$ that converge to λ_k. Let the eigenfunctions $\{u_{j,h}\}$ be an orthonormal basis for $M_h(\lambda_k)$ with $\|u_{j,h}\|_b = 1$. Write

$$\bar{\lambda}_{k,h} = \frac{1}{q}\sum_{j=k}^{k+q-1} \lambda_{j,h}, \bar{u}_{k,h} = \sum_{j=k}^{k+q-1} b(u_k, u_{j,h})u_{j,h} \tag{2.5}$$

and

$$\|(T - T_h) \mid_{M(\lambda_k)} \| = \max_{u \in M(\lambda_k), \|u\|_b = 1} \|Tu - T_h u\|_b. \tag{2.6}$$

Throughout this paper, we shall use the letter C to denote a generic positive constant independent of mesh size h, which may stand for different values at its different occurrences.

Lemma 2.1. **([17])** *Suppose* **C1** *is satisfied. Let* $(\lambda_{k,h}, u_{k,h})$ *be a nonconforming elemet eigenpair of* (2.2) *with* $\|u_{k,h}\|_b = 1$. *Then* $\lambda_{k,h} \to \lambda_k$ $(h \searrow 0)$ *and there is a function* $u_k \in M(\lambda_k)$ *with* $\|u_k\|_b = 1$ *such that*

$$| \lambda_{k,h} - \lambda_k | \leq C\|(T - T_h)|_{M(\lambda_k)}\|, \tag{2.7}$$

$$\|u_{k,h} - u_k\|_b \leq C\|(T - T_h)|_{M(\lambda_k)}\|, \tag{2.8}$$

$$\|u_{k,h} - u_k\|_h \leq \lambda_k\|Tu_k - T_h u_k\|_h + C\|(T - T_h)|_{M(\lambda_k)}\|. \tag{2.9}$$

Lemma 2.2. *Suppose* **C1** *is satisfied. Let* (λ_k, u_k) *be an eigenpair of* (2.1) *with* $\|u_k\|_b = 1$. *Then*

$$\begin{aligned}
&\|\bar{u}_{k,h} - u_k\|_b \leq C\|(T - T_h) \mid_{M(\lambda_k)} \|, \\
&\|\bar{u}_{k,h} - u_k\|_h \leq C\bigg(\|(T - T_h) \mid_{M(\lambda_k)} \| + \|Tu_k - T_h u_k\|_h\bigg).
\end{aligned}$$

Proof. Note that $\bar{u}_{k,h}$ is the orthogonal projection of u_k onto $M_h(\lambda_k)$. We can obtain

$$\begin{aligned}
\|\bar{u}_{k,h} - u_k\|_b &\leq \|u_{k,h} - u_k\|_b, \\
\|\bar{u}_{k,h} - u_k\|_h &\leq \|\bar{u}_{k,h} - u_{k,h}\|_h + \|u_{k,h} - u_k\|_h \\
&= \left\|\sum_{j=k}^{k+q-1} b(u_k - u_{k,h}, u_{j,h})u_{j,h}\right\|_h + \|u_{k,h} - u_k\|_h.
\end{aligned}$$

Combining (2.8) with (2.9), Lemma 2.2 is proved. □

3. Two Two-Grid Discretization Schemes

It is well-known that the Rayleigh quotient has higher accuracy. In fact, if

$$\|u_k\|_b = \|u_{k,H}\|_b = 1,$$

then

$$\begin{aligned}
&\left|\frac{b(Tu_{k,H},u_{k,H})}{b(u_{k,H},u_{k,H})}-\frac{1}{\lambda_k}\right|\\
=&\left|b(Tu_{k,H},u_{k,H})-\frac{1}{\lambda_k}\right|=|b(Tu_{k,H},u_{k,H})-b(Tu_k,u_k)|\\
=&\left|b(Tu_{k,H}-Tu_k,u_{k,H}-u_k)-\frac{1}{\lambda_k}\|u_{k,H}-u_k\|_b^2\right|\\
\le& C\|u_{k,H}-u_k\|_b^2. \qquad (3.1)
\end{aligned}$$

The problem is how to solve $Tu_{k,H}$. It is a natural method to take $Tu_{k,H}\approx T_hu_{k,H}$, which gives the following two-grid discretization scheme.

Scheme 1:

Step 1. *Solve an eigenvalue problem* (2.2) *on coarse mesh* π_H*: Find* $(\lambda_{k,H},u_{k,H})\in R\times V_H$ *such that* $\|u_{k,H}\|_b=1$ *and*

$$a_H(u_{k,H},v)=\lambda_{k,H}b(u_{k,H},v),\qquad \forall v\in V_H,$$

where $k\in\{1,2,\cdots,N_H=dimV_H\}$.

Step 2. *Solve a boundary value problem corresponding to* (2.1) *on fine mesh* π_h*: Find* $u^*_{k,h}\in V_h$*, such that*

$$a_h(u^*_{k,h},v)=\lambda_{k,H}b(u_{k,H},v),\ \forall v\in V_h. \qquad (3.2)$$

Step 3. *Compute the Rayleigh quotient*

$$\lambda_{k,r}=1/b(T_hu_{k,H},u_{k,H})\equiv\lambda^2_{k,H}/a_h(u^*_{k,h},u^*_{k,h}). \qquad (3.3)$$

Note that we have $u^*_{k,h}=\lambda_{k,H}T_hu_{k,H}$ *by the definition of* T_h *and* (3.2)*. We then use* $(\lambda_{k,r},u^*_{k,h})$ *as an approximation of* (λ_k,u_k).

The following scheme further develops the two-grid discretization scheme established by Xu and Zhou [13].

Scheme 2:

Step 1. *Solve an eigenvalue problem* (2.2) *on coarse mesh* π_H*: Find* $(\lambda_{k,H},u_{k,H})\in R\times V_H$ *such that* $\|u_{k,H}\|_b=1$ *and*

$$a_H(u_{k,H},v)=\lambda_{k,H}b(u_{k,H},v),\qquad \forall v\in V_H,$$

where $k\in\{1,2,\cdots,N_H=dimV_H\}$.

Step 2. *Solve a boundary value problem corresponding to* (2.1) *on fine mesh* π_h*: Find* $u^*_{k,h}\in V_h$*, such that*

$$a_h(u^*_{k,h},v)=\lambda_{k,H}b(u_{k,H},v),\ \forall v\in V_h.$$

Step 3. *Compute the Rayleigh quotient*

$$\lambda_{k,s} = a_h(u^*_{k,h}, u^*_{k,h})/b(u^*_{k,h}, u^*_{k,h}). \tag{3.4}$$

We then use $(\lambda_{k,s}, u^*_{k,h})$ as an approximation of (λ_k, u_k).

Theorem 3.1. *Suppose* **C1** *is satisfied. Let* $(\lambda_{k,r}, u^*_{k,h})$ *and* $(\lambda_{k,s}, u^*_{k,h})$ *be obtained by Scheme 1 and Scheme 2, respectively. Then there exists a function* $u_k \in M(\lambda_k)$ *with* $\|u_k\|_b = 1$ *such that*

$$\|u^*_{k,h} - u_k\|_h \le C\Big(\|Tu_k - T_h u_k\|_h + \|(T - T_H)|_{M(\lambda_k)}\|\Big), \tag{3.5}$$

$$\begin{aligned} |\lambda_{k,s} - \lambda_k| \le& C\|(T - T_h)|_{M(\lambda_k)}\| \\ &+ C\Big(\|Tu_k - T_h u_k\|_h + \|(T - T_H)|_{M(\lambda_k)}\|\Big)^2, \end{aligned} \tag{3.6}$$

$$|\lambda_{k,r} - \lambda_k| \le C\Big(\|Tu_k - T_h u_k\|_b + \|(T - T_H)|_{M(\lambda_k)}\|^2\Big). \tag{3.7}$$

Proof. Let $\bar\lambda_{k,h}$ and $\bar u_{k,h}$ be of the form (2.5). By Lemma 2.1 we can obtain for all $j = k, \cdots, k+q-1$,

$$|\bar\lambda_{k,h} - \lambda_k| + |\lambda_{j,h} - \bar\lambda_{k,h}| \le C\|(T - T_h)|_{M(\lambda_k)}\|. \tag{3.8}$$

By $u^*_{k,h} = \lambda_{k,H} T_h u_{k,H}$ we see that

$$\begin{aligned} &u^*_{k,h} - u_k = \lambda_{k,H} T_h u_{k,H} - \lambda_k T u_k \\ =&\lambda_{k,H}(T_h u_{k,H} - T_h u_k) + \lambda_{k,H}(T_h u_k - T u_k) + (\lambda_{k,H} - \lambda_k) T u_k. \end{aligned} \tag{3.9}$$

From **C1** we conclude that the $\{T_h\}$ is uniformly bounded. Thus

$$\|T_h(u_{k,H} - u_k)\|_b \le C\|u_{k,H} - u_k\|_b.$$

By the definition of T_h, Schwarz's inequality and (2.8) we have

$$\begin{aligned} &a_h(T_h(u_{k,H} - u_k), T_h(u_{k,H} - u_k)) \\ =&b(u_{k,H} - u_k, T_h(u_{k,H} - u_k)) \le \|u_{k,H} - u_k\|_b \|T_h(u_{k,H} - u_k)\|_b \\ \le& C\|u_{k,H} - u_k\|_b^2 \le C\|(T - T_H)|_{M(\lambda_k)}\|^2, \end{aligned}$$

and hence

$$\|T_h(u_{k,H} - u_k)\|_h \le C\|(T - T_H)|_{M(\lambda_k)}\|. \tag{3.10}$$

Using (3.9), (3.10) and (2.7) we get (3.5). From (3.9), (2.7) and (2.8) we get

$$\|u_k - u^*_{k,h}\|_b \le C\Big(\|Tu_k - T_h u_k\|_b + \|(T - T_H)|_{M(\lambda_k)}\|\Big). \tag{3.11}$$

From (3.5) and Lemma 2.1 we get

$$\begin{aligned} &\|u^*_{k,h} - \bar u_{k,h}\|_h \le \|u^*_{k,h} - u_k\|_h + \|u_k - \bar u_{k,h}\|_h \\ \le& C\Big(\|Tu_k - T_h u_k\|_h + \|(T - T_H)|_{M(\lambda_k)}\| + \|(T - T_h)|_{M(\lambda_k)}\|\Big), \end{aligned} \tag{3.12}$$

and combining (3.11) with Lemma 2.2 we get

$$\|u_{k,h}^* - \bar{u}_{k,h}\|_b \le C\Big(\big\|(T-T_h)\mid_{M(\lambda_k)}\| + \|(T-T_H)\mid_{M(\lambda_k)}\big\| \Big). \tag{3.13}$$

As $\bar{u}_{k,h}$ is of the form in (2.5), we conclude

$$\begin{aligned}
&a_h(u_{k,h}^*, \bar{u}_{k,h}) = a_h\left(u_{k,h}^*, \sum_{j=k}^{k+q-1} b(u_k, u_{j,h})u_{j,h}\right)\\
=&\sum_{j=k}^{k+q-1} b(u_k, u_{j,h})a_h(u_{k,h}^*, u_{j,h}) = \sum_{j=k}^{k+q-1} b(u_k, u_{j,h})\lambda_{j,h} b(u_{k,h}^*, u_{j,h})\\
=&\sum_{j=k}^{k+q-1} b(u_k, u_{j,h})(\lambda_{j,h} - \bar{\lambda}_{k,h} + \bar{\lambda}_{k,h}) b(u_{k,h}^*, u_{j,h})\\
=&\bar{\lambda}_{k,h} b(u_{k,h}^*, \bar{u}_{k,h}) + r_1,
\end{aligned}$$

where

$$r_1 = \sum_{j=k}^{k+q-1} b(u_k, u_{j,h})(\lambda_{j,h} - \bar{\lambda}_{k,h}) b(u_{k,h}^*, u_{j,h}).$$

Using a similar way we can obtain

$$a_h(\bar{u}_{k,h}, \bar{u}_{k,h}) = \bar{\lambda}_{k,h} b(\bar{u}_{k,h}, \bar{u}_{k,h}) + r_2,$$

where

$$r_2 = \sum_{j=k}^{k+q-1} b(u_k, u_{j,h})(\lambda_{j,h} - \bar{\lambda}_{k,h}) b(\bar{u}_{k,h}, u_{j,h}).$$

By the above two relations it follows that

$$\begin{aligned}
&\|u_{k,h}^* - \bar{u}_{k,h}\|_h^2 - \bar{\lambda}_{k,h}\|u_{k,h}^* - \bar{u}_{k,h}\|_b^2\\
=&a_h(u_{k,h}^*, u_{k,h}^*) - 2a_h(u_{k,h}^*, \bar{u}_{k,h}) + a_h(\bar{u}_{k,h}, \bar{u}_{k,h})\\
&- \bar{\lambda}_{k,h} b(u_{k,h}^*, u_{k,h}^*) + 2\bar{\lambda}_{k,h} b(u_{k,h}^*, \bar{u}_{k,h}) - \bar{\lambda}_{k,h} b(\bar{u}_{k,h}, \bar{u}_{k,h})\\
=&a_h(u_{k,h}^*, u_{k,h}^*) - \bar{\lambda}_{k,h} b(u_{k,h}^*, u_{k,h}^*) - 2r_1 + r_2,
\end{aligned}$$

and dividing by $b(u_{k,h}^*, u_{k,h}^*)$ on both sides of the above identity, we deduce

$$\lambda_{k,s} - \bar{\lambda}_{k,h} = \frac{\|u_{k,h}^* - \bar{u}_{k,h}\|_h^2 - \bar{\lambda}_{k,h}\|u_{k,h}^* - \bar{u}_{k,h}\|_b^2 + 2r_1 - r_2}{b(u_{k,h}^*, u_{k,h}^*)}.$$

It follows from (3.8) that

$$\mid 2r_1 \mid + \mid r_2 \mid \le C\|(T-T_h)\mid_{M(\lambda_k)}\|.$$

Write $\lambda_{k,s} - \lambda_k = \lambda_{k,s} - \bar{\lambda}_{k,h} + \bar{\lambda}_{k,h} - \lambda_k$. Using (3.12), (3.13) and (3.8) gives (3.6).

By **C1** we can deduce that $\|T_h - T\|_b \le C$, and hence

$$\begin{aligned}
&\left|\frac{1}{\lambda_{k,r}} - b(Tu_{k,H}, u_{k,H})\right| = |b(T_h u_{k,H}, u_{k,H}) - b(Tu_{k,H}, u_{k,H})| \\
&= |b((T_h - T)u_{k,H}, u_k) + b((T_h - T)u_{k,H}, u_{k,H} - u_k)| \\
&=|b((T_h - T)u_k, u_{k,H}) + b((T_h - T)(u_{k,H} - u_k), u_{k,H} - u_k) \\
&\quad + b((T_h - T)u_k, u_{k,H} - u_k)| \\
&\le C\|(T_h - T)u_k\|_b + C\|u_{k,H} - u_k\|_b^2.
\end{aligned}$$

By the above inequality, combining (3.1) with (2.8) we obtain (3.7). $\square$

Theorem 3.1 and Lemma 2.1 show that both $\lambda_{k,r}$ and $\lambda_{k,s}$ can achieve the order of convergence of the nonconforming finite element eigenvalue $\lambda_{k,h}$.

Theorem 3.2. *Suppose $(\lambda_{k,H}, u_{k,H})$ and $(\lambda_{k,r}, u_{k,h}^*)$ are obtained by Scheme 1. Then there exsits a function $u_k \in M(\lambda_k)$ with $\|u_k\|_b = 1$ such that*

$$\begin{aligned}
&\lambda_k - \lambda_{k,r} \\
&=2\frac{\lambda_{k,r}}{\lambda_{k,H}} a_h(u_k - I_h u_k, u_{k,h}^*) + \frac{\lambda_{k,r}}{\lambda_k \lambda_{k,H}^2}\|\lambda_{k,H} u_k - \lambda_k u_{k,h}^*\|_h^2 \\
&\quad - \lambda_{k,r}\|I_h u_k - u_{k,H}\|_b^2 + \lambda_{k,r}\Big(\|I_h u_k\|_b^2 - \|u_k\|_b^2\Big).
\end{aligned} \tag{3.14}$$

Proof. Since $\|u_k\|_b = \|u_{k,H}\|_b = 1$, we observe that

$$\frac{1}{\lambda_k} = b(Tu_k, u_k), \quad \frac{1}{\lambda_{k,r}} = b(T_h u_{k,H}, u_{k,H}).$$

By (2.3) and the identity $u_{k,h}^* = \lambda_{k,H} T_h u_{k,H}$, we have

$$\begin{aligned}
&\|Tu_k - T_h u_{k,H}\|_h^2 \\
&=\left\|\frac{1}{\lambda_k}u_k - \frac{1}{\lambda_{k,H}}u_{k,h}^*\right\|_h^2 = \left(\frac{1}{\lambda_k\lambda_{k,H}}\right)^2 \|\lambda_{k,H}u_k - \lambda_k u_{k,h}^*\|_h^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{\lambda_k} + \frac{1}{\lambda_{k,r}} \\
&=b(Tu_k, u_k) + b(T_h u_{k,H}, u_{k,H}) = a_h(Tu_k, Tu_k) + a_h(T_h u_{k,H}, T_h u_{k,H}) \\
&=\|Tu_k - T_h u_{k,H}\|_h^2 + 2a_h(Tu_k, T_h u_{k,H}) \\
&=\frac{2}{\lambda_k}a_h(u_k - I_h u_k, T_h u_{k,H}) + \|Tu_k - T_h u_{k,H}\|_h^2 + \frac{2}{\lambda_k}a_h(I_h u_k, T_h u_{k,H}) \\
&=\frac{2}{\lambda_k}a_h(u_k - I_h u_k, T_h u_{k,H}) + \|Tu_k - T_h u_{k,H}\|_h^2 + \frac{2}{\lambda_k}b(I_h u_k, u_{k,H}) \\
&=\frac{2}{\lambda_k\lambda_{k,H}}a_h(u_k - I_h u_k, u_{k,h}^*) + \left(\frac{1}{\lambda_k\lambda_{k,H}}\right)^2 \|\lambda_{k,H}u_k - \lambda_k u_{k,h}^*\|_h^2 \\
&\quad + \frac{1}{\lambda_k}\Big(- \|I_h u_k - u_{k,H}\|_b^2 + 2 + \|I_h u_k\|_b^2 - \|u_k\|_b^2\Big).
\end{aligned}$$

The conclusion follows by subtracting $\frac{2}{\lambda_k}$ from both sides. $\square$

Theorem 3.3. *Suppose* **C1** *and* **C2** *are satisfied. Let* $\lambda_{1,h}$ *be the first eigenvalue of* (2.2) *on* V_h. *If* $\lambda_{k,r}$ *and* $\lambda_{k,s}$ *are obtained by Scheme 1 and Scheme* 2*, respectively, then*

$$\lambda_{k,r} \geq \lambda_{k,s}, \tag{3.15}$$

$$\lambda_{1,r} \geq \lambda_{1,s} \geq \lambda_{1,h}. \tag{3.16}$$

Proof. By Theorem 3.1, we have $\|u^*_{k,h} - u_k\|_h \to 0$ $(h \searrow 0)$, and hence $a_h(u^*_{k,h}, u^*_{k,h}) > 0$ and $b(u^*_{k,h}, u^*_{k,h}) > 0$. Using (3.2)-(3.4) and Schwarz's inequality, we get

$$\begin{aligned}
\lambda_{k,r} - \lambda_{k,s} =& \frac{1}{b(T_h u_{k,H}, u_{k,H})} - \frac{a_h(u^*_{k,h}, u^*_{k,h})}{b(u^*_{k,h}, u^*_{k,h})} \\
=& \frac{1}{a_h(T_h u_{k,H}, T_h u_{k,H})} - \frac{b(u_{k,H}, T_h u_{k,H})}{b(T_h u_{k,H}, T_h u_{k,H})} \\
=& \frac{b(T_h u_{k,H}, T_h u_{k,H}) - b(u_{k,H}, T_h u_{k,H})^2}{a_h(T_h u_{k,H}, T_h u_{k,H}) b(T_h u_{k,H}, T_h u_{k,H})} \\
\geq& \frac{\|T_h u_{k,H}\|_b^2 - \|u_{k,H}\|_b^2 \|T_h u_{k,H}\|_b^2}{a_h(T_h u_{k,H}, T_h u_{k,H}) b(T_h u_{k,H}, T_h u_{k,H})} = 0,
\end{aligned}$$

which implies that (3.15) holds. It follows directly from the minimum principle (see, e.g., [2], p. 699) for eigenvalue problem (2.2) on π_h that $\lambda_{1,s} \geq \lambda_{1,h}$, which together with (3.15) yield (3.16). $\square$

Theorem 3.4. *Suppose* **C1** *and* **C2** *are satisfied. Let* $(\lambda_{k,H}, u_{k,H})$ *be an eigenpair of* (2.2) *with* $\|u_{k,H}\|_b = 1$, *and let* $(\lambda_{k,r}, u^*_{k,h})$ *and* $(\lambda_{k,s}, u^*_{k,h})$ *be obtained by Scheme* 1 *and Scheme* 2*, respectively. Then* $\lambda_{k,H} \to \lambda_k$ *and there exists a function* $u_k \in M(\lambda_k)$ *with* $\|u_k\|_b = 1$ *such that*

$$| \lambda_{k,H} - \lambda_k | \leq C H^{q_2}, \tag{3.17a}$$

$$\|u_{k,H} - u_k\|_b \leq C H^{q_2}, \tag{3.17b}$$

$$\|u_{k,H} - u_k\|_H \leq C H^{q_1}, \tag{3.17c}$$

$$\|u^*_{k,h} - u_k\|_h \leq C\big(h^{q_1} + H^{q_2}\big), \tag{3.17d}$$

$$| \lambda_{k,s} - \lambda_k | \leq C\big(h^{q_2} + h^{2q_1} + H^{2q_2}\big), \tag{3.17e}$$

$$| \lambda_{k,r} - \lambda_k | \leq C\big(h^{q_2} + H^{2q_2}\big). \tag{3.17f}$$

Proof. The desired results can be obtained from Lemma 2.1, Theorem 3.1 and **C2**. $\square$

4. Applications

We will need regular mesh in the following usual sense (see [3], p. 131):

Regular mesh: A family of meshes π_h is regular if there exists a constant $\sigma > 0$ such that

$$\frac{h_e}{\rho_e} \leq \sigma, \quad \text{for all} \quad e \in \bigcup_h \pi_h, \quad \text{and if} \quad h = \max_{e\in\pi_h} h_e \to 0,$$

where

$$h_e = \mathrm{diam}(e), \quad \rho_e = \sup\{\mathrm{diam(S)} : \ \text{S is a ball contained in } e\}.$$

Let $W_{s,p}(\Omega)$ be the usual Sobolev space with the norm $\|\cdot\|_{s,p}$ and the semi-norm $|\cdot|_{s,p}$. Write $W_{s,2}(\Omega) = H^s(\Omega)$, $\|\cdot\|_{s,2} = \|\cdot\|_s$, $|\cdot|_{s,2} = |\cdot|_s$ and $\|\cdot\|_{L_2(\Omega)} = \|\cdot\|_0$.

4.1. Wilson nonconforming element

Consider the following eigenvalue problem:

$$-\Delta u = \lambda u, \quad \text{in } \Omega; \quad u = 0, \quad \text{on } \partial\Omega. \tag{4.1}$$

The weak form of (4.1) is (2.1), where Ω is rectangular domain, $V = H_0^1(\Omega), W = L_2(\Omega)$,

$$a(u,v) = \int\limits_{\Omega} \bigtriangledown u \bigtriangledown v dx, \quad b(u,v) = \int\limits_{\Omega} uv dx,$$

and $\|u\|_b = \|u\|_0$. Let π_h be a rectangular mesh of domain Ω and let V_h be the Wilson nonconforming element space on π_h. The Wilson nonconforming element approximation of (4.1) is (2.2), where

$$a_h(u_h, v) = \sum_{e\in\pi_h} \int\limits_{e} \bigtriangledown u_h \bigtriangledown v dx.$$

Theorem 4.1. *Suppose both π_H and π_h are regular rectangular meshes. Let $(\lambda_{k,H}, u_{k,H})$ be a Wilson element eigenpair of* (4.1) *with $\|u_{k,H}\|_0 = 1$, let $(\lambda_{k,r}, u_{k,h}^*)$ and $(\lambda_{k,s}, u_{k,h}^*)$ be obtained by Scheme* 1 *and Scheme* 2 *for the Wilson element, respectively. Then there exists a function $u_k \in M(\lambda_k)$ with $\|u_k\|_0 = 1$ such that* (3.17) *holds with $q_1 = 1$ and $q_2 = 2$.*

Proof. From [3], $\forall f \in L_2(\Omega)$, we have

$$\|T_h f - Tf\|_h \le Ch\|Tf\|_2, \quad \|T_h f - Tf\|_0 \le Ch^2\|Tf\|_2,$$

for the Wilson element approximation of the associated source problems. And from well-known a priori estimate for elliptic problems on polygonal domain (see [4]), there exists a constant M independent of f and h, such that

$$\|Tf\|_2 \le M\|f\|_0 \quad \forall f \in L_2(\Omega).$$

Therefore, **C2** is satisfied with $q_1 = 1$ and $q_2 = 2$. Consequently,

$$\|T_h - T\|_0 = \sup_{f\in L_2(\Omega)} \frac{\|T_h f - Tf\|_0}{\|f\|_0} \le Ch^2 \to 0 \ \ (h \searrow 0),$$

i.e., **C1** is satisfied. The desired results are obtained from Theorem 3.4. □

Recently, Zhang et al. [18] proved that, when $\Omega = (0,a) \times (0,b)$ is a rectangular domain and π_h is a regular rectangular mesh, the Wilson nonconforming element eigenvalue provides lower bound of the exact eigenvalue for small enough mesh size. Yang [15] extended the method in [13] to Wilson noncconforming element, and showed by numerical experiments that the first eigenvalue of two-grid discretization scheme approximates exact eigenvalues from below and has much better accuracy than the first eigenvalue which is obtained by solving the eigenvalue problems on a fine mesh K^h directly. In the following theorem, we shall prove this result.

Theorem 4.2. *Suppose Ω is a rectangular domain, π_H and π_h are two regular rectangular meshes. Let λ_k be a simple eigenvalue, let$(\lambda_{k,r}, u_{k,h}^*)$ and $(\lambda_{k,s}, u_{k,h}^*)$ be obtained by Scheme* 1 *and Scheme* 2 *for the Wilson element, respectively. If $h = \mathcal{O}(H^{2-\delta})$ with arbitrarily given $\delta \in (0,1)$, then*

$$\lambda_{k,s} \le \lambda_{k,r} \le \lambda_k, \tag{4.2}$$

$$\lambda_{1,h} \le \lambda_{1,s} \le \lambda_{1,r} \le \lambda_1. \tag{4.3}$$

Proof. It follows from the proof of Theorem 4.1 that **C1** and **C2** are satisfied with $q_1 = 1$ and $q_2 = 2$. From Lemma 3.8 in [5], we have

$$a_h(u_k - I_h u_k, u_{k,h}^*) = \frac{1}{3}\sum_{e\in\pi_h}(h_e^2 + k_e^2)\int_e \partial_{11}u_k\partial_{22}u_k dx + \mathcal{O}(h^3 + h^2H^2),$$

where the eigenfunction is known as

$$u_k(x_1, x_2) = \sin\frac{k_1\pi}{a}x_1 \sin\frac{k_2\pi}{b}x_2 / \|\sin\frac{k_1\pi}{a}x_1 \sin\frac{k_2\pi}{b}x_2\|_0.$$

Therefore, $\partial_{11}u_k\partial_{22}u_k > 0$ on Ω, and we have

$$a_h(u_k - I_h u_k, u_{k,h}^*) = \mathcal{O}(h^2),$$
$$a_h(u_k - I_h u_k, u_{k,h}^*) > 0.$$

It follows from (3.17a) and (3.17f) that

$$2\frac{\lambda_{k,r}}{\lambda_{k,H}} \to 2 \ (H \to 0).$$

By the above three relations we get

$$2\frac{\lambda_{k,r}}{\lambda_{k,H}} a_h(u_k - I_h u_k, u_{k,h}^*) > 0.$$

By (3.17a) and (3.17d) we get

$$\frac{\lambda_{k,r}}{\lambda_k\lambda_{k,H}^2}\|\lambda_{k,H}u_k - \lambda_k u_{k,h}^*\|_h^2 \le C(h^2 + H^4).$$

By (3.17b) and the interpolation error estimate (see [3]), we get

$$\left|-\lambda_{k,r}\|I_h u_k - u_{k,H}\|_0^2\right| \le CH^4,$$
$$\left|\|I_h u_k\|_0^2 - \|u_k\|_0^2\right| = \left|\int (I_h u_k - u_k)(I_h u_k + u_k)dx\right| \le Ch^3.$$

Combining the above four relations and (3.14), when h is of lower order than H^2, the sign of $\lambda_k - \lambda_{k,r}$ is determined by the first and second term on the right hand side of (3.14). Therefore, the eigenvalue $\lambda_{k,r}$ gives lower bound of the exact eigenvalue λ_k for sufficiently small H. Consequently, from Theorem 3.3 we get (4.2) and (4.3). □

4.2. Crouzeix-Raviart Nonconforming Element

Consider eigenvalue problem (4.1) and its weak form (2.1), where $\Omega \subset R^2$ is a polygonal domain with the maximum internal angle ω.

If $\omega > \pi$, we denote $r_0 = \frac{\pi}{\omega}$, $r < r_0$ and sufficiently close to r_0, $p = 2/(2-r)$; and if $\omega < \pi$, we denote $r_0 = r = 1$ and $p = 2$.

Let π_H and π_h be regular triangular meshes of Ω, and V_h be the piecewise linear Crouzeix-Raviart nonconforming element space on π_h. The Crouzeix-Raviart nonconforming element approximation of (4.1) is (2.2).

Theorem 4.3. *Let* $(\lambda_{k,H}, u_{k,H})$ *be a Crouzeix-Raviart element eigenpair of* (4.1) *with* $\|u_{k,H}\|_0 = 1$*, and let* $(\lambda_{k,r}, u_{k,h}^*)$ *and* $(\lambda_{k,s}, u_{k,h}^*)$ *be obtained by using Scheme* 1 *and Scheme* 2 *for the Crouzeix-Raviart element, respectively. Then there exists a function* $u_k \in M(\lambda_k)$ *with* $\|u_k\|_0 = 1$ *such that* (3.17)*) holds with* $q_1 = r$ *and* $q_2 = 2r$.

Proof. It follows from known a priori estimate for elliptic problems on a polygonal domain (see [4]) that

$$Tf \in W_{2,p}(\Omega), \quad \|Tf\|_{2,p} \le M\|f\|_{0,p}, \forall f \in L_p(\Omega).$$

It is proved that u_k can be approximated in the $\|\cdot\|_h$ norm by functions in V_h with order h^r (see, for example [1]). Moreover, from the general theory of the nonconforming element approximation for the associated source problems it follows that **C2** can be satisfied with $q_1 = r$ and $q_2 = 2r$ (see [3, 7]). Consequently,

$$\|T_h - T\|_0 = \sup_{f \in L_2(\Omega)} \frac{\|T_h f - Tf\|_0}{\|f\|_0} \to 0,$$

i.e., **C1** is satisfied. Therefore, the desired results are obtained from Theorem 3.4. □

Armentano and Duran [1] proved that the Crouzeix-Raviart nonconforming element eigenvalue gives lower bound of the exact eigenvalue for small enough mesh size h when the exact eigenfunction is singular and $\|u_{k,h} - u_k\|_h \ge Ch^{r_0}$. Hence, it is natural to assume $\|u_{k,h}^* - u_k\|_h \ge Ch^{r_0}$ in the following theorem.

Theorem 4.4. *Under hypotheses of Theorem* 4.3*, and further suppose the exact eigenfunction is singular and*

$$\|u_{k,h}^* - u_k\|_h \ge Ch^{r_0}, r_0 < 1.$$

Then, if $h = \mathcal{O}(H^{2-\delta})$ *with arbitrarily given* $\delta \in (0, 1)$*, then*

$$\lambda_{k,s} \le \lambda_{k,r} \le \lambda_k, \tag{4.4}$$

$$\lambda_{1,h} \le \lambda_{1,s} \le \lambda_{1,r} \le \lambda_1. \tag{4.5}$$

Proof. It follows from the proof of Theorem 4.3 that **C1** and **C2** are satisfied with $q_1 = r$ and $q_2 = 2r$. By [1] we have

$$a_h(u_k - I_h u_k, u_{k,h}^*) = 0,$$

where I_h denotes the "edge average" interpolation operator. By (3.17b) and the interpolation error estimate (see [1]), we get

$$\left|-\lambda_{k,r}\|I_h u_k - u_{k,H}\|_0^2\right| \le C\left(h^{2+2r} + H^{4r}\right).$$

By (3.17a), (3.17d) and this assumption $\|u_{k,h}^* - u_k\|_h \ge Ch^{r_0}$, we have

$$\begin{aligned}
&\|\lambda_{k,H} u_k - \lambda_k u_{k,h}^*\|_h^2 \\
=&\|\lambda_{k,H} u_k - \lambda_k u_k + \lambda_k u_k - \lambda_k u_{k,h}^*\|_h^2 \\
=&(\lambda_{k,H} - \lambda_k)^2\|u_k\|_h^2 + \lambda_k^2\|u_k - u_{k,h}^*\|_h^2 \\
&+ 2a_h(\lambda_{k,H} u_k - \lambda_k u_k, \lambda_k u_k - \lambda_k u_{k,h}^*) \ge Ch^{2r_0}.
\end{aligned}$$

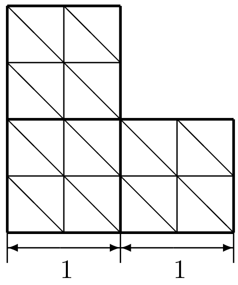

Fig. 4.1. The geometry of Example 1

Table 4.1: Numerical approximations for the first eigenvalue for Example 1

H	h	$\lambda_{1,H}$	$\lambda_{1,h}$	$\lambda_{1,r}$	$\lambda_{1,s}$
$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}$	9.13340	9.46120	9.48594	9.46222
$\frac{\sqrt{2}}{16}$	$\frac{\sqrt{2}}{64}$	9.57482	9.63049	9.63091	9.63052
$\frac{\sqrt{2}}{36}$	$\frac{\sqrt{2}}{216}$	9.61918	9.63797	9.63802	9.63798

Armentano and Duran [1] proved that

$$\begin{aligned}\left|\|I_h u_k\|_0^2 - \|u_k\|_0^2\right| &= \left|\int_\Omega (I_h u_k - u_k)(I_h u_k + u_k)dx\right| \\ &\le C\|u_k\|_{0,\infty}\|I_h u_k - u_k\|_{0,1} \le Ch^2.\end{aligned}$$

Combining the above four relations and (3.14), when $h = \mathcal{O}(H^{2-\delta})$ we conclude that the sign of $\lambda_k - \lambda_{k,r}$ is determined by the second term on the right hand side of (3.14). Therefore, the eigenvalue $\lambda_{k,r}$ gives lower bounds of the exact eigenvalue λ_k for sufficiently small H. Consequently, from Theorem 3.3 we get (4.4) and (4.5). $\square$

Example 1. Consider the eigenvalue problem (4.1).

Let $\Omega = [0,2]\times[0,1]\cup[0,1]\times[1,2]$ be the L-shaped domain, see Figure 4.1. For this domain, the first eigenvalue is $\lambda_1 = 9.6397\cdots$. In Figure 4.1 we show an initial mesh, and refine the initial mesh in a uniform way (each triangle is divided into four congruent triangles) to get meshes π_H and π_h. We compute the first eigenvalue by MATLAB 6.5. The numerical results are shown in Table 1.

From Table 4.1 we see that both $\lambda_{1,r}$ and $\lambda_{1,s}$ not only give lower bounds of the exact eigenvalue λ_1, but also have much better accuracy than that for the $\lambda_{1,h}$.

4.3. Adini Nonconforming Finite Element

Consider vibration plate problem (2.1), where $V = H_0^2(\Omega), W = L_2(\Omega)$,

$$a(u,v) = \int_\Omega \Big(\sigma\Delta u\Delta v + (1-\sigma)(2\partial_{12}u\partial_{12}v + \partial_{11}u\partial_{11}v + \partial_{22}u\partial_{22}v)\Big)dx,$$

$$b(u,v) = \int_\Omega uvdx, \quad \|u\|_b = \|u\|_0,$$

Ω is a polygonal domain in R^2 with boundary $\partial\Omega$, σ is the Poisson coefficient. It follows from [3] that the $a(\cdot,\cdot)$ is a symmetric, continuous and $H_0^2(\Omega)$-elliptic bilinear form on $H_0^2(\Omega)\times H_0^2(\Omega)$.

Let V_h be a nonconforming Adini element space. The nonconforming Adini element approximation of the vibration plate problem (2.1) reads (2.2), where

$$a_h(u_h,v)=\sum_{e\in\pi_h}\int_e\Big(\sigma\Delta u_h\Delta v+(1-\sigma)(2\partial_{12}u_h\partial_{12}v+\partial_{11}u_h\partial_{11}v+\partial_{22}u_h\partial_{22}v)\Big)dx.$$

Rannacher [6] discussed the nonconforming finite element approximation of the vibration plate problem (2.1), and proved the error estemates of the Adini rectangle element.

Based on Ciarlet [3], we next prove directly the error estemates of the Adini rectangle element by Lemma 2.1 for the vibration plate problem (2.1).

Theorem 4.5. *Suppose $M(\lambda_k)\subset H^3(\Omega)$, π_H and π_h are two regular rectangular meshes. Let $(\lambda_{k,H},u_{k,H})$ be an Adini element eigenpair with $\|u_{k,H}\|_0=1$, let $(\lambda_{k,r},u_{k,h}^*)$ and $(\lambda_{k,s},u_{k,h}^*)$ be obtained by Scheme 1 and Scheme 2 for the Adini element, respectively. Then there exists a function $u_k\in M(\lambda_k)$ with $\|u_k\|_0=1$ such that (3.17) holds with $q_1=1$ and $q_2=2$.*

Proof. From [3] it follows that **C2** is satisfied with $q_1=1$ and $q_2=2$. Hence, we have $\|T_h-T\|_0\to 0$, i.e., **C1** is satisfied. Therefore, the desired results follow by Theorem 3.4. □

The numerical example provided by Rannacher [6] showed that the Adini element eigenvalue gives lower bound, i.e., $\lambda_{k,h}<\lambda_k$, which is proved by Yang [16]. We now prove that both $\lambda_{k,s}$ and $\lambda_{k,r}$ also give lower bounds for small enough mesh size.

Theorem 4.6. *Suppose $M(\lambda_k)\subset H^4(\Omega)$, and both π_H and π_h are uniform rectangular meshes of a rectangular domain Ω. Let $(\lambda_{k,r},u_{k,h}^*)$ and $(\lambda_{k,s},u_{k,h}^*)$ be obtained by Scheme 1 and Scheme 2 for the Adini element, respectively. If $h=\mathcal{O}(H^{2-\delta})$ with arbitrarily given $\delta\in(0,1)$, then*

$$\lambda_{k,s}\le\lambda_{k,r}\le\lambda_k, \tag{4.6}$$
$$\lambda_{1,h}\le\lambda_{1,s}\le\lambda_{1,r}\le\lambda_1. \tag{4.7}$$

Proof. From [3] it follows that **C2** is satisfied with $q_1=q_2=2$. It follows from the proof of Theorem 4.5 that **C1** is satisfied. Let $D_h=a_h(u_k,u_{k,h}^*)-\lambda_k b(u_k,u_{k,h}^*)$.Then

$$\begin{aligned}&a_h(u_k-I_hu_k,u_{k,h}^*)\\=&a_h(u_k,u_{k,h}^*)-\lambda_kb(u_k,u_{k,h}^*)+\lambda_kb(u_k,u_{k,h}^*)-a_h(I_hu_k,u_{k,h}^*)\\=&D_h+\lambda_kb(u_k,\lambda_{k,H}T_hu_{k,H})-b(I_hu_k,\lambda_{k,H}u_{k,H})\\=&D_h+\lambda_k\lambda_{k,H}b(u_k,T_hu_{k,H})-\lambda_{k,H}b(u_k,u_{k,H})-\lambda_{k,H}b(I_hu_k-u_k,u_{k,H})\\=&D_h+\lambda_k\lambda_{k,H}b(T_hu_k-Tu_k,u_{k,H})-\lambda_{k,H}b(I_hu_k-u_k,u_{k,H}).\end{aligned}\tag{4.8}$$

From [16] it follows that

$$\begin{aligned}&D_h=\mathcal{O}(h^2),\quad D_h\ge 0;\\&\lambda_{j,h}-\lambda_k=\mathcal{O}(h^2),\quad \lambda_k-\lambda_{j,h}\ge 0\quad j=k,k+1,\cdots,k+q-1.\end{aligned}$$

Write $r=b(T_hu_k-Tu_k,u_{k,H}-\bar u_{k,h})$. From **C2**, (3.17b) and Lemma 2.2, we have $|\,r\,|\le CH^2h^2$.

Consequently, from (2.3) and (2.4) we have

$$\begin{aligned}
&b(T_h u_k - Tu_k, u_{k,H}) = b(T_h u_k - Tu_k, \bar{u}_{k,h}) + r\\
=&b\left(T_h u_k - \frac{1}{\lambda_k}u_k, \sum_{j=k}^{k+q-1} b(u_k, u_{j,h})u_{j,h}\right) + r\\
=&b\left(u_k, \sum_{j=k}^{k+q-1} \frac{\lambda_k - \lambda_{j,h}}{\lambda_{j,h}\lambda_k} b(u_k, u_{j,h})u_{j,h}\right) + r\\
=&\sum_{j=k}^{k+q-1} \frac{\lambda_k - \lambda_{j,h}}{\lambda_{j,h}\lambda_k} b(u_k, u_{j,h})^2 + r \geq 0 \quad (H \searrow 0).
\end{aligned}$$

By the interpolation error estimate (see [3]) we have

$$| \, b(I_h u_k - u_k, u_{k,H}) \, | \leq Ch^4.$$

It is derived from combining (4.8) with the above analysis that

$$a_h(u_k - I_h u_k, u_{k,h}^*) > 0.$$

By (3.17a) and (3.17d) we have

$$0 \leq \frac{\lambda_{k,r}}{\lambda_k \lambda_{k,H}^2} \|\lambda_{k,H} u_k - \lambda_k u_{k,h}^*\|_h^2 \leq CH^4.$$

By (3.17b) and the interpolation error estimate (see [3]), we have

$$\begin{aligned}
&\left|-\lambda_{k,r}\|I_h u_k - u_{k,H}\|_0^2\right| \leq CH^4,\\
&\left|\|I_h u_k\|_0^2 - \|u_k\|_0^2\right| = \left|\int (I_h u_k - u_k)(I_h u_k + u_k)dx\right| \leq Ch^4.
\end{aligned}$$

Combining the above four relations with (3.14), we conclude that when h is of lower order than H^2 the sign of $\lambda_k - \lambda_{k,r}$ is determined by the first term on the right hand side of (3.14). Therefore, the eigenvalue $\lambda_{k,r}$ gives lower bounds of the exact eigenvalue λ_k for sufficiently small H. As a result, from Theorem 3.3 we get (4.6) and (4.7). $\square$

5. Concluding Remarks

The EQ_1^{rot} element proposed by Lin, Tobiska and Zhou in 2005 was applied to solving eigenvalue problems in Lin-Lin's book [5], which proved that when Ω is a rectangular domain and π_h is a uniform rectangular mesh, the EQ_1^{rot} element eigenvalues give lower bounds of the exact eigenvalues for small enough mesh size. A similar analysis indicates that it is possible to apply the two-grid method to the EQ_1^{rot} element.

Recently, Morley element, Adini element, Bogner-Fox-Schmit element and Zienkiewicz-type element have been extended to arbitrary dimensions by Wang, Shi and Xu [8,9]. Based on their work, it is possible to propose and analyze two-grid discretization schemes of the nonconforming finite elements for eigenvalue problems with arbitrarily dimensions.

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