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A MULTIVARIATE MULTIQUADRIC QUASI-INTERPOLATION WITH QUADRIC REPRODUCTION*

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Abstract

In this paper, by using multivariate divided differences to approximate the partial derivative and superposition, we extend the multivariate quasi-interpolation scheme based on dimension-splitting technique which can reproduce linear polynomials to the scheme quadric polynomials. Furthermore, we give the approximation error of the modified scheme. Our multivariate multiquadric quasi-interpolation scheme only requires information of location points but not that of the derivatives of approximated function. Finally, numerical experiments demonstrate that the approximation rate of our scheme is significantly improved which is consistent with the theoretical results.

Mathematics subject classification: 41A05, 41A25.
Key words: Quasi-interpolation, Multiquadric functions, Polynomial reproduction, $\mathcal{P}_n$-exact A-discretization of $\mathcal{D}^\alpha$, Approximation error.

1. Introduction

The approximation of multivariate functions from scattered data is an important theme in numerical mathematics. One of the methods to attack this problem is quasi-interpolation. For a set of functional values $\{f(X_j)\}_{1\le j\le n}$ taken on a set of nodes $\Xi = \{X_1, X_2, \cdots, X_n\} \subseteq \mathbb{R}^d$, the form of quasi-interpolation function $Q_f(X)$ corresponding to $f(X)$ is as follows

$$Q_f(X) = \sum_{j=1}^{n} f(X_j)\varphi_j(X), \tag{1.1}$$

where $\{\varphi_j(X)\}$ is a set of quasi-interpolation basis functions. Using quasi-interpolation there is no need to solve large algebraic systems. The approximation properties of quasi-interpolants in the case that X_j are the nodes of a uniform grid are well-understood. For example, the quasi-interpolant

$$\sum_{j=1}^{n} f(jh)\varphi\Big(\frac{X-hj}{h}\Big)$$

can be studied via the theory of principal shift-invariant spaces, which has been developed in several articles by de Boor et al. (see, e.g., [1,2]). Here φ is supposed to be a compactly

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supported or rapidly decaying function. Based on the Strang-Fix condition for φ, which is equivalent to polynomial reproduction, convergence and approximation orders for several classes of basis functions were obtained (see also [3-5]). Scattered data quasi-interpolation by functions, which reproduces polynomials, has been studied by Buhmann *et al.* [6], Dyn and Ron [7], Wu and Schaback [8], Feng and Li [9], Wu and Liu [10], and Wu and Xiong [11].

Beast and Powell [12] first proposed a univeriate quasi-interpolation formula where φ_i in (1) is a linear combination of the Hardy's MQ basis [13]

$$\phi_i(x) = \sqrt{(x-x_i)^2 + c^2}, \quad x, x_i \in \mathbb{R}$$

and low order polynomials. Their formula requires the derivative informations of f at the endpoints, which is not convenient for practical purposes. Wu and Schaback [8] proposed another quasi-interpolation formula with modifications at the endpoints. Wu-Schaback's formula is given by

$$\mathcal{L}_D f(x) = \sum_{i=0}^{n} f_i \alpha_i(x), \tag{1.2}$$

where $f_i, i = 0, \cdots, n$ are the values of $f(x)$ at nodes $\{x_i\}$ and the interpolation kernel $\alpha_i(x)$ is also formed from linear combinations of the MQ basis functions, plus a constant, and linear polynomial:

$$\begin{aligned}
\alpha_0(x) &= \frac{1}{2} + \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)}, && (1.3a)\\
\alpha_1(x) &= \frac{\phi_2(x) - \phi_1(x)}{2(x_2 - x_1)} - \frac{\phi_1(x) - (x - x_0)}{2(x_1 - x_0)}, && (1.3b)\\
\alpha_i(x) &= \frac{\phi_{i+1}(x) - \phi_i(x)}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad 2 \le i \le n-2, && (1.3c)\\
\alpha_{n-1}(x) &= \frac{(x_n - x) - \phi_{n-1}(x)}{2(x_n - x_{n-1})} - \frac{\phi_{n-1}(x) - \phi_{n-2}(x)}{2(x_{n-1} - x_{n-2})}, && (1.3d)\\
\alpha_n(x) &= \frac{1}{2} + \frac{\phi_{n-1}(x) - (x_n - x)}{2(x_n - x_{n-1})}. && (1.3e)
\end{aligned}$$

It is shown that (1.2) preserves monotonicity and convexity, and converges with a rate of $\mathcal{O}(h^{2.5} \log h)$ as $c = \mathcal{O}(h)$.

Ling [14] extended the univariate quasi-interpolation formula (1.2) to multidimensions using the dimension-splitting multiquadric basis function approach. Given data $\{(x_i, y_j, f_{ij}), i = 0, 1, \cdots, n, j = 0, 1, \cdots, m\}$, the form of dimension-splitting quasi-interpolation for MQ basis function is

$$\Phi_1 f(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} f_{ij} \alpha_i(x) \beta_j(y), \tag{1.4}$$

where $\alpha_i(x), i = 0, 1, \cdots, n$ are given by (1.3). Along that y direction, the basis functions $\beta_j(y)$

are defined as follows

$$\beta_0(y) = \frac{1}{2} + \frac{\phi_1(y) - (y - y_0)}{2(y_1 - y_0)}, \tag{1.5a}$$

$$\beta_1(y) = \frac{\phi_2(y) - \phi_1(y)}{2(y_2 - y_1)} - \frac{\phi_1(y) - (y - y_0)}{2(y_1 - y_0)}, \tag{1.5b}$$

$$\beta_j(y) = \frac{\phi_{j+1}(y) - \phi_j(y)}{2(y_{j+1} - y_j)} - \frac{\phi_j(y) - \phi_{j-1}(y)}{2(y_j - y_{j-1})}, \quad 2 \le j \le m-2, \tag{1.5c}$$

$$\beta_{m-1}(y) = \frac{(y_m - y) - \phi_{m-1}(y)}{2(y_m - y_{m-1})} - \frac{\phi_{m-1}(y) - \phi_{m-2}(y)}{2(y_{m-1} - y_{m-2})}, \tag{1.5d}$$

$$\beta_m(y) = \frac{1}{2} + \frac{\phi_{m-1}(y) - (y_m - y)}{2(y_m - y_{m-1})}. \tag{1.5e}$$

Ling did not give the error estimate of $\Phi_1 f(x,y)$ to $f(x,y)$. We may verify that the form of (1.4) has the property of linear reproduction.

Theorem 1.1. *The scheme $\Phi_1 f$ satisfies the property of linear reproduction.*

Proof. In virtue of the property of constant and linear reproduction of (1.2), we have

$$\sum_{i=0}^{n} \alpha_i(x) = 1, \quad \sum_{i=0}^{n} x_i \alpha_i(x) = x. \tag{1.6}$$

Similarly

$$\sum_{j=0}^{m} \beta_j(y) = 1, \quad \sum_{j=0}^{m} y_j \beta_j(y) = y. \tag{1.7}$$

Then when $f(x,y) = c$ with c being a constant, we have

$$\sum_{i=0}^{n}\sum_{j=0}^{m} c\alpha_i(x)\beta_j(y) = c\Big(\sum_{i=0}^{n} \alpha_i(x)\Big)\Big(\sum_{j=0}^{m} \beta_j(y)\Big) = c; \tag{1.8}$$

when $f(x,y) = x$,

$$\sum_{i=0}^{n}\sum_{j=0}^{m} x_i\alpha_i(x)\beta_j(y) = \Big(\sum_{i=0}^{n} x_i\alpha_i(x)\Big)\Big(\sum_{j=0}^{m} \beta_j(y)\Big) = x;$$

similarly when $f(x,y) = y$,we have $(\Phi_1 f)(x,y) = y$. □

From Theorem 1.1, we see that the scheme $\Phi_1 f$ can only reproduce linear polynomial, consequently the approximation rate requires increasing. In this paper, we modify the scheme $\Phi_1 f$ and give a quasi-interpolation operator reproducing quadric polynomial and give the approximation error of the proposed scheme and find when $c = \mathcal{O}(h)$, the approximation rate can reach up to $\mathcal{O}(h^3)$, $h = \max\{h_1, h_2\}$, $2h_1 = \max_{0\le i\le n-1} |x_{i+1} - x_i|$, $2h_2 = \max_{0\le j\le m-1} |y_{j+1} - y_j|$. In addition, the proposed scheme doesn't require the derivatives of f. Some numerical experiments are shown that the approximate rate of $\Phi_2 f$ is far higher than that of $\Phi_1 f$.

The organization of the paper is as follows: Section 2 gives some notions; Section 3 presents the construction of a quasi-interpolation operator with quadric reproduction; Section 4 makes its error estimate; Section 5 is numerical experiments; Section 6 is the conclusion part.

2. Preliminaries

Definition 2.1. *Let $k \in \mathbb{N}, c > 0$. Multiquadric functions of degree $2k-1(2k$ order$)$ are defined by*

$$\phi(X;2k) = \Big(\|X\|_2^2 + c^2\Big)^{(2k-1)/2}, \quad X = (x_1, \cdots, x_d) \in \mathbb{R}^d, \tag{2.1}$$

where c is called shape parameter, $\mathbb{R}^d$ is d-dimensional Euclidean space, $\|\cdot\|$ is Euclidean norm, and

$$\phi_{j,2k}(X) := \phi(X - X_j; 2k) \tag{2.2}$$

is a shift of $\phi(X;2k)$ centered at X_j.

Definition 2.2. *For given point set $\{X_i = (x_{i1}, \cdots, x_{id})\}_{i=0}^n \subset \Omega$ and a set of values $\{f(X_i)\}_{i=0}^n$, Ω is a bounded domain on $\mathbb{R}^d$, the quasi-interpolation $(Lf)(X)$ of a function $f : \mathbb{R}^d \to \mathbb{R}$ is defined as follows:*

$$(\mathcal{L}f)(X) = \sum_{j=0}^{n} f(X_j)\alpha_j(X), \quad X \in \mathbb{R}^d, \tag{2.3}$$

where $\alpha_j(X)$ are quasi-interpolation basis functions.

Definition 2.3. *For any real-coefficient polynomial $p(X)$ of degree m , if $(\mathcal{L}p)(X) = p(X)$, then quasi-interpolation $(\mathcal{L}f)(X)$ is called to be satisfying the reproduction property of polynomials of degree $\le m$.*

Definition 2.4. ([15]) *Suppose $\mathcal{F} = \{f | f : \mathbb{R}^d \to \mathbb{R}\}$, A is a discrete subset of $\mathbb{R}^d$, $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d) \in Z_+^d$, $D^\alpha = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_d}$ is the derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $\mathcal{P}_n = \mathcal{P}_n(\mathbb{R}^d)$ is the set of multivariate polynomials of degree $\le n$. An operator $\mathcal{D}_A^\alpha : \mathcal{F} \to \mathcal{F}$ is said to be a $\mathcal{P}_n$-exact A-discretization of $\mathcal{D}^\alpha$ if*

(a) There exists a real vector $\lambda = (\lambda_a)_{a \in A}$ such that, for any $f \in \mathcal{F}$,

$$(\mathcal{D}_A^\alpha f)(X) = \sum_{a \in A} \lambda_a f(X + a); \tag{2.4}$$

(b) For any $p \in \mathcal{P}_n$, $\mathcal{D}_A^\alpha p = \mathcal{D}^\alpha p$.

In such a situation, we also say that $\mathcal{D}_A^\alpha f$ is a $\mathcal{P}_n$-exact A-discretization of $\mathcal{D}^\alpha f$. If the points in set A are properly posed for $\mathcal{P}_n$, then $\mathcal{D}_A^\alpha$ is determined uniquely.

3. Quasi-Interpolation Operators with Quadric Reproduction

For given data $\{(x_i, y_j, f_{ij}), i = 0, 1, \cdots, n, j = 0, 1, \cdots, m\}$, we define a quasi-interpolation operator as follows:

$$(\tilde{\Phi}_2 f)(x,y) = \sum_{i=0}^{n} \sum_{j=0}^{m} \Big(f_{ij} + \frac{1}{2}(x - x_i) f_x(x_i, y_j) + \frac{1}{2}(y - y_j) f_y(x_i, y_j)\Big) \alpha_i(x) \beta_j(y). \tag{3.1}$$

Theorem 3.1. *Quasi-interpolation operator $\tilde{\Phi}_2 f$ satisfies quadric polynomial reproduction property.*

Proof. By Theorem 1, we know that $\Phi_1 f$ reproduces linear polynomial, i.e.,

$$(\Phi_1 f)(x,y) = ax+by+c, \quad \text{if} \quad f(x,y)=ax+by+c, \quad a,\ b,\ c \ \text{are constants.}$$

Then when $f(x,y)=xy$, we have

$$\begin{aligned}(\tilde{\Phi}_2 f)(x,y) &= \sum_{i=0}^{n}\sum_{j=0}^{m}\Big(x_iy_j+\frac{1}{2}(x-x_i)y_j+\frac{1}{2}(y-y_j)x_i\Big)\alpha_i(x)\beta_j(y)\\ &= \sum_{i=0}^{n}\sum_{j=0}^{m}(\frac{1}{2}xy_j+\frac{1}{2}x_iy)\alpha_i(x)\beta_j(y)=xy,\end{aligned}$$

and when $f(x,y)=x^2$, we have

$$(\tilde{\Phi}_2 f)(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{m}\Big(x_i^2+\frac{1}{2}(x-x_i)(2x_i)\Big)\alpha_i(x)\beta_j(y)=x^2.$$

For $f(x,y)=y^2$, we can similarly prove that $(\tilde{\Phi}_2 f)(x,y)=y^2$. □

Although the quasi-interpolation operator $\tilde{\Phi}_2 f$ satisfies quadric polynomial reproduction property, it requires the first derivatives of the approximated function $f(x,y)$ in the process of using, which are very difficult to measure in practice. Hence, we shall use multivariate divided difference operator $D_A^{\alpha} f$ defined in Definition 4 to replace first-order partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ in $\tilde{\Phi}_2 f$. In the following, we introduce the specific computing formula of multivariate divided differences which approximate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Suppose the point set $A=\{Q_1(x_1,y_1),Q_2(x_2,y_2),\cdots,Q_6(x_6,y_6)\}$ is posed for $\mathcal{P}_2$, that is to say, there doesn't exist a nonzero polynomial in $\mathcal{P}_2$ which vanishes on all the points of A, where $\mathcal{P}_2$ is the bivariate polynomial space of polynomials of total degree ≤ 2. The computing formula of divided difference $f[A]^{(1,0)}$ defined by paper [15, Sec.3] is given by

$$f[A]^{(1,0)} = \frac{1}{(1,0)!}D_A^{(1,0)}f(0)=\sum_{i=1}^{6}\lambda_i f(Q_i), \tag{3.2}$$

where $(\alpha_1,\alpha_2)!=\alpha_1!\alpha_2!$ and coefficient λ_i is determined by the following equations

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1\\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6\\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6\\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6\\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2\\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 \end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\\ \lambda_6\end{pmatrix}=\begin{pmatrix}0\\ 1\\ 0\\ 0\\ 0\\ 0\end{pmatrix}. \tag{3.3}$$

Therefore, we have

$$D_A^{(1,0)}f(X)=\sum_{i=1}^{6}\lambda_i f(Q_i+X), \quad X=(x,y)\in\mathbb{R}^2.$$

According to (7) and (8), the computing formula of divided difference $f[A_X]^{(1,0)}$ on six points $A_X = \{Q_1 - X, Q_2 - X, Q_3 - X, Q_4 - X, Q_5 - X, Q_6 - X\}$ is given by

$$f[A_X]^{(1,0)} = \frac{1}{(1,0)!} D_{A_X}^{(1,0)} f(0) = \sum_{i=1}^{6} \lambda_{i,X} f(Q_i - X),$$

where coefficients $\{\lambda_{i,X}\}_{i=1}^{6}$ are determined by the following equations

$$M \begin{pmatrix} \lambda_{1,X} \\ \lambda_{2,X} \\ \lambda_{3,X} \\ \lambda_{4,X} \\ \lambda_{5,X} \\ \lambda_{6,X} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{3.4}$$

where

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1-x & x_2-x & x_3-x & x_4-x & x_5-x & x_6-x \\ y_1-y & y_2-y & y_3-y & y_4-y & y_5-y & y_6-y \\ (x_1-x)(y_1-y) & (x_2-x)(y_2-y) & (x_3-x)(y_3-y) & (x_4-x)(y_4-y) & (x_5-x)(y_5-y) & (x_6-x)(y_6-y) \\ (x_1-x)^2 & (x_2-x)^2 & (x_3-x)^2 & (x_4-x)^2 & (x_5-x)^2 & (x_6-x)^2 \\ (y_1-y)^2 & (y_2-y)^2 & (y_3-y)^2 & (y_4-y)^2 & (y_5-y)^2 & (y_6-y)^2 \end{pmatrix}. \tag{3.5}$$

Therefore, we have

$$D_{A_X}^{(1,0)} f(X) = \sum_{i=1}^{6} \lambda_{i,X} f(Q_i). \tag{3.6}$$

Similarly, on the point set $A = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$, the computing formula of divided difference $f[A]^{(0,1)}$ defined by [15] is given by

$$f[A]^{(0,1)} = \frac{1}{(0,1)!} D_{A}^{(0,1)} f(0) = \sum_{i=1}^{6} \eta_i f(Q_i), \tag{3.7}$$

where coefficients $\{\eta_i\}_{i=1}^{6}$ are determined by the following equations

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ y_1 & y_2 & y_3 & y_4 & y_5 & y_6 \\ x_1y_1 & x_2y_2 & x_3y_3 & x_4y_4 & x_5y_5 & x_6y_6 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 & x_5^2 & x_6^2 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 & y_5^2 & y_6^2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \\ \eta_6 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{3.8}$$

Therefore, we have

$$D_{A}^{(0,1)} f(X) = \sum_{i=1}^{6} \eta_i f(Q_i + X). \tag{3.9}$$

According to (3.6) and (3.7), the computing formula of divided difference $f[A_X]^{(0,1)}$ on six points $A_X = \{Q_1 - X, Q_2 - X, Q_3 - X, Q_4 - X, Q_5 - X, Q_6 - X\}$ is given by

$$f[A_X]^{(0,1)} = \frac{1}{(0,1)!} D_{A_X}^{(0,1)} f(0) = \sum_{i=1}^{6} \eta_{i,X} f(Q_i - X), \tag{3.10}$$

where coefficients $\{\eta_{i,X}\}_{i=1}^{6}$ are determined by the following equations

$$M\begin{pmatrix}\eta_{1,X}\\ \eta_{2,X}\\ \eta_{3,X}\\ \eta_{4,X}\\ \eta_{5,X}\\ \eta_{6,X}\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\\0\\0\end{pmatrix}, \tag{3.11}$$

where the matrix M is defined by (3.4). Therefore, we have

$$D_{A_X}^{(0,1)}f(X)=\sum_{i=1}^{6}\eta_{i,X}f(Q_i). \tag{3.12}$$

According to [15] and Definition 2.4 we know that

$$D_{A_X}^{(1,0)}f(X)\approx\frac{\partial f}{\partial x}(X),\quad D_{A_X}^{(0,1)}f(X)\approx\frac{\partial f}{\partial y}(X),\quad X\in\mathbb{R}^2, \tag{3.13}$$

and when $f(x,y)=1,x,y,x^2,xy,y^2$, we have

$$D_{A_X}^{(1,0)}f(X)=\frac{\partial f}{\partial x}(X),\quad D_{A_X}^{(0,1)}f(X)=\frac{\partial f}{\partial y}(X). \tag{3.14}$$

Now writing $X_{ij}=(x_i,y_j)$, we use $D_{A_{X_{ij}}}^{(1,0)}f(x_i,y_j)$ and $D_{A_{X_{ij}}}^{(0,1)}f(x_i,y_j)$ to substitute $f_x(x_i,y_j)$ and $f_y(x_i,y_j)$ in the scheme $\tilde{\Phi}_2 f$ in (3.1) respectively, then get the following scheme $\Phi_2 f$

$$\begin{aligned}(\Phi_2 f)(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(f_{ij}+&\frac{1}{2}(x-x_i)D_{A_{X_{ij}}}^{(1,0)}f(x_i,y_j)\\ &+\frac{1}{2}(y-y_j)D_{A_{X_{ij}}}^{(0,1)}f(x_i,y_j)\Big)\alpha_i(x)\beta_j(y).\end{aligned} \tag{3.15}$$

Theorem 3.2. *Quasi-interpolation operator $\Phi_2 f$ satisfies quadric polynomial reproduction property.*

Proof. When $f(x,y)=1$, by (3.14) and the linear reproduction property of $\Phi_1 f$, we have

$$(\Phi_2 f)(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{m}\alpha_i(x)\beta_j(y)\equiv 1; \tag{3.16}$$

when $f(x,y)=x$, according to (3.14), we have

$$(\Phi_2 f)(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(x_i+\frac{1}{2}(x-x_i)\Big)\alpha_i(x)\beta_j(y)=\sum_{i=0}^{n}\frac{1}{2}(x+x_i)\alpha_i(x)=x; \tag{3.17}$$

in view of (3.14) and the linear reproduction property of $\Phi_1 f$, when $f(x,y)=y,x^2,xy,y^2$, we can obtain

$$(\Phi_2 f)(x,y)=f(x,y). \tag{3.18}$$

Now we have proven $\Phi_2 f$ satisfies quadric polynomial reproduction property and complete the proof. □

4. Estimating the Approximation Error of $\Phi_2 f$

Suppose Ω is a bounded domain which contains the point set $\{(x_i, y_j), i = 0, 1, \cdots, n, j = 0, 1, \cdots, m\}$ in $\mathbb{R}^2$. Note

$$\| D^k f \|_\infty = \max_{|\beta|=k} \Big(\sup_{X\in\Omega} |(D^\beta f)(X)| \Big),$$

$2h_1 = \max_{0\le i\le n-1} |x_{i+1} - x_i|$, $2h_2 = \max_{0\le j\le m-1} |y_{j+1} - y_j|$. For simplicity, assume $x_0 < x_1 < \cdots < x_n$, $y_0 < y_1 < \cdots < y_m$. Let $\phi_{-1}(x) = |x - x_{-1}|$, $\phi_0(x) = |x - x_0|$, $\phi_n(x) = |x - x_n|$, $\phi_{n+1}(x) = |x - x_{n+1}|$, $x_{-1} < x_0$, $x_{n+1} > x_n$, $x \in [x_0, x_n]$. Then $\mathcal{L}_D f$ can be written as follows:

$$(\mathcal{L}_D f)(x) = \sum_{i=0}^{n} f(x_i)\alpha_i(x),$$

where

$$\alpha_i(x) = \frac{\phi_{i+1}(x) - \phi_i(x)}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}(x)}{2(x_i - x_{i-1})}, \quad 0 \le i \le n. \tag{4.1}$$

By considering $\alpha_i(x)$ as the second divided difference of $\phi_i(x)$, we have

$$\begin{aligned}\alpha_i(x) &= \frac{\phi_{i+1}(x) - \phi_i(x)}{2(x_{i+1} - x_i)} - \frac{\phi_i(x) - \phi_{i-1}(x)}{2(x_i - x_{i-1})} \\ &= \frac{1}{4}\frac{c^2}{[(x-\xi_i)^2 + c^2]^{3/2}}(x_{i+1} - x_{i-1}), \quad \xi_i \in (x_{i-1}, x_{i+1})\end{aligned} \tag{4.2}$$

when $x \ne \xi_i$, we have

$$\alpha_i(x) \le \frac{1}{2}c^2(x_{i+1} - x_{i-1})|x - \xi_i|^{-3} \le 2c^2 h_1 |x - \xi_i|^{-3}, \quad \xi_i \in (x_{i-1}, x_{i+1}).$$

Let

$$N_1 = \left[\frac{1}{2h_1}\right] + 1, \quad N_2 = \left[\frac{1}{2h_2}\right] + 1, Q_\rho(u) = (u - \rho, u + \rho], \quad u \in [0, 1], \rho > 0,$$

and

$$T_k = Q_h(x - 2kh) \cup Q_h(x + 2kh), \quad k = 0, 1, \cdots, N,$$

where $[\cdot]$ denotes the integer part of the argument. The set $\cup_{k_1=-N_1}^{N_1} Q_{k_1}(x + 2k_1h_1)$ is a covering of $[x_0, x_n]$ with half open intervals (the set $\cup_{k_2=-N_2}^{N_2} Q_{k_2}(x + 2k_2h_2)$ is a covering of $[y_0, y_m]$ with half open intervals). Therefore, for each $i \in \{0, 1, \cdots, n\}$ ($j \in \{0, 1, \cdots, m\}$) there exists a unique $k_1 \in \{0, 1, \cdots, N_1\}$ ($k_2 \in \{0, 1, \cdots, N_2\}$) such that $x_i \in T_{k_1}$ ($y_j \in T_{k_2}$), and the following inequalities hold

$$\begin{aligned}
&(2k_1 - 1)h_1 \le |x - x_i| \le (2k_1 + 1)h_1, && \text{for} \quad k_1 \ge 1;\\
&\big(2(k_1 - 1) - 1\big)h_1 \le |x - \xi_i| \le \big(2(k_1 + 1) + 1\big)h_1, \quad \forall \xi_i \in (x_{i-1}, x_{i+1}), && \text{for} \quad k_1 \ge 2.\\
&(2k_2 - 1)h_2 \le |y - y_j| \le (2k_2 + 1)h_2, && \text{for} \quad k_2 \ge 1;\\
&\big(2(k_2 - 1) - 1\big)h_2 \le |y - \xi_j| \le \big(2(k_2 + 1) + 1\big)h_2, \quad \forall \xi_j \in (y_{j-1}, y_{j+1}), && \text{for} \quad k_2 \ge 2.
\end{aligned}$$

Let $T_x = \{x_0, \cdots, x_n\}$, $T_y = \{y_0, \cdots, y_m\}$ and use $|Q_h(x) \cap T_x|$ to denote the number of intersection point of $Q_h(x)$ and T_x. We define

$$\overline{M}_x = \max_{x\in[x_0,x_n]} |Q_{h_1}(x) \cap T_x|, \quad \overline{M}_y = \max_{y\in[y_0,y_m]} |Q_{h_2}(x) \cap T_y|,$$

we have

$$\begin{cases}1 \le |T_0 \cap T_x| \le \overline{M}_1, \\ 1 \le |T_{k_1} \cap T_x| \le 2\overline{M}_1, \;\; k_1 = 1, 2, \cdots, N_1.\end{cases} \qquad \begin{cases}1 \le |T_0 \cap T_y| \le \overline{M}_2, \\ 1 \le |T_{k_2} \cap T_y| \le 2\overline{M}_2, \;\; k_2 = 1, 2, \cdots, N_2.\end{cases}$$

Theorem 4.1. *For any function $f(x, y) \in C^3(\Omega)$, if its relevant order of derivatives are bounded on Ω, then*

$$\begin{aligned}\|(\Phi_2 f)(x, y) - f(x, y)\|_\infty = \mathcal{O}\bigg(& c^{-2}\Big(h_1^3 h_2^2 + h_1^2 h_2^3\Big) + c^{-1}\Big(h_1^4 + h_2^4\Big) + c^2 \\ &+ c\Big(h_1^3 h_2^{-1} + h_1^2 h_2^{-1} + h_2^2 h_1^{-1} + h_1^{-1} h_2^3\Big) + c^2\Big(h_1^2 + h_2^2\Big) + c^4 h_1^{-1} h_2^{-1}\bigg). \end{aligned} \tag{4.3}$$

Proof.

$$\begin{aligned}\Big|(\Phi_2 f)(x, y) - f(x, y)\Big| &= \Big|(\Phi_2 f)(x, y) - (\tilde{\Phi}_2 f)(x, y) + (\tilde{\Phi}_2 f)(x, y) - f(x, y)\Big| \\ &\le \Big|(\Phi_2 f)(x, y) - (\tilde{\Phi}_2 f)(x, y)\Big| + \Big|(\tilde{\Phi}_2 f)(x, y) - f(x, y)\Big|.\end{aligned}$$

The point set $A = \{Q_1, \cdots, Q_6\}$ for computing $D^{(1,0)}_{A_{X_{ij}}} f(x_i, y_j)$ and $D^{(0,1)}_{A_{X_{ij}}} f(x_i, y_j)$ is chosen by the lattice in Fig. 5.1 given in Section 5. In this case, by [15], we have

$$\Big|D^{(1,0)}_{A_{X_{ij}}} f(x_i, y_j) - f_x(x_i, y_j)\Big| \le \frac{32}{3} C_1 h_1^3 \|D^3 f\|_{\infty,\Omega}, \tag{4.4a}$$

$$\Big|D^{(0,1)}_{A_{X_{ij}}} f(x_i, y_j) - f_y(x_i, y_j)\Big| \le \frac{32}{3} C_2 h_2^3 \|D^3 f\|_{\infty,\Omega}, \tag{4.4b}$$

where constants C_1, C_2 are related to the point set A. In following error estimation we still use the same symbol C_1, C_2 to denote these constants in upper bound, but their values maybe different at different places. Note that,

$$\begin{aligned}&\Big|(\Phi_2 f)(x, y) - (\tilde{\Phi}_2 f)(x, y)\Big| \\ \le& \Big|\sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{1}{2}(x - x_i)\big(D^{(1,0)}_{A_{X_{ij}}} f(x_i, y_j) - f_x(x_i, y_j)\big) \\ &\quad + \frac{1}{2}(y - y_j)\big(D^{(0,1)}_{A_{X_{ij}}} f(x_i, y_j) - f_y(x_i, y_j)\big)\Big)\alpha_i(x)\beta_j(y) \\ \le& \sum_{i=0}^{n}\sum_{j=0}^{m}\Big(\frac{1}{2}|x - x_i|\frac{32}{3} C_1 h_1^3 \|D^3 f\|_{\infty,\Omega} + \frac{1}{2}|y - y_j|\frac{32}{3} C_2 h_2^3 \|D^3 f\|_{\infty,\Omega}\Big)\alpha_i(x)\beta_j(y) \\ \le& \frac{16}{3} C_1 h_1^3 \|D^3 f\|_{\infty,\Omega} \sum_{i=0}^{n} |x - x_i|\alpha_i(x) + \frac{16}{3} C_2 h_2^3 \|D^3 f\|_{\infty,\Omega} \sum_{j=0}^{m} |y - y_j|\beta_j(y) \\ \le& \frac{16}{3} C_1 h_1^3 \|D^3 f\|_{\infty,\Omega}\Big(\sum_{k_1=0}^{1}\sum_{x_i \in T_{k_1}} \alpha_i(x)|x - x_i| + \sum_{k_1=2}^{N_1}\sum_{x_i \in T_{k_1}} \alpha_i(x)|x - x_i|\Big) \\ &+ \frac{16}{3} C_2 h_2^3 \|D^3 f\|_{\infty,\Omega}\Big(\sum_{k_2=0}^{1}\sum_{y_j \in T_{k_2}} \beta_j(y)|y - y_j| + \sum_{k_2=2}^{N_2}\sum_{y_j \in T_{k_2}} \beta_j(y)|y - y_j|\Big)\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{16}{3} C_1 h_1^3 \|D^3 f\|_{\infty,\Omega} \Big(2\overline{M}_1 c^{-1} h_1^2 + 2\overline{M}_1 \sum_{k_1=2}^{N_1} c^2 h_1^{-2} (2k_1-3)^{-3} (2k_1+1) h_1 \Big) \\
&\quad + \frac{16}{3} C_2 h_2^3 \|D^3 f\|_{\infty,\Omega} \Big(2\overline{M}_2 c^{-1} h_2^2 + 2\overline{M}_2 \sum_{k_2=2}^{N_2} c^2 h_2^{-2} (2k_2-3)^{-3} (2k_2+1) h_2 \Big) \\
&\leq C_1 (c^{-1} h_1^5 + c^2 h_1^2) + C_2 (c^{-1} h_2^5 + c^2 h_2^2). \qquad (4.5)
\end{aligned}
$$

It follows from [16], that

$$
\begin{aligned}
&\Big| (\tilde{\Phi}_2 f)(x,y) - f(x,y) \Big| \\
=&\Big| \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_i(x) \beta_j(y) \int_0^1 D^3_{(x-x_i, y-y_j)} f(tx+(1-t)x_i, ty+(1-t)y_j) t(1-t) dt \Big| \\
=&\Big| \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{m} \alpha_i(x) \beta_j(y) \Big((x-x_i)^3 \int_0^1 f_{xxx} \cdot t(1-t) dt + 3(x-x_i)^2 (y-y_j) \int_0^1 f_{xxy} \cdot t(1-t) dt \\
&\quad + 3(x-x_i)(y-y_j)^2 \int_0^1 f_{xyy} \cdot t(1-t) dt + (y-y_j)^3 \int_0^1 f_{yyy} \cdot t(1-t) dt \Big) \Big| \\
\leq& C_1 \Big(\sum_{i=0}^{n} \alpha_i(x) |x-x_i|^3 \Big) + C_2 \Big(\sum_{i=0}^{n} \alpha_i(x) |x-x_i|^2 \Big) \Big(\sum_{j=0}^{m} \beta_j(y) |y-y_j| \Big) \\
&\quad + C_3 \Big(\sum_{i=0}^{n} \alpha_i(x) |x-x_i| \Big) \Big(\sum_{j=0}^{m} \beta_j(y) |y-y_j|^2 \Big) + C_4 \Big(\sum_{j=0}^{m} \beta_j(y) |y-y_j|^3 \Big) \\
\leq& C_1 \Big(2\overline{M}_1 c^{-1} h_1^4 + 2\overline{M}_1 \sum_{k_1=2}^{N_1} c^2 h_1^{-2} (2k_1-3)^{-3} (2k_1+1)^3 h_1^3 \Big) \\
&\quad + C_2 \Big(2\overline{M}_1 c^{-1} h_1^3 + 2\overline{M}_1 \sum_{k_1=2}^{N_1} c^2 h_1^{-2} (2k_1-3)^{-3} (2k_1+1)^2 h_1^2 \Big) \\
&\quad \times \Big(2\overline{M}_2 c^{-1} h_2^2 + 2\overline{M}_2 \sum_{k_2=2}^{N_2} c^2 h_2^{-2} (2k_2-3)^{-3} (2k_2+1) h_2 \Big) \\
&\quad + C_3 \Big(2\overline{M}_1 c^{-1} h_1^2 + 2\overline{M}_1 \sum_{k_1=2}^{N_1} c^2 h_1^{-2} (2k_1-3)^{-2} (2k_1+1) h_1 \Big) \\
&\quad \times \Big(2\overline{M}_2 c^{-1} h_2^3 + 2\overline{M}_2 \sum_{k_2=2}^{N_2} c^2 h_2^{-2} (2k_2-3)^{-3} (2k_2+1)^2 h_2^2 \Big) \\
&\quad + C_4 \Big(2\overline{M}_2 c^{-1} h_2^4 + 2\overline{M}_2 \sum_{k_2=2}^{N_2} c^2 h_2^{-2} (2k_2-3)^{-3} (2k_2+1)^3 h_2^3 \Big) \\
\leq& \mathcal{O}\Big(c^{-1} (h_1^4 + h_2^4) + c^2 \Big) \\
&\quad + \mathcal{O}\Big(c^{-2} (h_1^3 h_2^2 + h_1^2 h_2^3) + c (h_1^3 h_2^{-1} + h_1^{-1} h_2^3) + c (h_2^2 h_1^{-1} + h_1^2 h_2^{-1}) + c^4 h_1^{-1} h_2 - 1 \Big).
\end{aligned}
$$

Combining the above results yields (4.3). □

5. Numerical Examples

In this section, first of all we verify that $(\Phi_2 f)(x,y)$ satisfies quadric polynomial reproduction property, and then suppose $f(x,y) = (x-\frac{1}{2})^2 \sin y$, $(x,y) \in [0,1]^2$ is the approximated function, we sample it and use these sampling points to generate $(\Phi_1 f)(x,y)$ and $(\Phi_2 f)(x,y)$. Next we use $(\Phi_1 f)(x,y)$ and $(\Phi_2 f)(x,y)$ to approximate to $f(x,y)$ and choose different h_1, h_2, c to compare the approximation errors of $(\Phi_1 f)(x,y)$ and $(\Phi_2 f)(x,y)$ in the infinite norm $\|\cdot\|_\infty$. Before working to start, we introduce the selection of properly posed point set $A = \{Q_1, Q_2, \cdots, Q_6\}$ for computing $D^{(1,0)}_{A_{X_{ij}}} f(x_i, y_j)$ and $D^{(0,1)}_{A_{X_{ij}}} f(x_i, y_j)$. The principle of selecting lattice A is as follows: on the one hand, we require A to be a properly posed point set for $\mathcal{P}_2$; on the other hand, we require A to be as close to X_{ij} as possible. Based on the above two points, we choose point set A by Fig. 5.1.

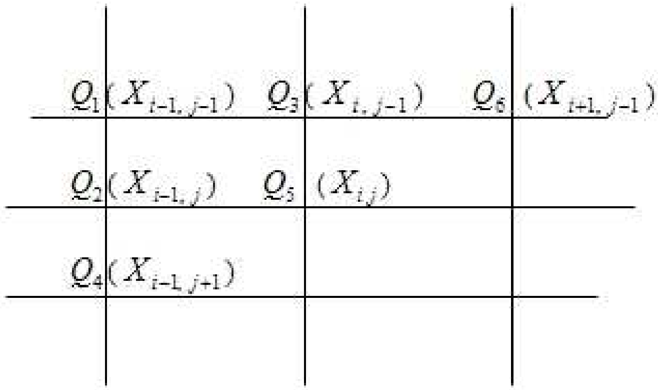

Fig. 5.1. Choosing point set $A = \{Q_1, Q_2, \cdots, Q_6\}$ for computing $D^{(1,0)}_{A_{X_{ij}}} f(x_i, y_j)$ and $D^{(0,1)}_{A_{X_{ij}}} f(x_i, y_j)$.

From Table 5.1, we believe that the scheme $(\Phi_2 f)(x,y)$ satisfies quadric polynomial reproduction property. However, $(\Phi_1 f)(x,y)$ can't reproduce quadric polynomial.

From Table 5.2, we find that when $h_1 = h_2 = h$, the errors $\|\cdot - f(x,y)\|_\infty$ of the schemes $(\Phi_1 f)(x,y)$ and $(\Phi_2 f)(x,y)$ reduce with the decrease of h and c , but the error of $(\Phi_2 f)(x,y)$ is much smaller than that of $(\Phi_1 f)(x,y)$. Moreover, we conclude that the approximation rate of two schemes is dependent on the shape parameter c.

Table 5.1: The approximated function is $f(x,y) = x^2 + y^2 + 3xy + 3x + 5y + 6$, $(x,y) \in [0,1]^2$, $h = h_1 = h_2$.

c	h	$\|f(x,y) - (\Phi_1 f)(x,y)\|_\infty$	$\|f(x,y) - (\Phi_2 f)(x,y)\|_\infty$
0.1	0.2	6.38×10^{-2}	1.0658×10^{-14}
0.01	0.2	4.2×10^{-3}	8.8818×10^{-15}
0.1	0.1	5.69×10^{-2}	2.4869×10^{-14}
0.01	0.1	2.4×10^{-3}	1.5987×10^{-14}
0.1	0.04	5.55×10^{-2}	8.3489×10^{-14}
0.01	0.04	1.4×10^{-3}	9.9476×10^{-14}

Table 5.2: The approximated function is $f(x,y) = (x-\frac{1}{2})^2 \sin y$, $(x,y) \in [0,1]^2$, $h = h_1 = h_2$.

c	h	$\|f(x,y) - (\Phi_1 f)(x,y)\|_\infty$	$\|f(x,y) - (\Phi_2 f)(x,y)\|_\infty$
0.1	0.2	2.67×10^{-2}	3.9259×10^{-4}
0.01	0.2	1.8×10^{-3}	2.3758×10^{-5}
0.001	0.2	1.6934×10^{-4}	2.3718×10^{-6}
0.1	0.1	2.38×10^{-2}	2.9125×10^{-4}
0.01	0.1	1.0×10^{-3}	4.2850×10^{-5}
0.001	0.1	8.5875×10^{-5}	2.6717×10^{-7}
0.1	0.04	2.31×10^{-2}	2.9209×10^{-4}
0.01	0.04	5.9022×10^{-4}	2.5963×10^{-5}
0.001	0.04	3.6211×10^{-5}	3.2220×10^{-8}

Table 5.3: The approximated function is $f(x,y) = (x-\frac{1}{2})^2 \sin y$, $(x,y) \in [0,1]^2$, $h_1 \neq h_2$.

c	h_1	h_2	$\|f(x,y) - (\Phi_1 f)(x,y)\|_\infty$	$\|f(x,y) - (\Phi_2 f)(x,y)\|_\infty$
0.1	0.2	0.1	2.67×10^{-2}	3.0154×10^{-4}
0.01	0.2	0.1	1.8×10^{-3}	7.0302×10^{-6}
0.001	0.2	0.1	1.6934×10^{-4}	5.8762×10^{-7}
0.1	0.2	0.04	2.66×10^{-2}	2.9134×10^{-4}
0.01	0.2	0.04	1.8×10^{-3}	3.0058×10^{-6}
0.001	0.2	0.04	1.6934×10^{-4}	9.5075×10^{-8}
0.1	0.1	0.04	2.38×10^{-2}	2.9130×10^{-4}
0.01	0.1	0.04	1.0×10^{-3}	2.6879×10^{-6}
0.001	0.1	0.04	8.5873×10^{-5}	5.2754×10^{-8}
0.1	0.04	0.2	2.32×10^{-2}	2.5988×10^{-4}
0.01	0.04	0.2	5.9057×10^{-4}	6.4311×10^{-6}
0.001	0.04	0.2	3.6214×10^{-5}	4.9039×10^{-7}
0.1	0.04	0.1	2.32×10^{-2}	2.8443×10^{-4}
0.01	0.04	0.1	5.9042×10^{-4}	2.8298×10^{-6}
0.001	0.04	0.1	3.6213×10^{-5}	1.1418×10^{-7}

From Table 5.3, we find that when $h_1 \neq h_2$, the errors $\| \cdot - f(x,y)\|_\infty$ of the schemes $(\Phi_1 f)(x,y)$ and $(\Phi_2 f)(x,y)$ reduce with the decrease of h_1, h_2 and c, but the error of $(\Phi_2 f)(x,y)$ is still much smaller than the error of $(\Phi_1 f)(x,y)$. Moreover, if h_1 becomes smaller, the error of $(\Phi_1 f)(x,y)$ reduce more quickly than the error of $(\Phi_2 f)(x,y)$. On the contrary, both of them reduce slowly with the decrease of h_2.

6. Conclusion

In the paper, by using multivariate divided differences to approximate the partial derivative and superposition modifying idea, we extend the bivariate quasi-interpolation scheme which dimension-splitting multiquadric proposed by Ling [14] to the scheme which reproduces quadric polynomials. Furthermore, we give approximation error of the modified scheme. Our multivariate multiquadric quasi-interpolation scheme only requires information of location points, and not require the derivatives of approximated function. From some numerical experiments we find that the approximation rate of our scheme is higher than that of Ling's scheme, which is consistent with our theoretical prediction. Due to the ability in quadric reproduction, our scheme can be used to serve CAGD and the numerical solution of PDEs. Our work may be also extended to three space dimensions. Note that the object of this paper is the scheme for dimension-splitting, not the scheme based on completely scattered data. Multivariate quasi-interpolation scheme based on completely scattered data will be investigated in our future research.

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