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A Truly Boundary-Only Meshfree Method Applied to Kirchhoff Plate Bending Problems

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Abstract. The boundary particle method (BPM) is a truly boundary-only collocation scheme, whose basis function is the high-order nonsingular general solution or singular fundamental solution, based on the recursive composite multiple reciprocity method (RC-MRM). The RC-MRM employs the high-order composite differential operator to solve a much wider variety of inhomogeneous problems with boundary-only collocation nodes while significantly reducing computational cost via a recursive algorithm. In this study, we simulate the Kirchhoff plate bending problems by the BPM based on the RC-MRM. Numerical results show that this approach produces accurate solutions of plates subjected to various loadings with boundary-only discretization.

AMS subject classifications: 35J40,41A30,65N35,74K20,74S30
Key words: Boundary particle method, recursive composite multiple reciprocity method, Kirchhoff plate, boundary-only, meshfree.

1 Introduction

In recent decades, the boundary-type meshfree methods, such as method of fundamental solution (MFS) [1–3], boundary knot method (BKM) [4], boundary collocation method (BCM) [5], regularized meshless method (RMM) [6, 7] and boundary node method (BNM) [8, 9], have attracted a lot of attention in the numerical solution of various partial differential equations. All the above-mentioned boundary methods can solve homogeneous problems with boundary-only discretization. However, these methods require inner nodes in conjunction with the other techniques to handle inhomogeneous problems, such as quasi-Monte-Carlo method [10], dual reciprocity method (DRM) [11] and multiple reciprocity method (MRM) [12].

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Since 1980s the DRM and MRM have been emerging as the two most promising techniques to handle inhomogeneous problems in conjunction with the boundary type methods [11–13]. The striking advantage of the MRM over the DRM is that it does not require using inner nodes at all for the particular solution. To take advantage of truly boundary-only merit of the MRM, Chen [14, 15] developed the MRM-based meshfree boundary particle method (BPM). However, the standard MRM is computationally expensive in the construction of the interpolation matrix and has limited feasibility for general inhomogeneous problems due to its use of high-order Laplacian operators in the annihilation process [12]. Chen and Jin [16, 17] presented the recursive composite multiple reciprocity method (RC-MRM), which employs the high-order composite differential operators to vanish the inhomogeneous term of various types. The RC-MRM significantly expands the application territory of the BPM to a much wider variety of inhomogeneous problems. In addition, the RC-MRM includes a recursive algorithm to dramatically reduce the total computing cost.

This paper is organized as follows. Section 2 introduces the BPM based on RC-MRM through its discretization to the Kirchhoff plate bending problems. The efficiency and utility of this new technique are numerically examined in Section 3. Section 4 concludes this paper with some remarks and opening issues.

2 RC-MRM based BPM for plate bending

Without lose of generality, this section introduces the BPM through its discretization to the Kirchhoff plate problems.

2.1 Plate bending

The deflection of a thin plate under a distributed loading is governed by the governing equation

$$\nabla^4 w = \frac{q}{D}, \tag{2.1}$$

where w is the deflection of the middle surface of plate, ∇^4 denotes the biharmonic operator, and $D{=}Eh^3/\big(12(1-\nu^2)\big)$ represents the flexural rigidity.

At every boundary point, the two boundary conditions have to be satisfied, which are a combination of the following conditions: displacement, normal slope, bending moment, and effective shear force. In this study, the following three types of boundary conditions are encountered: (1) Clamped edge, denoted by C in this paper: $w{=}0$, $\theta_n{=}0$, where w and θ_n denote the displacement and normal slope condition, respectively. (2) Simply supported edge, denoted by S in this paper: $w{=}0$, $M_n{=}0$, where M_n represents the bending moment condition. (3) Free edge, denoted by F in this paper: $M_n{=}0$, $V_n{=}0$, where V_n expresses the effective shear force.

The above boundary conditions can be expressed in terms of the deflection w as follows.

Normal slope:

$$\theta_n = \frac{\partial w}{\partial n} = \frac{\partial w}{\partial x}\frac{dx}{dn} + \frac{\partial w}{\partial y}\frac{dy}{dn} = \frac{\partial w}{\partial x}\cos\alpha + \frac{\partial w}{\partial y}\sin\alpha, \tag{2.2}$$

Normal bending moment:

$$M_n = -D\Big\{\nu\nabla^2 w + (1-\nu)\Big(\cos^2\alpha\frac{\partial^2 w}{\partial x^2} + \sin^2\alpha\frac{\partial^2 w}{\partial y^2} + \sin 2\alpha\frac{\partial^2 w}{\partial x\partial y}\Big)\Big\}, \tag{2.3}$$

Normal effective shear:

$$\begin{aligned} V_n = -D\Big\{&(\cos\alpha\frac{\partial}{\partial x} + \sin\alpha\frac{\partial}{\partial y})\nabla^2 w + (1-\nu)(-\sin\alpha\frac{\partial}{\partial x} + \cos\alpha\frac{\partial}{\partial y}) \\ &\cdot\Big(\frac{1}{2}\sin 2\alpha(\frac{\partial^2 w}{\partial y^2} - \frac{\partial^2 w}{\partial x^2}) + \cos 2\alpha\frac{\partial^2 w}{\partial x\partial y}\Big)\Big\}, \end{aligned} \tag{2.4}$$

where n=[$\cos\alpha, \sin\alpha$] is the unit outward normal vector.

2.2 Recursive composite multiple reciprocity based BPM

The recursive composite multiple reciprocity technique uses the high-order composite differential operator to evaluate the particular solutions and employs a recursive algorithm to significantly reduce computing cost .The details about the BPM based on RC-MRM can be found in reference [16, 17]. To illustrate the BPM application to plate bending, we consider a fully clamped plate subjected to a trigonometric loading q=$100q_0\cos(\lambda x)$ as follow:

$$\triangle^2 w = \frac{100q_0}{D}\cos(\lambda x), \qquad x\in\Omega, \tag{2.5a}$$

$$w = 0, \quad \theta_n = 0, \qquad x\in\partial\Omega. \tag{2.5b}$$

By implementing the RC-MRM approach, the above inhomogeneous problem is transformed to the following homogeneous problem.

$$(\triangle + \lambda^2)\triangle^2 w = 0, \qquad x\in\Omega, \tag{2.6a}$$

$$\triangle^2 w = \frac{100q_0}{D}\cos(\lambda x), \qquad x\in\partial\Omega, \tag{2.6b}$$

$$w = 0, \quad \theta_n = 0, \qquad x\in\partial\Omega. \tag{2.6c}$$

Hence the deflection w is approximated by a linear combination of the kernel function of high-order composite differential operators of $(\triangle + \lambda^2)\triangle^2$ and $\triangle^2$. Note that the kernel function represents either the fundamental solution [1], general solution [4, 5], T-function [18] or de-singular fundamental solution [6, 7], which satisfies the high-order composite differential operator equation. This study uses higher order singular fundamental solutions [12, 19] and nonsingular general solutions [14] of the composite differential operators. However, the Laplace equation has no nonsingular general solution. Hon and Wu [20] propose the translate-invariant 2D Laplacian harmonic function to efficiently solve the 2D Laplace problems. Chen and Jin [16] found and verified the harmonic function of $(m+1)$-order Laplace operator $\triangle^{m+1}$

$$H^m(x_i, y_i) = r_{ik}^{2m} \exp\big(-c(x_{ik}^2 - y_{ik}^2)\big)\cos(2cx_{ik}y_{ik}), \tag{2.7}$$

where c is the shape parameter, and

$$r_{ik} = \sqrt{(x_i - x_k)^2 + (y_i - y_k)^2}, \quad x_{ik} = x_i - x_k, \quad y_{ik} = y_i - y_k.$$

This study uses the nonsingular harmonic function as the high-order Laplacian general solution.

3 Numerical results and discussions

In this section we present several numerical examples to investigate the efficiency, accuracy, and convergence of the RC-MRM based BPM on plate bending problems.

The mechanics parameters in the tested plates are E=2.1 × 10^{11}, h=0.01, ν=0.3, q_0=10^6, NN is the number of boundary collocation points. The average relative error is defined as

$$Lerr = \Big(\frac{1}{NT}\sum_{i=1}^{NT}\Big|\frac{w(i) - \bar{w}(i)}{\bar{w}(i)}\Big|^2\Big)^{\frac{1}{2}}, \tag{3.1}$$

where $\bar{w}(i)$ represents the deflection at x_i in [21], $w(i)$ denotes the numerical solution via nonsingular or singular formulation at x_i, and NT is the total number of testing points in inner domain and on the boundary. Unless otherwise specified, NT is taken to be 2601 for all plate bending problems in this paper.

In addition, the deflections presented in a tabular form are normalized by

$$W_i = \frac{100D \times w(i)}{q_0 \times L^4}, \quad \bar{W}_i = \frac{100D \times \bar{w}(i)}{q_0 \times L^4},$$

where L is the length of plate. The relative error is calculated by

$$rerr = \Big|\frac{W_i - \bar{W}_i}{\bar{W}_i}\Big|. \tag{3.2}$$

In the BPM using singular fundamental solution, fictitious boundary outside the physical domain is a circle with radius R=2, whose center coincides with the center of the plate.

The BPM using nonsingular kernel function avoids such a perplexing issue of arbitrary fictitious boundary. On the other hand, the appropriate choice of the shape parameter c in nonsingular harmonic function $H^m(x_i, y_i)$, however, is also problematic

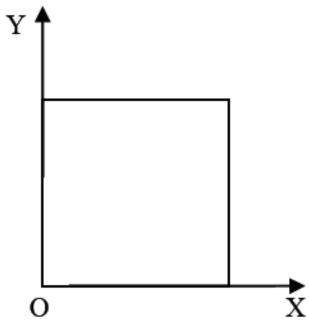

Figure 1: The coordinates in unit square plate.

and requires particular care. Through trial-error experiments, we find an empirical formula:

$$c = \frac{NN+4}{32}. \tag{3.3}$$

The coordinates of a unit square plate are displayed in Fig. 1.

3.1 Square plates

Firstly, considering a simply-supported rectangular plate (SSSS) subjected to uniform loading, we investigate the convergent rate and stability of the RC-MRM based BPM and also examine the relationships between accuracy, conditioning number and the other computational parameters such as fictitious radius R or shape parameter c.

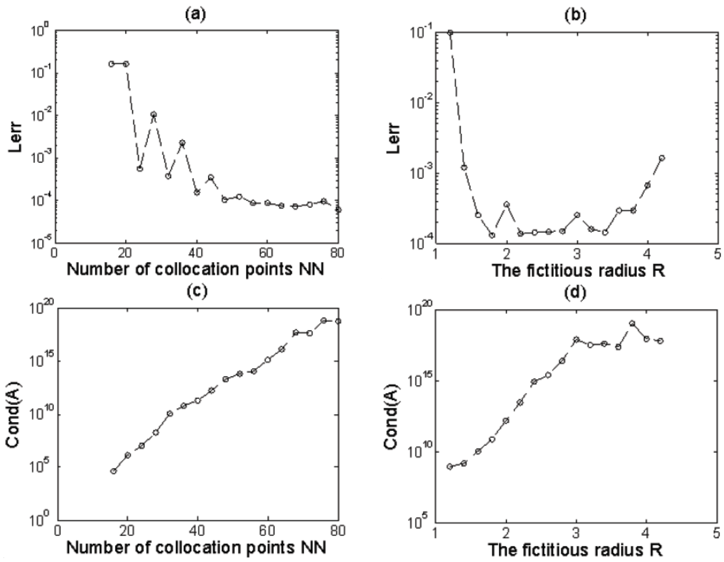

Figure 2: Convergence and stability of the BPM using singular fundamental solution. (a) The error Lerr with respect to NN (R=2); (b) the error Lerr for the fictitious radius R (NN=44); (c) the condition number Cond (A) with respect to NN (R=2); (d) the condition number Cond (A) with the fictitious radius R (NN=44).

Fig. 2(a) shows the convergence curve obtained by the BPM with singular fundamental solution. Roughly speaking, the numerical accuracy improves in an oscillatory way with an increasing number of boundary collocation points NN, and then appear far less improvement with the further increasing NN. It's observed that only 40 or so boundary collocation points suffice reasonable accuracy. Fig. 2(b) depicts the accuracy variation with respect to the increasing fictitious radius R. The numerical accuracy becomes obviously higher with the increase of R to some extent, before it starts to become worse. We note that the better accuracy somewhat goes with the worse ill-conditioning interpolation matrix, also reported in the other RBF literature. This is contradictory to the common wisdom in numerical calculations. Figs. 2(c) and 2(d) display the condition number Cond of the BPM interpolation matrix $A=(A_{ij})$ with respect to NN and R ,respectively. There are several ways to mitigate the effect of bad conditioning, such as the domain decomposition method [23], preconditioning technique based on approximate cardinal basis function, the fast multiple method [24], regularization methods (e.g., the truncated singular value decomposition (TSVD)) [25, 26]. It is noted that in some cases presented below, the TSVD with GCV function choice criterion is employed to obtain accurate and stable results. And the MATLAB SVD code developed by Hansen [26] for the discrete ill-posed problem has been adopted in our computations.

By the BPM using the nonsingular general solution, Figs. 3(a) and 3(b) display the accuracy variation with respect to the increasing NN and c, respectively. Figs. 3(c) and 3(d) show the curves of the condition number with respect to NN and c, respectively. It's observed that the present method obtains best accuracy between 30 and 60 bound-

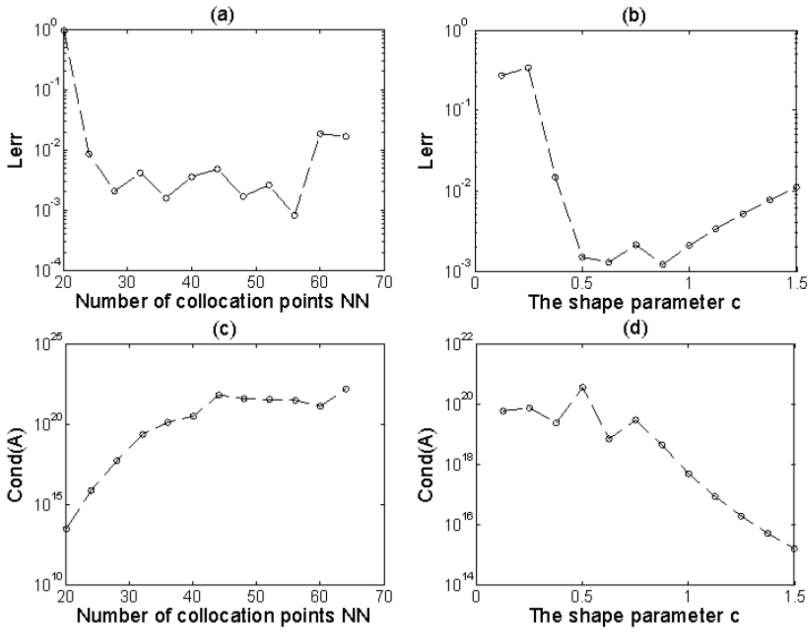

Figure 3: Convergence and stability for the method with nonsingular formulation. (a) The error $Lerr$ with respect to NN; (b) the error $Lerr$ for the shape parameter c (NN=44); (c) the condition number Cond (A) with respect to NN; (d) the condition number Cond (A) for the shape parameter c (NN=44).

ary nodes, so does the accuracy variation with respect to c. The best shape parameter c is found between 0.5 and 1.0. In addition, it is noted that the curve of the condition number declines with the increasing of c. In the following, we investigate the accuracy of the present RC-MRM based BPM solution of plates subjected to varied loadings.

(1) Uniform loading

Table 1: The maximum deflection (W) and its relative error (rerr) and average relative errors (Lerr) of square plates under different boundary conditions (28 boundary nodes).

Boundary Condition	[21]	BPM (nonsingular formulation)			BPM (singular formulation)		
	$\bar{W}$	W	rerr	Lerr	W	rerr	Lerr
CCCC	0.12653	0.12650	2.37E-04	/	0.12650	2.37E-04	/
SSSS	0.40624	0.40600	5.91E-04	2.10E-03	0.40620	9.85E-05	1.20E-03
CSCS	0.19171	0.19170	5.22E-05	1.50E-03	0.19170	5.22E-05	9.20E-03
SSSF	1.28600	1.15880	9.89E-02	5.35E-04	1.17980	8.26E-02	8.40E-03

Note: The deflection $\bar{W}$ is given by Timoshenko [21]. The parameters are R=2 and c=$(NN+4)/32$.

Table 1 illustrates the BPM solution of square plates of different boundary conditions subjected to a uniform loading in comparison with the Timoshenko solution [21]. It is observed that the present results agree well with Timoshenko solutions. As in the other collocation methods, numerical errors tend to be worse around boundary. It is noted that the relative error of the maximum deflection of SSSF plate (three simply-supported and one free boundary conditions) is worse than those of the other boundary condition cases. The reason is that its maximum deflection occurs on the free edge. Both BPM results by using employing nonsingular and singular formulation agree well with the exact solution. However, the nonsingular BPM avoids arbitrary fictitious boundary outside physical domain required by the singular BPM.

(2) Nonuniform loading

A unit square plate with various loadings is calculated as below:

• Hydrostatic Pressure Loading: $q(x,y)$=$q_0(x)$.

By using the RC-MRM, the hydrostatic pressure loading term is differentiated to zero by differential operator $\triangle$. The deflection w is approximated by a linear combination of the kernel functions of high-order differential operators of $\triangle^3$ and $\triangle^2$.

• Exponential Loading: $q(x,y)$=$2q_0\exp(5(x-y))$.

By using the RC-MRM, the exponential loading term is vanished by differential oper-

Table 2: The deflection and the corresponding relative error of SSSS (fully simply-supported) square plate with hydrostatic pressure loading (28 boundary knots).

Coordinate	[21]	Nonsingular Formulation		Singular Formulation	
	$\bar{W}$	W	rerr	W	rerr
0.25, 0.5	0.1310	0.1311	7.63E-04	0.1311	7.63E-04
0.5, 0.5	0.2320	0.2031	4.93E-04	0.2031	4.93E-04
0.6, 0.5	0.2010	0.2027	8.46E-03	2.2027	8.46E-03
0.75, 0.5	0.1620	0.1662	2.59E-02	0.1627	4.32E-03
0.55, 0.5 (max)	0.2060	0.2054	2.91E-03	0.2054	2.91E-03

Note: The deflection $\bar{W}$ is reported in [21]. The parameters are R=2 and c=$(NN+4)/32$.

Table 3: The deflection and the corresponding relative errors of CCCC (fully clamped) square plate with various loading (36 boundary knots). NF: Nonsingular formulation. SF: singular formulation.

Loading	Coordinate	$\bar{W}$ [21]	W (NF)	W (SF)
$q_0 x$	0.50, 0.50	0.06300	0.06320	0.06330
$2q_0 \exp(5(x-y))$	0.64, 0.36	0.67423	0.67440	0.67500
$100q_0 \cos(10x)$	0.50, 0.50	1.71750	1.71120	1.71077
$100q_0 \cos(10x)$	0.66, 0.32	0.32058	0.32130	0.32135

Note: The deflection $\bar{W}$ is calculated by the method in [22].

ator $(\triangle - \lambda^2)$, where $\lambda=5\sqrt{2}$. The deflection is approximated by a linear combination of the kernel functions of high-order differential operators of $(\triangle - \lambda^2)\triangle^2$ and $\triangle^2$.

- Trigonometric Loading: $q(x,y)=100q_0\cos(10x)$.

By using the RC-MRM, the trigonometric loading term is vanished by differential operator $(\triangle + \lambda^2)$, where $\lambda=10$. The deflection w is approximated by a linear combination of the kernel functions of differential operators of $(\triangle + \lambda^2)\triangle^2$ and $\triangle^2$.

Table 2 gives the results of the SSSS square plate with hydrostatic pressure loading. Table 3 shows the result of the CCCC square plate with various loading. According to Tables 2 and 3, the present results obtained by nonsingular and singular BPM formulations both agree well with those obtained by the other approaches [21,22]. Figs. 4 and 5 represent the deflection contour plots of the CCCC plate with exponential and trigonometric loadings, respectively.

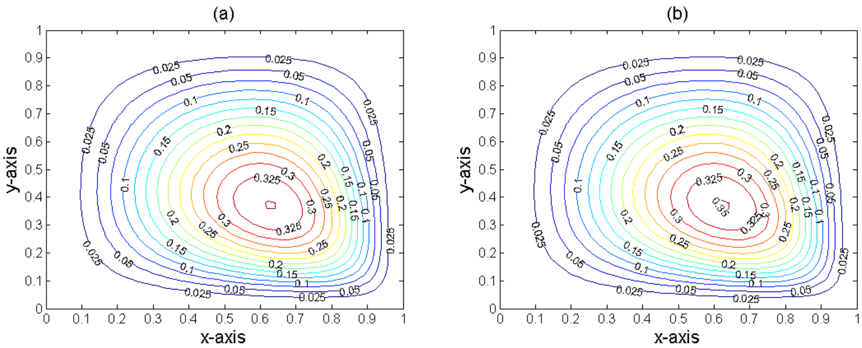

Figure 4: The deflection contour plots of the CCCC plate with exponential loading by (a) nonsingular and (b) singular BPM formulations.

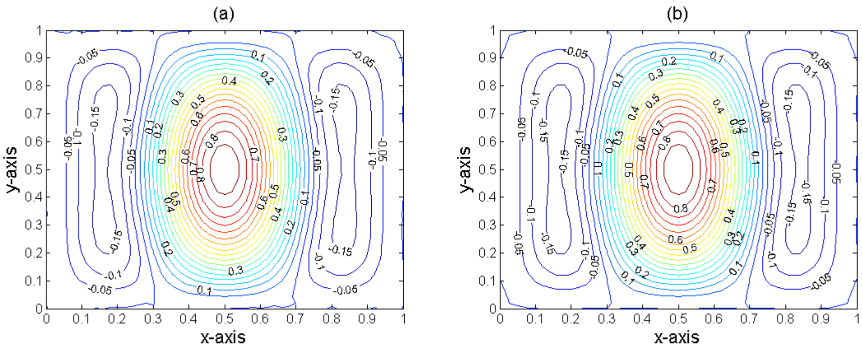

Figure 5: The deflection contour plots of the CCCC plate with trigonometric loading by (a) nonsingular and (b) singular BPM formulations.

3.2 Plates of different boundary shapes

Fig. 6 shows a parallelogram CCCC plate with skew angle θ. Table 4 displays the maximum deflection of a uniformly loaded parallelogram CCCC plate by using the BPM. It is observed that both the numerical solutions with nonsingular and singular formulations agree well with the solution obtained by the reference method [22]. The maximum deflections obtained by the BPM have accuracy of three to four significant digits compared with the results obtained by the method presented in [22].

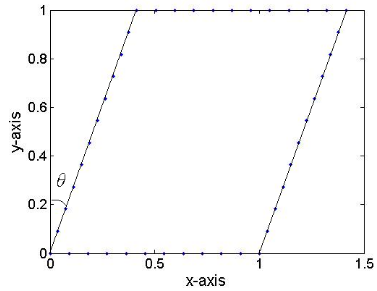

Figure 6: Parallelogram plate with skew angle θ.

We consider the clamped circular plate subjected to a uniform loading, with center at $(0,0)$ and radius r=1. Here the number of testing points is 7845 to examine the numerical accuracy. According to Table 5, the results obtained by the nonsingular BPM have reasonable accuracy, while the results by the singular BPM reach extremely high accuracy. In order to depict the error distribution, Fig. 7 presents the relative error at line y=0. It is observed that maximum errors occur on the boundary as in the other collocation methods.

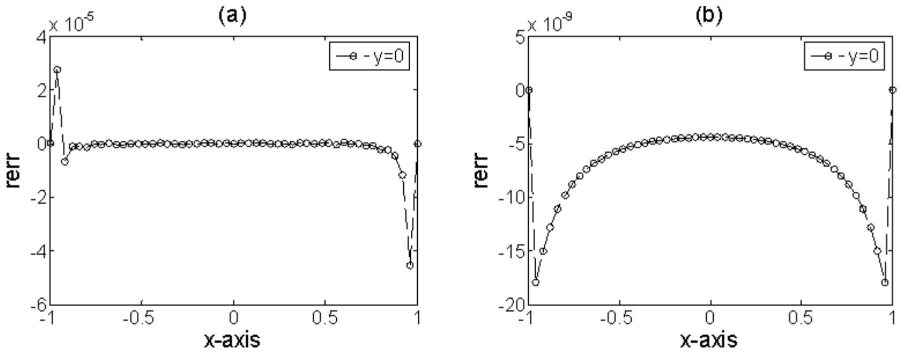

Figure 7: Relative error distribution of circular plate at line y=0 used by (a) nonsingular and (b) singular BPM formulations.

Table 4: The maximum deflection and its relative error of the CCCC parallelogram plate with different skew angles (44 boundary nodes). θ is skew angle. NF: Nonsingular formulation. SF: singular formulation.

θ	Coordinate	$\bar{W}$ [21]	W (NF)	W (SF)
0	0.5, 0.5	0.12653	0.12646	0.12650
15	0.634, 0.5	0.12011	0.12010	0.12002
30	0.7887, 0.5	0.09866	0.09860	0.09864
45	1.0, 0.5	0.05874	0.05871	0.05871

Note: The deflection $\bar{W}$ is calculated by the method in [22].

Table 5: The maximum deflection and its relative errors and average relative errors of circular plate subjected to a uniform loading (30 boundary knots, fictitious boundary radius R=3).

Boundary Condition	[21]	Nonsingular Formulation			Singular Formulation		
	$\bar{W}$	W	rerr	Lerr	W	rerr	Lerr
Clamped	1.5625	1.5625	0	1.50E-04	1.5625	0	2.61E-08

Note: The deflection $\bar{W}$ is given by Timoshenko [21]. The parameters are R=2 and $c=(NN+4)/32$.

4 Conclusions

This paper makes the first attempt to calculate Kirchhoff plate bending problems by employing the RC-MRM based BPM. The convergence rate and stability of this method are carefully examined. Based on the results and discussions in Section 3, the following conclusions are drawn:

- The present BPM produces numerical solution of high accuracy with boundary-only discretization. No any inner nodes are required for inhomogeneous problems of different types. In addition, the results by using the nonsingular and singular BPM have similar order of accuracy.
- Computational efficiency of the present BPM is examined. For most cases tested in this study, only a few boundary particles can produce highly accurate solution.
- Convergence curves appear somewhat oscillatory because the conditioning number of the BPM interpolation matrix quickly increases with increasing boundary nodes.
- The Laplacian harmonic function requires determining a problem-dependent parameter c, which is a reminiscence of the shape parameter in well-known radial basis function MQ.
- This paper has tested the BPM to the Kirchhoff plate bending problems subjected to a few typical loadings. For the other types of loading function, we may find a suitable composite operator to reduce it to zero. If it is not workable, we can express it by a sum of polynomial or trigonometric function series, and then the present BPM can simply be implemented to solve these problems with the boundary-only discretization.

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