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Approximate Boundary Conditions for Patch Antennas Mounted on Thin Dielectric Layers

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Abstract. In this paper we discuss scattering problems inherent in curved microstrip structures mounted on thin dielectric structures. We provide approximate boundary conditions for such structures in the framework of integral equations.

Key words: Approximate boundary conditions; boundary integral method; thin dielectric layer; patch antenna.

1 Introduction

The solution of the scattering problem of a plane wave by a metallic target coated by a thin dielectric substrate requires to solve a system of integral equations coupling the solution in the gain and the surrounding medium. Then, the resulting method is both time and memory consuming. Moreover, since the scale of the spatial change in electromagnetic fields in the direction of the thickness of such a thin layer is considerably different from that in the transverse directions, such a direct formulation suffers from numerical instabilities if the dielectric layer is too thin. An alternative to this approach consists in approximately simulating the interior propagation phenomenon by the way of a boundary conditions, the so-called approximate boundary conditions, set on the surface of the obstacle and next to solve the associated scattering problem. See [3, 4, 10, 12].

In this paper we discuss scattering problems inherent in curved microstrip structures mounted on thin dielectric layers. These structures are widely used in printed-circuit technology, microwave integrated circuits, and the antenna industry [7, 9, 11, 14]. It has

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been difficult to analyze electromagnetic fields around such structures. Indeed, the classical approximate boundary conditions do not provide an accurate approximation of the electromagnetic fields due to the fact that the presence of the microstrip patch causes a change in the relation between the electromagnetic fields at the dielectric interface.

This paper extends the concept of approximate boundary conditions to microstrip structures and gives a detailed mathematical derivation of an approximate boundary condition for a microstrip patch in the framework of integral equations.

2 Problem formulation

Let Ω be a bounded domain in $\mathbb{R}^2$, with a connected $\mathcal{C}^{2,\alpha}, 0 < \alpha < 1$, boundary. For $h > 0$, let $\Omega_d := \{x \in \mathbb{R}^2 \setminus \overline{\Omega} : \mathrm{dist}(x, \partial\Omega) < h\}$ and $\Omega_e := \mathbb{R}^2 \setminus \overline{(\Omega \cup \Omega_d)}$. We assume that the outer part of $\partial\Omega_d$ consists of two disjoint parts Γ and Γ_e so that $\partial\Omega_d = \Gamma \cup \Gamma_e \cup \partial\Omega$. Put $\Gamma_0 := \{x \in \partial\Omega : x + h\nu_x \in \Gamma\}$, where ν denotes the outward normal to $\partial\Omega$. The domain Ω_d represents the thin dielectric structure while Γ represents the antenna patch mounted on it.

The profiles of electric permeability and permittivity are given by

$$\mu_h(x) = \begin{cases} \mu_d, & x \in \Omega_d, \\ \mu_e, & x \in \Omega_e, \end{cases}$$

and

$$\epsilon_h(x) = \begin{cases} \epsilon_d, & x \in \Omega_d, \\ \epsilon_e, & x \in \Omega_e, \end{cases}$$

respectively, where μ_d, μ_e, ϵ_d and ϵ_e are positive constants. If we allow the degenerate case $h = 0$, then the functions $\mu_h(x)$ and $\epsilon_h(x)$ are equal to the constants μ_e and ϵ_e.

Let $k_d := \omega\sqrt{\mu_d \epsilon_d}$ and define k_e likewise. For a given incident wave E_i, let E_h^Γ denote the solution to the scattering problem

$$\nabla \cdot \frac{1}{\mu_h} \nabla E_h^\Gamma(x) + \omega^2 \epsilon_h E_h^\Gamma(x) = 0, \quad x \in \Omega_e \cup \Omega_d,$$

with the radiation condition

$$\lim_{|x|\to\infty} \sqrt{|x|} \left(\frac{\partial (E_h^\Gamma - E_i)(x)}{\partial |x|} - i\omega k_e (E_h^\Gamma - E_i)(x) \right) = 0,$$

and the Dirichlet boundary condition

$$E_h^\Gamma = 0 \quad \text{on } \partial\Omega \cup \Gamma. \tag{2.1}$$

Then the scattering problem in the presence of the patch can be written as

$$\begin{cases} (\Delta + k_e^2)E_h^\Gamma = 0 & \text{in } \Omega_e, \\ (\Delta + k_d^2)E_h^\Gamma = 0 & \text{in } \Omega_d, \\ E_h^\Gamma = 0 & \text{on } \Gamma \cup \partial\Omega, \\ E_h^\Gamma|_+ = E_h^\Gamma|_- & \text{on } \partial\Omega_e, \\ \dfrac{1}{\mu_e}\dfrac{\partial E_h^\Gamma}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_d}\dfrac{\partial E_h^\Gamma}{\partial \nu}\Big|_- & \text{on } \Gamma_e, \\ \lim\limits_{|x|\to\infty} \sqrt{|x|}\left(\dfrac{\partial (E_h^\Gamma - E_i)(x)}{\partial |x|} - ik_e(E_h^\Gamma - E_i)(x)\right) = 0. \end{cases} \tag{2.2}$$

Here and throughout this paper, $E|_+$ and $E|_-$ denote the limits from inside and outside the given domain, respectively.

Let E_h be the solution without the patch, that is, the boundary condition (2.1) is replaced with $u = 0$ on $\partial\Omega$. Then E_h is the solution to

$$\begin{cases} (\Delta + k_e^2)E_h = 0 & \text{in } \Omega_e, \\ (\Delta + k_d^2)E_h = 0 & \text{in } \Omega_d, \\ E_h = 0 & \text{on } \partial\Omega \\ E_h|_+ = E_h|_- & \text{on } \partial\Omega_e, \\ \dfrac{1}{\mu_e}\dfrac{\partial E_h}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_d}\dfrac{\partial E_h}{\partial \nu}\Big|_- & \text{on } \partial\Omega_e, \\ \lim\limits_{|x|\to\infty} \sqrt{|x|}\left(\dfrac{\partial (E_h - E_i)(x)}{\partial |x|} - ik_e(E_h - E_i)(x)\right) = 0. \end{cases} \tag{2.3}$$

The main objective of this paper is to present a schematic way based on a boundary integral method to derive the leading-order term in the asymptotic expansions of E_h^Γ as h goes to zero. Because of the changes in the electromagnetic fields around the microstrip patch, E_h^Γ can not be approximated inside the thin layer by a linear function in the normal direction. This causes the most serious difficulty in deriving approximate boundary conditions for a patch antenna mounted on a thin dielectric layer.

3 Asymptotic formula for the solution without patch

We start with deriving an asymptotic expansion of E_h as h goes to zero. Let Φ_e be the fundamental solution for the Helmholtz operator $\Delta + k_e^2$, that is,

$$\Phi_e(x, y) = -\frac{i}{4} H_0^{(1)}(k_e|x - y|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind of order 0, and let Φ_d be the one for $\Delta + k_d^2$.

For a bounded smooth domain D in $\mathbb{R}^2$, let $\mathcal{S}_D^e$ and $\mathcal{D}_D^e$ be the single and double layer potential defined by

$$\mathcal{S}_D^e \varphi(x) = \int_{\partial D} \Phi_e(x,y)\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^2,$$
$$\mathcal{D}_D^e \varphi(x) = \int_{\partial D} \frac{\partial \Phi_e(x,y)}{\partial \nu_y}\varphi(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \partial D.$$

It is well-known that

$$\frac{\partial \mathcal{S}_D^e \varphi}{\partial \nu_x}\Big|_{\pm}(x) = \Big(\pm \frac{1}{2} I + (\mathcal{K}_D^e)^* \Big)\varphi(x) \quad a.e.\ x \in \partial D,$$
$$\mathcal{D}_D^e \varphi\Big|_{\pm}(x) = \Big(\mp \frac{1}{2} I + \mathcal{K}_D^e \Big)\varphi(x) \quad a.e.\ x \in \partial D,$$

where $\mathcal{K}_D^e : L^2(\partial D) \to L^2(\partial D)$ is the operator defined by

$$\mathcal{K}_D^e \varphi(x) = \text{p.v.} \int_{\partial D} \frac{\partial \Phi_e(x,y)}{\partial \nu_y}\varphi(y)d\sigma(y),$$

and $(\mathcal{K}_D^e)^*$ is the L^2-adjoint of $\mathcal{K}_D^e$ and is given by

$$(\mathcal{K}_D^e)^* \varphi(x) = \text{p.v.} \int_{\partial D} \frac{\partial \Phi_e(x,y)}{\partial \nu_x}\varphi(y)d\sigma(y).$$

Those operators with the superscripts e replaced with d denote the corresponding layer potentials defined with Φ_d.

Let the space $H^1(\partial\Omega_e)$ be the set of functions $f \in L^2(\partial\Omega_e)$ such that $\partial f/\partial\tau \in L^2(\partial\Omega_e)$, where $\partial/\partial\tau$ denotes the tangential derivative on $\partial\Omega_e$. The following lemma is essentially from [1].

Lemma 3.1. *Suppose that k_d^2 is not a Dirichlet eigenvalue for $-\Delta$ in Ω and k_e^2 is not a Dirichlet eigenvalue for $-\Delta$ in $\mathbb{R}^2 \setminus \overline{\Omega}_e$. For each $(F,G) \in H^1(\partial\Omega_e) \times L^2(\partial\Omega_e)$, there exists a unique solution $(f, g_1, g_2) \in L^2(\partial\Omega_e) \times L^2(\partial\Omega_e) \times L^2(\partial\Omega)$ to the integral equation*

$$\begin{cases} \mathcal{S}_{\Omega_e}^e f - \mathcal{S}_{\Omega_e}^d g_1 - \mathcal{S}_{\Omega}^d g_2 = F & \text{on } \partial\Omega_e, \\ \dfrac{1}{\mu_e}\dfrac{\partial \mathcal{S}_{\Omega_e}^e f}{\partial \nu}\Big|_+ - \dfrac{1}{\mu_d}\dfrac{\partial \mathcal{S}_{\Omega_e}^d g_1}{\partial \nu}\Big|_- - \dfrac{1}{\mu_d}\dfrac{\partial \mathcal{S}_{\Omega}^d g_2}{\partial \nu} = G & \text{on } \partial\Omega_e, \\ \mathcal{S}_{\Omega_e}^d g_1 + \mathcal{S}_{\Omega}^d g_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Moreover, there exists a constant C independent of F and G such that

$$\|f\|_{L^2(\partial\Omega_e)} + \|g_1\|_{L^2(\partial\Omega_e)} + \|g_2\|_{L^2(\partial\Omega)} \le C\Big(\|F\|_{H^1(\partial\Omega_e)} + \|G\|_{L^2(\partial\Omega_e)}\Big). \tag{3.2}$$

Proof. We sketch a proof of this lemma. Since k_d^2 is not eigenvalue for $-\Delta$ in Ω, $\mathcal{S}_\Omega^d$ has an inverse $(\mathcal{S}_\Omega^d)^{-1} : H^1(\partial\Omega) \to L^2(\partial\Omega)$. It follows from the third condition in (3.1) that

$$g_2 = -(\mathcal{S}_\Omega^d)^{-1}\Big((\mathcal{S}_{\Omega_e}^d g_1)|_{\partial\Omega}\Big) \quad \text{on } \partial\Omega.$$

Let $X := L^2(\partial\Omega_e) \times L^2(\partial\Omega_e)$ and $Y := H^1(\partial\Omega_e) \times L^2(\partial\Omega_e)$, and define the operator $T : X \to Y$ by

$$T(f,g_1) := \Bigg(\mathcal{S}_{\Omega_e}^e f - \mathcal{S}_{\Omega_e}^d g_1 - \mathcal{S}_\Omega^d(\mathcal{S}_\Omega^d)^{-1}\big((\mathcal{S}_{\Omega_e}^d g_1)|_{\partial\Omega}\big), \\ \frac{1}{\mu_e}\frac{\partial \mathcal{S}_{\Omega_e}^e f}{\partial\nu}\Big|_+ - \frac{1}{\mu_d}\frac{\partial \mathcal{S}_{\Omega_e}^d g_1}{\partial\nu}\Big|_- + \frac{1}{\mu_d}\frac{\partial \mathcal{S}_\Omega^d(\mathcal{S}_\Omega^d)^{-1}\big((\mathcal{S}_{\Omega_e}^d g_1)|_{\partial\Omega}\big)}{\partial\nu}\Bigg).$$

Then (3.1) can be rewritten as $T(f,g_1) = (F,G)$. We also introduce $T_0 : X \to Y$ defined by

$$T_0(f,g_1) := \Bigg(\mathcal{S}_{\Omega_e}^e f - \mathcal{S}_{\Omega_e}^d g_1, \frac{1}{\mu_e}\frac{\partial \mathcal{S}_{\Omega_e}^e f}{\partial\nu}\Big|_+ - \frac{1}{\mu_d}\frac{\partial \mathcal{S}_{\Omega_e}^d g_1}{\partial\nu}\Big|_-\Bigg).$$

We see that $T - T_0$ is a compact operator form X into Y. It is proved in [1] that T_0 is invertible provided that k_e^2 is not a Dirichlet eigenvalue for $-\Delta$ in $\mathbb{R}^2 \setminus \overline{\Omega}_e$. Thus by the Fredholm alternative, it is enough to prove that T is injective. Suppose that $T(f,g_1) = 0$. Then the function u defined by

$$u(x) = \begin{cases} \mathcal{S}_{\Omega_e}^e f(x) & \text{for } x \in \Omega_e \\ \mathcal{S}_{\Omega_e}^d g_1(x) + \mathcal{S}_\Omega^d g_2(x) & \text{for } x \in \Omega_d \end{cases}$$

is the unique solution of the transmission problem

$$\begin{cases} (\Delta + k_e^2)u = 0 & \text{in } \Omega_e, \\ (\Delta + k_d^2)u = 0 & \text{in } \Omega_d, \\ u|_+ = u|_- & \text{on } \partial\Omega_e, \\ \dfrac{1}{\mu_e}\dfrac{\partial u}{\partial\nu}\Big|_+ = \dfrac{1}{\mu_d}\dfrac{\partial u}{\partial\nu}\Big|_- & \text{on } \partial\Omega_e, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with the radiation condition

$$\lim_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial u(x)}{\partial|x|} - ik_e u(x)\right) = 0.$$

By the uniqueness of a solution to the interface problem for the Helmholtz equation, we conclude that $f = g_1 = 0$ and hence $g_2 = 0$. The estimate (3.2) is a consequence of solvability and the closed graph theorem. This completes the proof. $\square$

In the next lemma we give a representation of the solution of (2.3). From now on, we assume that k_d^2 is not a Dirichlet eigenvalue for $-\Delta$ in Ω and k_e^2 is not a Dirichlet eigenvalue for $-\Delta$ in $\mathbb{R}^2 \setminus \overline{\Omega}_e$.

Lemma 3.2. *Let $(\varphi, \psi_1, \psi_2) \in L^2(\partial\Omega_e) \times L^2(\partial\Omega_e) \times L^2(\partial\Omega)$ be the unique solution of*

$$\begin{cases} \mathcal{S}^e_{\Omega_e}\varphi - \mathcal{S}^d_{\Omega_e}\psi_1 - \mathcal{S}^d_{\Omega}\psi_2 = -E_i & \text{on } \partial\Omega_e, \\ \dfrac{1}{\mu_e}\dfrac{\partial(\mathcal{S}^e_{\Omega_e}\varphi)}{\partial\nu}\Big|_+ - \dfrac{1}{\mu_d}\dfrac{\partial(\mathcal{S}^d_{\Omega_e}\psi_1)}{\partial\nu}\Big|_- - \dfrac{1}{\mu_d}\dfrac{\partial(\mathcal{S}^d_{\Omega}\psi_2)}{\partial\nu} = -\dfrac{1}{\mu_e}\dfrac{\partial E_i}{\partial\nu} & \text{on } \partial\Omega_e, \\ \mathcal{S}^d_{\Omega_e}\psi_1 + \mathcal{S}^d_{\Omega}\psi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Then the solution E_h to (2.3) can be represented as

$$E_h(x) = \begin{cases} E_i(x) + \mathcal{S}^e_{\Omega_e}\varphi(x), & x \in \Omega_e, \\ \mathcal{S}^d_{\Omega_e}\psi_1(x) + \mathcal{S}^d_{\Omega}\psi_2(x), & x \in \Omega_d. \end{cases} \tag{3.4}$$

Proof. Note that E_h defined by (3.4) satisfies the differential equations and the transmission conditions on $\partial\Omega_e$ given in (2.3). The uniqueness of a solution to (2.3) proves the claim. $\square$

The following lemma is essentially from [12].

Lemma 3.3. *For $h > 0$ small enough and for any $\phi \in \mathcal{C}^1(\partial\Omega)$, the following expansions hold uniformly in $x \in \partial\Omega$:*

$$\mathcal{S}^e_{\Omega}\phi(x + h\nu_x) = \mathcal{S}^e_{\Omega}\phi(x) + h\frac{\partial\mathcal{S}^e_{\Omega}\phi}{\partial\nu_x}\Big|_+(x) + \mathcal{O}(h^2), \tag{3.5}$$

$$\frac{\partial\mathcal{S}^e_{\Omega}\phi}{\partial\nu_x}(x + h\nu_x) = \frac{\partial\mathcal{S}^e_{\Omega}\phi}{\partial\nu_x}\Big|_+(x) + \mathcal{O}(h), \tag{3.6}$$

where $\mathcal{O}(h^2)$ and $\mathcal{O}(h)$ terms depend on $\|\phi\|_{\mathcal{C}^1(\partial\Omega)}$. Moreover, if $\phi \in L^2(\partial\Omega_e)$, then the following expansions hold uniformly in $x \in \partial\Omega$:

$$\mathcal{S}^e_{\Omega_e}\phi(x + h\nu_x) = \mathcal{S}^e_{\Omega}\hat{\phi}(x) + h\Big((\mathcal{K}^e_{\Omega})^*\hat{\phi}(x) + \mathcal{K}^e_{\Omega}\hat{\phi}(x) + \mathcal{S}^e_{\Omega}(\rho\hat{\phi})(x)\Big) + \mathcal{O}(h^2), \tag{3.7}$$

$$\mathcal{S}^e_{\Omega_e}\phi(x) = \mathcal{S}^e_{\Omega}\hat{\phi}(x) + h\Big((\frac{1}{2}I + \mathcal{K}^e_{\Omega})\hat{\phi}(x) + \mathcal{S}^e_{\Omega}(\rho\hat{\phi})(x)\Big) + \mathcal{O}(h^2), \tag{3.8}$$

where $\hat{\phi}(x) := \phi(x + h\nu_x)$, $x \in \partial\Omega$, and $\rho(x)$ is the curvature at the point $x \in \partial\Omega$. The $\mathcal{O}(h^2)$ terms depends on $\|\phi\|_{L^2(\partial\Omega_e)}$. The same formulae hold for $\mathcal{S}^d_{\Omega}\phi$ and $\mathcal{S}^d_{\Omega_e}\phi$.

Proof. Since $\partial\Omega$ is $\mathcal{C}^{2,\alpha}$, $\mathcal{S}_\Omega^e\phi \in \mathcal{C}^2(\overline{\Omega}_e)$ if $\phi \in \mathcal{C}^1(\partial\Omega)$. Thus (3.5) and (3.6) are simply Taylor expansions.

To derive (3.7), we note that from [5] it follows that

$$\begin{aligned}
&\Phi_e(x+h\nu_x, y+h\nu_y)\\
&= -\frac{i}{4}H_1^0(k_e|(x-y)+h(\nu_x-\nu_y)|)\\
&= \frac{1}{2\pi}\log|(x-y)+h(\nu_x-\nu_y)| + \tau_e\\
&\quad + \sum_{m=1}^{\infty}\Big[b_m \log k_e|x-y+h(\nu_x-\nu_y)| + c_m\Big]\Big[k_e|(x-y)+h(\nu_x-\nu_y)|\Big]^{2m},
\end{aligned}$$

where the constant $\tau_e := \frac{1}{2\pi}\log k_e + \gamma - \frac{i}{4}$ and γ is the Euler constant. Using the formula

$$\log(1+r) = -\sum_{n=1}^{\infty}\frac{(-r)^n}{n} \quad (|r|<1), \tag{3.9}$$

we obtain that

$$\begin{aligned}
&\Phi_e(x+h\nu_x, y+h\nu_y)\\
&= \frac{1}{4\pi}\log|x-y|^2 + \frac{1}{4\pi}\log\Big[1 + h\frac{2(x-y)\cdot(\nu_x-\nu_y)+h|\nu_x-\nu_y|^2}{|x-y|^2}\Big] + \tau_e\\
&+\sum_{m=1}^{\infty}\Big[b_m\log k_e + c_m + \frac{b_m}{2}\log|x-y|^2 + \frac{b_m}{2}\log\Big(1+h\frac{2(x-y)\cdot(\nu_x-\nu_y)+h|\nu_x-\nu_y|^2}{|x-y|^2}\Big)\Big]\\
&\quad\times (k_e)^{2m}\sum_{\ell=0}^{m}\Big[\frac{m!}{\ell!(m-\ell)!}|x-y|^{2\ell}\Big(2h(x-y)\cdot(\nu_x-\nu_y)+h^2|\nu_x-\nu_y|^2\Big)^{m-\ell}\Big]\\
&= \Phi_e(x,y) + \frac{h}{2\pi}\frac{\langle x-y, \nu_x-\nu_y\rangle}{|x-y|^2}\\
&\quad + h\sum_{m=1}^{\infty}\Big(b_m\log k_e|x-y| + c_m\Big)(k_e)^{2m}2m|x-y|^{2m-2}(x-y)\cdot(\nu_x-\nu_y)\\
&\quad + h\sum_{m=1}^{\infty}\Big(b_m\frac{(x-y)\cdot(\nu_x-\nu_y)}{|x-y|^2}k_e^{2m}|x-y|^{2m}\Big) + \mathcal{O}(h^2)\\
&= \Phi_e(x,y) + h\frac{\partial\Phi_e(x,y)}{\partial\nu_x} + h\frac{\partial\Phi_e(x,y)}{\partial\nu_y} + \mathcal{O}(h^2).
\end{aligned}$$

If $d\sigma_e$ denotes the surface measure on $\partial\Omega_e$, then

$$d\sigma_e(y+h\nu_y) = (1+h\rho(y))d\sigma(y) + \mathcal{O}(h^2), \quad y\in\partial\Omega, \tag{3.10}$$

as was shown in [2]. Thus we obtain (3.7).

To obtain (3.8), we use duality. It follows from (3.5) and (3.10) that for any $f \in L^2(\partial\Omega)$,

$$
\begin{aligned}
&\int_{\partial\Omega} \mathcal{S}^e_{\Omega_e}\phi(x)f(x)d\sigma \\
&= \int_{\partial\Omega_e} \phi(\tilde{y})\mathcal{S}^e_{\Omega}f(\tilde{y})d\sigma(\tilde{y}) \\
&= \int_{\partial\Omega} \phi(y+h\nu_y)(\mathcal{S}^e_{\Omega}f)(y+h\nu_y)d\sigma(y+h\nu_y) \\
&= \int_{\partial\Omega} \hat{\phi}(y)\left[\mathcal{S}^e_{\Omega}f(y) + h\left(\frac{1}{2}I + (\mathcal{K}^e_{\Omega})^*\right)f(x)\right](1+h\rho(y))d\sigma(y) + \mathcal{O}(h^2).
\end{aligned}
$$

Thus we get (3.8). This completes the proof. □

We are now in position to derive from (3.2) the following asymptotic expansion. Let $\hat{f}(x) := f(x+h\nu_x)$ for $x \in \partial\Omega$ as before.

Lemma 3.4. *Let $(\varphi, \psi_1, \psi_2) \in L^2(\partial\Omega_e)\times L^2(\partial\Omega_e)\times L^2(\partial\Omega)$ be the unique solution to* (3.3)*. As $h \to 0$, the triple $(\hat{\varphi}, \hat{\psi}_1, \psi_2)$ converges to $(\varphi_0, \psi_1^0, \psi_2^0)$ in $H^1(\partial\Omega)$ where $(\varphi_0, \psi_1^0, \psi_2^0)$ is the unique solution to the integral equation*

$$
\begin{cases}
\mathcal{S}^e_{\Omega}\varphi_0 = -E_i, \\
\psi_1^0 + \psi_2^0 = 0, \\
\psi_1^0 = -\dfrac{\mu_d}{2\mu_e}\varphi_0 - \dfrac{\mu_d}{\mu_e}\mathcal{K}^{*,e}_{\Omega}\varphi_0 - \dfrac{\mu_d}{\mu_e}\dfrac{\partial E_i}{\partial\nu},
\end{cases}
\quad \text{on } \partial\Omega. \tag{3.11}
$$

Proof. Using Lemma 3.3 and taking the limit in (3.3) as $h \to 0$, it follows that

$$
\begin{cases}
\mathcal{S}^e_{\Omega}\hat{\varphi} - \mathcal{S}^d_{\Omega}\hat{\psi}_1 - \mathcal{S}^d_{\Omega}\psi_2 = -E_i, \\
\dfrac{1}{\mu_e}\dfrac{\partial(\mathcal{S}^e_{\Omega}\hat{\varphi})}{\partial\nu}\Big|_+ - \dfrac{1}{\mu_d}\dfrac{\partial(\mathcal{S}^d_{\Omega}\hat{\psi}_1)}{\partial\nu}\Big|_- - \dfrac{1}{\mu_d}\dfrac{\partial(\mathcal{S}^d_{\Omega}\psi_2)}{\partial\nu}\Big|_+ = -\dfrac{1}{\mu_e}\dfrac{\partial E_i}{\partial\nu}, \\
\mathcal{S}^d_{\Omega}\hat{\psi}_1 + \mathcal{S}^d_{\Omega}\psi_2 = 0,
\end{cases}
\quad \text{on } \partial\Omega. \tag{3.12}
$$

Let φ_0, ψ_1^0 and ψ_2^0 be the solutions satisfying above equations. Since $\mathcal{S}^d_{\Omega}$ is invertible, it follows from the third equation in (3.12) that

$$\psi_1^0 + \psi_2^0 = 0 \quad \text{on } \partial\Omega.$$

It also follows from the second equation in (3.12) that

$$\frac{1}{2}(\psi_1^0 - \psi_2^0) = -\frac{\mu_d}{2\mu_e}\varphi_0 - \frac{\mu_d}{\mu_e}\mathcal{K}^{*,e}_{\Omega}\varphi_0 - \frac{\mu_d}{\mu_e}\frac{\partial E_i}{\partial\nu}.$$

Therefore the proof is completed. □

Define E_0 by

$$E_0(x) = \mathcal{S}_\Omega^e \varphi_0(x) + E_i(x), \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}. \tag{3.13}$$

Note that E_0 is the solution to the following scattering problem:

$$\begin{cases} (\Delta + k_e^2) E_0 = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ E_0 = 0 \quad \text{on } \partial\Omega, \\ \displaystyle\lim_{|x|\to\infty} \sqrt{|x|} \Big(\frac{\partial (E_0 - E_i)(x)}{\partial |x|} - i k_e (E_0 - E_i)(x) \Big) = 0. \end{cases} \tag{3.14}$$

Lemma 3.5. *Let φ and φ_0 be as in the previous lemma, and define $\varphi^{1,h}$ by*

$$\varphi^{1,h} := \frac{\hat{\varphi} - \varphi_0}{h}.$$

Then as $h \to 0$, $\varphi^{1,h}$ converges to φ_1 in $L^2(\partial\Omega)$ which satisfies

$$\mathcal{S}_\Omega^e \varphi_1(x) = \Big(\frac{\mu_d}{2\mu_e} I + \Big(\frac{\mu_d}{\mu_e} - 1 \Big) (\mathcal{K}_\Omega^e)^* - \mathcal{K}_\Omega^e - \mathcal{S}_\Omega^e M_\rho \Big) \varphi_0 + \Big(\frac{\mu_d}{\mu_e} - 1 \Big) \frac{\partial E_i}{\partial \nu} \quad \text{on } \partial\Omega, \tag{3.15}$$

where M_ρ is the multiplication operator by ρ.

Proof. Subtracting the third equations in (3.3) and (3.12), we get, for $x \in \partial\Omega$,

$$\begin{aligned} 0 &= [\mathcal{S}_{\Omega_e}^d \psi_1(x) - \mathcal{S}_\Omega^d \psi_1^0(x)] + [\mathcal{S}_\Omega^d \psi_2(x) - \mathcal{S}_\Omega^d \psi_2^0(x)] \\ &= [\mathcal{S}_{\Omega_e}^d \psi_1(x) - \mathcal{S}_\Omega^d \hat{\psi}_1(x)] + \mathcal{S}_\Omega^d (\hat{\psi}_1 - \psi_1^0)(x) + \mathcal{S}_\Omega^d (\psi_2 - \psi_2^0)(x). \end{aligned}$$

If we define $(\psi_1^{1,h}, \psi_2^{1,h}) = (\frac{\hat{\psi}_1 - \psi_1^0}{h}, \frac{\psi_2 - \psi_2^0}{h})$ and divide above equation by h, then we get

$$0 = \frac{\mathcal{S}_{\Omega_e}^d \psi_1(x) - \mathcal{S}_\Omega^d \hat{\psi}_1(x)}{h} + \mathcal{S}_\Omega^d \Big(\psi_1^{1,h} + \psi_2^{1,h} \Big)(x), \quad x \in \partial\Omega.$$

Sending $h \to 0$, it follows from Lemma 3.3 that

$$\psi_1^1 + \psi_2^1 := \lim_{h \to 0} \Big(\psi_1^{1,h} + \psi_2^{1,h} \Big) = -(\mathcal{S}_\Omega^d)^{-1} \Big(\frac{1}{2} I + \mathcal{K}_\Omega^d + \mathcal{S}_\Omega^d M_\rho \Big) (\psi_1^0). \tag{3.16}$$

By virtue of Lemma 3.3, the first equation in (3.3) takes the form

$$\begin{aligned} &\mathcal{S}_\Omega^e \hat{\varphi} + h \Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho \Big) \hat{\varphi} - \mathcal{S}_\Omega^d \hat{\psi}_1 - h \Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho \Big) \hat{\psi}_1 \\ &- \mathcal{S}_\Omega^d \psi_2 - h \Big(\frac{1}{2} I + \mathcal{K}_\Omega^d \Big) \psi_2 = -E_i - h \frac{\partial E_i}{\partial \nu} + \mathcal{O}(h^2), \end{aligned}$$

on $\partial\Omega$. By subtracting the above equation from the first equation in (3.12), we get

$$\mathcal{S}_\Omega^e \varphi^{1,h} = \mathcal{S}_\Omega^d\Big(\psi_1^{1,h} + \psi_2^{1,h}\Big) - \Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho\Big)\hat{\varphi} + \Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho\Big)\hat{\psi}_1 + \Big(\frac{1}{2}I + \mathcal{K}_\Omega^d\Big)\psi_2 - \frac{\partial E_i}{\partial \nu} + \mathcal{O}(h^2).$$

Observe from (3.16) that the right-hand side of above equation converges in $L^2(\partial\Omega)$, as $h \to 0$, to

$$-\Big(\frac{1}{2}I + \mathcal{K}_\Omega^d + \mathcal{S}_\Omega^d M_\rho\Big)(\psi_1^0) - \Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho\Big)\varphi_0 + \Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho\Big)\psi_1^0 + \Big(\frac{1}{2}I + \mathcal{K}_\Omega^d\Big)\psi_2^0 - \frac{\partial E_i}{\partial \nu}.$$

It now follows from (3.11) that $\varphi^{1,h}$ converges to φ_1 in $H^1(\partial\Omega)$ as $h \to 0$ and φ_1 satisfies (3.15). This completes the proof. □

In view of the third equation in Lemma 3.3 and Lemma 3.5, we get

$$\begin{aligned}
\mathcal{S}_{\Omega_e}^e \varphi(x + h\nu_x) &= \mathcal{S}_\Omega^e\varphi_0(x) + h\mathcal{S}_\Omega^e\varphi_1(x) + h\Big((\mathcal{K}_\Omega^e)^* + \mathcal{K}_\Omega^e + \mathcal{S}_\Omega^e M_\rho\Big)\varphi_0(x) + o(h)\\
&= \mathcal{S}_\Omega^e\varphi_0(x) + h\Big[\frac{\mu_d}{\mu_e}\Big(\frac{\partial(\mathcal{S}_\Omega^e\varphi_0)}{\partial\nu}\Big|_+(x) + \frac{\partial E_i}{\partial\nu}(x)\Big) - \frac{\partial E_i}{\partial\nu}(x)\Big] + o(h)\\
&= \mathcal{S}_\Omega^e\varphi_0(x) + h\Big[\frac{\mu_d}{\mu_e}\frac{\partial E_0}{\partial\nu}\Big|_+(x) - \frac{\partial E_i}{\partial\nu}(x)\Big] + o(h),
\end{aligned}$$

for $x \in \partial\Omega$ and hence,

$$E_h(x + h\nu_x) = h\frac{\mu_d}{\mu_e}\frac{\partial E_0}{\partial\nu}(x) + o(h) \quad \text{for } x \in \partial\Omega. \tag{3.17}$$

Thus we get the following theorem.

Theorem 3.1. *Let E_0 be given by (3.13) and let E_1 be the solution to*

$$\begin{cases}
(\Delta + k_e^2)E_1 = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{\Omega},\\
E_1 = (\frac{\mu_d}{\mu_e} - 1)\frac{\partial E_0}{\partial\nu} \quad \text{on } \partial\Omega,\\
\lim\limits_{|x|\to\infty} \sqrt{|x|}\left(\frac{\partial E_1(x)}{\partial|x|} - ik_eE_1(x)\right) = 0.
\end{cases}$$

Then the following asymptotic expansion for E_h:

$$E_h(x) = E_0(x) + hE_1(x) + o(h), \tag{3.18}$$

holds uniformly in any bounded subset of Ω_e.

We should emphasize the fact that the (first-order) asymptotic expansion derived in Theorem 3.1 is independent of the electric permittivity of the thin layer. The effect of the profile of the electric permittivity is of higher-order.

4 Representation for E_h^Γ

We begin this section by proving the following uniqueness result.

Lemma 4.1. *Problem* (2.2) *has at most one solution in* $H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Omega})$.

Proof. Let E_h^Γ be the solution of (2.2) corresponding to $E_i \equiv 0$. Then we see that $E_h^\Gamma|_+ = E_h^\Gamma|_- = 0$ on Γ. Applying Green's formula to a domain B_R of radius R containing Ω_d, and using the transmission conditions, we have

$$\begin{aligned}
&\int_{\Omega_d} \left(\frac{1}{\mu_d} |\nabla E_h^\Gamma|^2 - \omega^2 \epsilon_d |E_h^\Gamma|^2 \right) + \int_{\Omega_e \cap B_R} \left(\frac{1}{\mu_e} |\nabla E_h^\Gamma|^2 - \omega^2 \epsilon_e |E_h^\Gamma|^2 \right) \\
&= \int_{\Omega_d} \left(-\frac{1}{\mu_d} \overline{E_h^\Gamma} \Delta E_h^\Gamma - \omega^2 \epsilon_d |E_h^\Gamma|^2 \right) + \int_{\partial \Omega_e} \left(\frac{1}{\mu_d} \overline{E_h^\Gamma} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_- - \frac{1}{\mu_e} \overline{E_h^\Gamma} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_+ \right) \\
&\quad + \int_{\Omega_e \cap B_R} \left(-\frac{1}{\mu_e} \overline{E_h^\Gamma} \Delta E_h^\Gamma - \omega^2 \epsilon_e |E_h^\Gamma|^2 \right) + \int_{\partial B_R} \frac{1}{\mu_e} \overline{E_h^\Gamma} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_- \\
&= \int_{\partial B_R} \frac{1}{\mu_e} \overline{E_h^\Gamma} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_- + \int_\Gamma \left(\frac{1}{\mu_d} \overline{E_h^\Gamma} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_- - \frac{1}{\mu_e} \overline{E_h^\Gamma} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_+ \right) \\
&= \int_{\partial B_R} \frac{\overline{E_h^\Gamma}}{\mu_e} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_- .
\end{aligned}$$

Taking the imaginary part of both sides, we obtain

$$0 = \mathrm{Im} \int_{\partial B_R} \frac{\overline{E_h^\Gamma}}{\mu_e} \frac{\partial E_h^\Gamma}{\partial \nu}\Big|_- .$$

Using Rellich's lemma and the unique continuation principle, we arrive at $E_h^\Gamma\Big|_{\Omega_d} = E_h^\Gamma\Big|_{\Omega_e} = 0$. This completes the proof. □

Define the Dirichlet-to-Neumann map $\Lambda : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ by

$$\Lambda(g) = \frac{\partial u}{\partial \nu}\Big|_{\partial B_R},$$

where u is the unique solution to the exterior Dirichlet problem for the Helmholtz equation in $\mathbb{R}^d \setminus \overline{B}_R$ with the Dirichlet boundary data g on ∂B_R which satisfies the radiation condition. The following result from [8] is of use to us.

Lemma 4.2. *There exists an operator* $\Lambda_0 : H^{\frac{1}{2}}(\partial B_R) \to H^{-\frac{1}{2}}(\partial B_R)$ *such that*

$$\int_{\partial B_R} \overline{\varphi} \Lambda_0 \varphi \leq 0,$$

and $\Lambda - \Lambda_0$ *is a compact operator from* $H^{\frac{1}{2}}(\partial B_R)$ *to* $H^{-\frac{1}{2}}(\partial B_R)$.

Using arguments similar to those in [6], we can obtain the following lemma.

Lemma 4.3. *For any incident field E_i, there exists a unique solution E_h^Γ in $H^1_{loc}(\mathbb{R}^2 \setminus \overline{\Omega})$ to* (2.2).

Proof. Let $H^1_\Gamma(B_R \setminus \overline{\Omega}) := \Big\{ \varphi \in H^1(B_R \setminus \overline{\Omega}) \mid \varphi = 0 \text{ on } \Gamma \cup \partial\Omega \Big\}$. We formulate the following variational problem which is equivalent to solving (2.2): Find $u \in H^1_\Gamma(B_R \setminus \overline{\Omega})$ such that

$$\begin{aligned}
&\int_{\Omega_d} \Big(\frac{\nabla\overline{\varphi} \cdot \nabla u}{\mu_d} - \omega^2 \epsilon_d \overline{\varphi} u \Big) + \int_{\Omega_e \cap B_R} \Big(\frac{\nabla\overline{\varphi} \cdot \nabla u}{\mu_e} - \omega^2 \epsilon_e \overline{\varphi} u \Big) \\
&= \int_{\partial B_R} \frac{\overline{\varphi}}{\mu_e} \frac{\partial u}{\partial \nu} \\
&= \int_{\partial B_R} \frac{\overline{\varphi}}{\mu_e} \Lambda(u + E_i) - \int_{\partial B_R} \frac{\overline{\varphi}}{\mu_e} \frac{\partial E_i}{\partial \nu},
\end{aligned}$$

for any function $\varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega})$. Define

$$\begin{aligned}
A_1(u, \varphi) &:= \int_{\Omega_d} \Big(\frac{\nabla\overline{\varphi} \cdot \nabla u}{\mu_d} + \overline{\varphi} u \Big) + \int_{\Omega_e \cap B_R} \Big(\frac{\nabla\overline{\varphi} \cdot \nabla u}{\mu_e} + \overline{\varphi} u \Big) - \int_{\partial B_R} \frac{\overline{\varphi}}{\mu_e} \Lambda_0(u), \\
A_2(u, \varphi) &:= \int_{\partial B_R} \frac{\overline{\varphi}}{\mu_e} (\Lambda_0 - \Lambda)(u) - \int_{\Omega_d} (\omega^2 \epsilon_d + 1) \overline{\varphi} u - \int_{\Omega_e \cap B_R} (\omega^2 \epsilon_e + 1) \overline{\varphi} u,
\end{aligned}$$

and

$$L(\varphi) := \int_{\partial B_R} \frac{\overline{\varphi}}{\mu_e} \Big(\Lambda(E_i) - \frac{\partial E_i}{\partial \nu} \Big),$$

for $u, \varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega})$. The problem (2.2) can be rewritten as follows:

$$A_1(E_h^\Gamma - E_i \chi(\Omega_e), \varphi) + A_2(E_h^\Gamma - E_i \chi(\Omega_e), \varphi) = L(\varphi) \quad \text{for all } \varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega}). \tag{4.1}$$

Using Lemma 4.2, we get

$$\begin{aligned}
\mathrm{Re}(A_1(u,u)) &= \int_{\Omega_d} \Big(\frac{|\nabla u|^2}{\mu_d} + |u|^2 \Big) + \int_{\Omega_e \cap B_R} \Big(\frac{|\nabla u|^2}{\mu_e} + |u|^2 \Big) - \mathrm{Re} \int_{\partial B_R} \frac{\bar{u}}{\mu_e} \Lambda_0(u) \\
&\geq C_2 \|u\|^2_{H^1(B_R \setminus \bar{\Omega})}.
\end{aligned}$$

Furthermore, $|A_1(u, \varphi)| \leq C \|u\|^2_{H^1(B_R \setminus \bar{\Omega})} \|\varphi\|^2_{H^1(B_R \setminus \bar{\Omega})}$. Thus, by the Lax-Milgram theorem and the Riesz Representation theorem, there is a bounded linear operator T on $H^1_\Gamma(B_R \setminus \overline{\Omega})$ having a bounded inverse such that $(Tu, \varphi) = A_1(u, \varphi)$ for all $u, \varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega})$ where $(\cdot, \cdot)$ is the inner product on $H^1_\Gamma(B_R \setminus \overline{\Omega})$.

We define the operator K on $H^1_\Gamma(B_R \setminus \overline{\Omega})$ by

$$(Ku, \varphi) = A_2(u, \varphi), \quad \text{for all } u, \varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega}).$$

The compact embedding of $H^1(B_R \setminus \overline{\Omega})$ into $L^2(B_R \setminus \overline{\Omega})$ and the compactness of $\Lambda - \Lambda_0$ from Lemma 4.2 imply that the operator K is compact. In short, we have

$$((T+K)u, \varphi) = A_1(u, \varphi) + A_2(u, \varphi), \quad \text{for all } \varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega}),$$

with T invertible and K compact. If $(T+K)u = 0$, then

$$A_1(u, \varphi) + A_2(u, \varphi) = 0, \quad \text{for all } \varphi \in H^1_\Gamma(B_R \setminus \overline{\Omega}),$$

and hence by Lemma 4.1 $u = 0$. Using the Fredholm alternative, we have existence of a solution to (4.1). This completes the proof. □

Let $G_h(x,y)$, $x, y \in \mathbb{R}^2 \setminus \overline{\Omega}$, be the Green's function satisfying

$$\begin{cases} \Big(\Delta + k_e^2\Big) G_h = \delta_y & \text{in } \Omega_e, \\ \Big(\Delta + k_d^2\Big) G_h = \delta_y & \text{in } \Omega_d, \\ \Big| \dfrac{\partial G_h}{\partial \nu} - i k_e G_h \Big| = \mathcal{O}\Big(\dfrac{1}{\sqrt{|x|}}\Big), & \\ \dfrac{1}{\mu_d} \dfrac{\partial G_h}{\partial \nu}\Big|_- = \dfrac{1}{\mu_e} \dfrac{\partial G_h}{\partial \nu}\Big|_+ & \text{on } \partial\Omega_e, \\ G_h\Big|_- = G_h\Big|_+ & \text{on } \partial\Omega_e, \\ G_h = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Then we have the following representation for the solution of (2.2):

$$E_h^\Gamma(x) = E_h(x) + \int_\Gamma G_h(x,y) \psi_\Gamma(y) d\sigma_y, \tag{4.3}$$

where ψ_Γ is given by

$$\psi_\Gamma(x) := \frac{\partial E_h^\Gamma}{\partial \nu_y}\Big|_+ (x) - \frac{\mu_e}{\mu_d} \frac{\partial E_h^\Gamma}{\partial \nu_y}\Big|_- (x).$$

Indeed, by the divergence theorem, we get for $x \in \Omega_e$ that

$$
\begin{aligned}
&\frac{1}{\mu_e}(E_h^\Gamma(x) - E_h(x)) \\
&= \int_{\Omega_e} \frac{1}{\mu_e}(\Delta + k_e^2)G_h(x,y)(E_h^\Gamma - E_h)(y)d\sigma(y) \\
&\qquad + \int_{\Omega_d} \frac{1}{\mu_d}(\Delta + k_d^2)G_h(x,y)(E_h^\Gamma - E_h)(y)d\sigma(y) \\
&= \int_{\partial\Omega_e} \frac{1}{\mu_d}\frac{\partial G_h}{\partial \nu_y}\Big|_-(x,y)(E_h^\Gamma(y) - E_h(y)) - \frac{1}{\mu_e}\frac{\partial G_h}{\partial \nu_y}\Big|_+(x,y)(E_h^\Gamma(y) - E_h(y))d\sigma(y) \\
&\qquad + \int_{\partial\Omega_e} \frac{G_h(x,y)}{\mu_e}\frac{\partial(E_h^\Gamma - E_h)}{\partial \nu_y}\Big|_+(y) - \frac{G_h(x,y)}{\mu_d}\frac{\partial(E_h^\Gamma - E_h)}{\partial \nu_y}\Big|_-(y)d\sigma(y) \\
&= \int_\Gamma G_h(x,y)\left(\frac{1}{\mu_e}\frac{\partial E_h^\Gamma}{\partial \nu_y}\Big|_+(y) - \frac{1}{\mu_d}\frac{\partial E_h^\Gamma}{\partial \nu_y}\Big|_-(y)\right)d\sigma(y).
\end{aligned}
$$

We now derive an asymptotic formula for E_h^Γ. Since $G_h(x,y)$ is the Green's function of (4.2), for any continuous function f on $\partial\Omega$ the function u defined by

$$
u(x) := \int_{\partial\Omega} \frac{\partial G_h(x,y)}{\partial \nu_y} f(y)d\sigma(y), \quad x \in \mathbb{R}^2 \setminus \overline{\Omega}
$$

is the solution to

$$
\begin{cases}
(\Delta + k_e^2)u = 0 & \text{in } \Omega_e, \\
(\Delta + k_d^2)u = 0 & \text{in } \Omega_d, \\
u|_+ = u|_- & \text{on } \partial\Omega_e, \\
\dfrac{1}{\mu_e}\dfrac{\partial u}{\partial \nu}\Big|_+ = \dfrac{1}{\mu_d}\dfrac{\partial u}{\partial \nu}\Big|_- & \text{on } \partial\Omega_e, \\
u = f & \text{on } \partial\Omega,
\end{cases}
$$

with the radiation condition. In particular,

$$
\lim_{x \to x_0 \in \partial\Omega} \int_{\partial\Omega} \frac{\partial G_h(x,y)}{\partial \nu_y} f(y)d\sigma(y) = f(x_0) \tag{4.4}
$$

if f is continuous at x_0. Therefore, we get

$$
E_h(x) - E_i(x) = -\int_{\partial\Omega} \frac{\partial G_h(x,y)}{\partial \nu_y} E_i(y)d\sigma(y). \tag{4.5}
$$

Let G_0 be the Green's function for $\Delta + k_e^2$ in $\mathbb{R}^2 \setminus \overline{\Omega}$, i.e.,

$$
\begin{cases}
(\Delta_x + k_e^2)G_0(x,y) = \delta_y & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\
G_0(x,y) = 0 & x \in \partial\Omega,\ y \in \mathbb{R}^2 \setminus \overline{\Omega},
\end{cases} \tag{4.6}
$$

together with the radiation condition. Note that (4.4) holds with G_h replaced with G_0. Since E_0 is the solution to (3.14), we also have

$$E_0(x) - E_i(x) = -\int_{\partial\Omega} \frac{\partial G_0(x,y)}{\partial \nu_y} E_i(y) d\sigma(y). \tag{4.7}$$

It then follows from (3.18), (4.5), and (4.7) that

$$\int_{\partial\Omega} \frac{\partial G_h(x,y)}{\partial \nu_y} E_i(y) d\sigma(y) = \int_{\partial\Omega} \frac{\partial G_0(x,y)}{\partial \nu_y} E_i(y) d\sigma(y) + \mathcal{O}(h).$$

Since this identity holds for any incidence field E_i, we have

$$\frac{\partial G_h(x,y)}{\partial \nu_y} = \frac{\partial G_0(x,y)}{\partial \nu_y} + \mathcal{O}(h), \tag{4.8}$$

which holds uniformly for x in a bounded subset of Ω_e and $y \in \partial\Omega$.

By Taylor expansion, we get, for $y \in \partial\Omega$,

$$G_h(x, y + h\nu_y) = G_h(x,y) + h\frac{\partial G_h}{\partial \nu_y}(x,y) + o(h) = h\frac{\partial G_h}{\partial \nu_y}(x,y) + o(h). \tag{4.9}$$

It then follows from (4.8) that for $x \in \Omega_e$,

$$\begin{aligned}\int_\Gamma G_h(x,y)\psi_\Gamma(y)d\sigma(y) &= \int_{\Gamma_0} G_h(x, y+h\nu_y)\psi_\Gamma(y+h\nu_y)(1+h\rho(y))d\sigma(y) \\ &= h\int_{\Gamma_0} \frac{\partial G_0}{\partial \nu_y}(x,y)\widetilde{\psi}(y)d\sigma(y) + o(h),\end{aligned}$$

where

$$\widetilde{\psi}(x) = \psi(x + h\nu_x) \quad \text{for } x \in \Gamma_0. \tag{4.10}$$

If $x = x_0 + h\nu_{x_0}$ for some $x_0 \in \Gamma_0$, then

$$\frac{\partial G_0}{\partial \nu_y}(x_0 + h\nu_{x_0}, y) = \frac{\partial G_0}{\partial \nu_y}(x_0, y) + \mathcal{O}(h),$$

and hence, we have

$$\begin{aligned}\int_\Gamma G_h(x,y)\psi_\Gamma(y)d\sigma(y) &= h\int_{\Gamma_0} \frac{\partial G_0}{\partial \nu_y}(x_0,y)\widetilde{\psi}(y)d\sigma(y) + o(h) \\ &= h\widetilde{\psi}(x_0) + o(h),\end{aligned} \tag{4.11}$$

where the last equality holds thanks to (4.4).

Note that by Theorem 3.1

$$E_h(x + h\nu_x) = h\frac{\mu_d}{\mu_e}\frac{\partial E_0}{\partial \nu}(x) + o(h), \quad x \in \Gamma_0.$$

It then follows from (4.11) and the condition on the patch

$$\int_{\Gamma} G_h(x,y)\psi_{\Gamma}(y)d\sigma(y) = -E_h \quad \text{on } \Gamma$$

that

$$\widetilde{\psi}(x) = -\frac{\mu_d}{\mu_e}\frac{\partial E_0}{\partial \nu}(x), \quad x \in \Gamma_0.$$

We finally arrive at the following result.

Theorem 4.1. *We have the following asymptotic formula for* E_h^{Γ}*:*

$$E_h^{\Gamma}(x) - E_h(x) = -h\frac{\mu_d}{\mu_e}\int_{\Gamma_0}\frac{\partial G_0}{\partial \nu_y}(x,y)\frac{\partial E_0}{\partial \nu}(y)d\sigma(y) + o(h), \tag{4.12}$$

which holds uniformly in any bounded subset of Ω_e *for* h *small enough.*

5 Numerical experiments

In this section, we perform numerical experiments to demonstrate the validity of the approximation formulae (3.18) and (4.12). We first provide an explicit form for G_h defined by (4.2) when $\partial\Omega$ and $\partial\Omega_e$ are the disks centered at the origin with radius r and $r+h$. If we write

$$G_e(x,y) = \Phi_e(x,y) + H_e(x,y) = -\frac{i}{4}H_0^{(1)}(k_e|x-y|) + H_e(x,y),$$
$$G_d(x,y) = \Phi_d(x,y) + H_d(x,y) = -\frac{i}{4}H_0^{(1)}(k_d|x-y|) + H_d(x,y),$$

then the functions H_e and H_d satisfy

$$(\Delta + k_e^2)H_e(x,y) = 0 \quad \text{for } x,y \in \Omega_e, \tag{5.1}$$
$$(\Delta + k_d^2)H_d(x,y) = 0 \quad \text{for } x,y \in \Omega_d, \tag{5.2}$$

In addition, by (4.2), H_e and H_d are subject to the transmission conditions:

$$H_e(x,y) - H_d(x,y) = \Phi_d(x,y) - \Phi_e(x,y) \quad \text{for } x,y \in \partial\Omega_e, \tag{5.3}$$
$$\frac{1}{\mu_e}\frac{\partial H_e(x,y)}{\partial \nu_x} - \frac{1}{\mu_d}\frac{\partial H_d(x,y)}{\partial \nu_x} = \frac{1}{\mu_d}\frac{\partial \Phi_d}{\partial \nu_x}\Big|_{-}(x,y) - \frac{1}{\mu_e}\frac{\partial \Phi_e}{\partial \nu_x}\Big|_{+}(x,y) \quad \text{on } x,y \in \partial\Omega, \tag{5.4}$$
$$H_d(x,y) = -\Phi_d(x,y) \quad \text{for } x \in \Omega_d \text{ and } y \in \partial\Omega, \tag{5.5}$$

and the radiation condition:

$$\lim_{|x|\to\infty}\sqrt{|x|}\left(\frac{\partial H_e(x,y)}{\partial |x|} - ik_e H_e(x,y)\right) = 0.$$

Since H_e is the solution of (5.1) satisfying the radiation condition, we have

$$H_e(x,y) = \sum_{n=0}^{\infty} a_n^e H_n^{(1)}(k_e|x|) H_n^{(1)}(k_e|y|) \cos n(\theta_x - \theta_y),$$

for $x = |x|e^{i\theta_x}$ and $y = |y|e^{i\theta_y}$. In addition, because of (5.5), we have

$$H_d(x,y) = \sum_{n=0}^{\infty} \Big(a_n^d J_n(k_d|x|) J_n(k_d|y|) + b_n^d J_n(k_d|x|) Y_n(k_d|y|) + b_n^d Y_n(k_d|x|) J_n(k_d|y|) + c_n^d Y_n(k_d|x|) Y_n(k_d|y|) \Big) \cos n(\theta_x - \theta_y).$$

Here the functions $J_n(t)$ and $Y_n(t)$ are the spherical Bessel functions and the spherical Neumann functions of order n satisfying

$$t^2 f''(t) + t f'(t) + [t^2 - n^2] f(t) = 0.$$

We now find the complex constants $a_n^e, a_n^d, b_n^d, c_n^d$ for $n = 0, 1, \ldots$. Set $\ell(m) = 1$ if $m \neq 0$ and $\ell(0) = 2$. Using the condition (5.5) and the fact that for $x \in \Omega_d$ and $y \in \partial\Omega$

$$H_0^{(1)}(k|x-y|) = H_0^{(1)}(k|x|) J_0(k|y|) + 2 \sum_{n=1}^{\infty} H_0^{(1)}(k|x|) J_n(k|y|) \cos n\theta,$$

we derive

$$a_m^d = -b_m^d \frac{Y_0(k_d r)}{J_0(k_d r)} + \frac{i}{2\ell(m)}, \tag{5.6}$$

$$c_m^d = -b_m^d \frac{J_m(k_d r)}{Y_m(k_d r)} - \frac{1}{2\ell(m)} \frac{J_m(k_d r)}{Y_m(k_d r)}, \tag{5.7}$$

for $m \geq 1$. Multiplying (5.3) and (5.4) by $\cos m(\theta_x - \theta_y)/(2\pi^2)$ for $m = 0, 1, \ldots$, and integrating over $\partial\Omega_e$ yield

$$\begin{aligned} &\frac{1}{2\pi^2} \int_{\partial\Omega_e} \int_{\partial\Omega_e} \Big(H_e(x,y) - H_d(x,y) \Big) \cos m(\theta_x - \theta_y) d\sigma(x) d\sigma(y) \\ =& \frac{1}{2\pi^2} \int_{\partial\Omega_e} \int_{\partial\Omega_e} \Big(\Phi_d(x,y) - \Phi_e(x,y) \Big) \cos m(\theta_x - \theta_y) d\sigma(x) d\sigma(y), \end{aligned}$$

and

$$\begin{aligned} &\frac{1}{2\pi^2} \int_{\partial\Omega_e} \int_{\partial\Omega_e} \Big(\frac{1}{\mu_e} \frac{\partial H_e(x,y)}{\partial \nu_x} - \frac{1}{\mu_d} \frac{\partial H_d(x,y)}{\partial \nu_x} \Big) \cos m(\theta_x - \theta_y) d\sigma(x) d\sigma(y) \\ =& \frac{1}{2\pi^2} \int_{\partial\Omega_e} \int_{\partial\Omega_e} \Big(\frac{1}{\mu_d} \frac{\partial \Phi_d(x,y)}{\partial \nu_x} - \frac{1}{\mu_e} \frac{\partial \Phi_e(x,y)}{\partial \nu_x} \Big) \cos m(\theta_x - \theta_y) d\sigma(x) d\sigma(y). \end{aligned}$$

It then follows that, for $m = 0, 1, \dots,$

$$\begin{aligned} & a_m^e H_m^{(1)}(k_e(r+h))^2 \\ & \quad - a_m^d J_m(k_d(r+h))^2 - 2b_m^d J_m(k_d(r+h)) Y_m(k_d(r+h)) - c_m^d Y_m(k_d(r+h))^2 \\ & = \frac{1}{2(r+h)^2\pi^2\ell(m)} \int_{\partial\Omega_e}\int_{\partial\Omega_e} \Big(\Phi_d(x,y) - \Phi_e(x,y)\Big) \cos m(\theta_x - \theta_y) d\sigma(x) d\sigma(y), \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} & a_m^e \frac{H_m^{(1)}(k_e(r+h))}{\mu_e}\Big(\frac{mH_m^{(1)}(k_e(r+h))}{r+h} - k_e H_{m+1}^{(1)}(k_d(r+h))\Big) \\ & \quad - a_m^d \frac{J_m(k_d(r+h))}{\mu_d}\Big(\frac{mJ_m(k_d(r+h))}{r+h} - k_d J_{m+1}(k_d(r+h))\Big) \\ & \quad - b_m^d \left(\frac{J_m(k_d(r+h))}{\mu_d}\Big(\frac{mY_m(k_d(r+h))}{r+h} - k_d Y_{m+1}\Big)\right. \\ & \qquad \left. + \frac{Y_m(k_d(r+h))}{\mu_d}\Big(\frac{mJ_m(k_d(r+h))}{r+h} - k_d J_{m+1}\Big)\right) \\ & \quad - c_m^d \frac{Y_m(k_d(r+h))}{\mu_d}\Big(\frac{mY_m(k_d(r+h))}{r+h} - k_d Y_{m+1}\Big) \\ & = \frac{1}{2(r+h)^2\pi^2\ell(m)} \int_{\partial\Omega_e}\int_{\partial\Omega_e} \left(\frac{1}{\mu_d}\frac{\partial\Phi_d(x,y)}{\partial\nu_x} - \frac{1}{\mu_e}\frac{\partial\Phi_e(x,y)}{\partial\nu_x}\right) \cos m(\theta_x - \theta_y) d\sigma(x) d\sigma(y). \end{aligned} \tag{5.9}$$

Using (5.6)-(5.9), we get the values of $a_m^e, a_m^d, b_m^d, c_m^d$ for $m = 0, 1, \dots$.

Finally, we illustrate our approximate boundary condition in the following numerical experiments. Our configuration involves two circular disks of radii 0.5 and 0.49 so that $h = 0.01$. Corresponding dielectric permittivities ϵ_e and ϵ_d are equated to 2 and 6 and magnetic permeabilities μ_e and μ_d are equated to 4 and 3. The frequency is fixed to $\omega = 1$ and $E_i = e^{ik_e x\cdot d}$ with $d = (1, 0)$.

We solve for the solutions $\tilde{E}_h$ and $\tilde{E}_h^\Gamma$ using our approximate boundary conditions, and E_h and E_h^Γ using a boundary integral method. To accomplish this, we discretize the integral equations at the node points on $\partial\Omega$ and on $\partial\Omega_e$ given by

$$\xi_n^d = r\left(\cos\frac{2\pi(n-1)}{N}, \sin\frac{2\pi(n-1)}{N}\right) \quad \text{on } \partial\Omega,$$

and

$$\xi_n^e = (r+h)\left(\cos\frac{2\pi(n-1)}{N}, \sin\frac{2\pi(n-1)}{N}\right) \quad \text{on } \partial\Omega_e,$$

for $n = 1, 2, \dots, N$, with $N = 256$.

Example 1. In this example, we compare E_h (computed using a boundary integral method) and $\tilde{E}_h$ (using the approximate boundary condition in Theorem 3.1) without

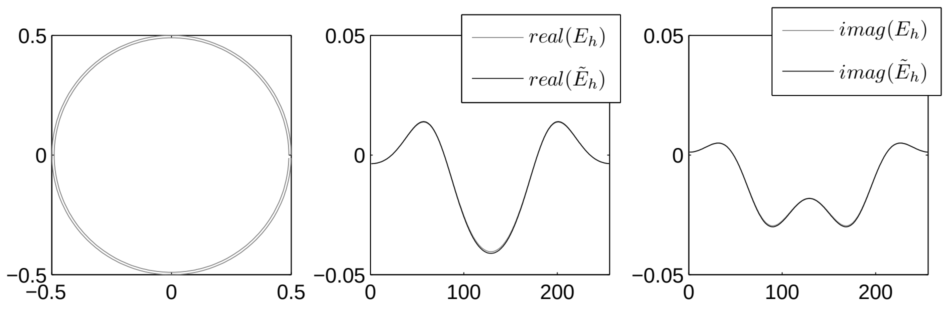

Figure 1: The case without patch.

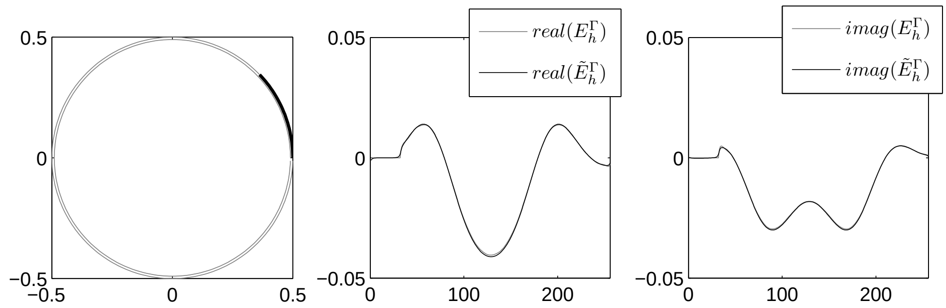

Figure 2: The case with patch.

patch. Fig. 1 shows the numerical results. In the first diagram, the grey line is the real part of E_h and the black line is the real part of $\tilde{E}_h$ on $\partial\Omega_e$. The second diagram shows the imaginary parts of E_h and $\tilde{E}_h$ on $\partial\Omega_e$. The errors computed in L^2 and L^∞ are $||E_h - \tilde{E}_h||_{L^2(\partial\Omega_e)} = 3.4091e\text{-}4$ and $\max|E_h - \tilde{E}_h| = 7.1113e\text{-}4$, respectively, which are of order h^2.

Example 2. In the second example we consider the case with patch. The configuration is the same as in Example 1. In the first diagram in Fig. 2, the black-line represents the patch Γ. The mesh points on Γ are given by

$$\xi_n^{e,\Gamma} = (r+h)\Big(\cos\frac{2\pi(n-1)}{N_\Gamma}, \sin\frac{2\pi(n-1)}{N_\Gamma}\Big),$$

for $n = 1, 2, \ldots, N_\Gamma$, with $N_\Gamma = 32$. The second and third diagrams express the difference between the real and imaginary parts of E_h^Γ (the exact field which is computed by solving the integral equation (4.3)) and $\tilde{E}_h^\Gamma$ (solved using the approximate boundary condition in Theorem 4.1) on $\partial\Omega_e$. The errors are $||E_h - \tilde{E}_h||_{L^2(\partial\Omega_e)} = 3.9992e\text{-}4$ and $\max|E_h - \tilde{E}_h| = 2.1e\text{-}3$. Relatively large errors occur at the end points of the patch.

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References

[1] H. Ammari and H. Kang, Boundary layer techniques for solving the Helmholtz equation in the presence of small inhomogeneities, J. Math. Anal. Appl., 296 (2004), 190–208.
[2] H. Ammari, H. Kang, M. Lim and H. Zribi, Conductivity interface problems. Part I: Small perturbations of an interface, preprint.
[3] H. Ammari, H. Kang and F. Santosa, Scattering of electromagnetic waves by thin dielectric structures, SIAM J. Math. Anal., to appear.
[4] A. Bendali and K. Lemrabet, The effect of a thin coating on the scattering of a time-harmonic wave for the Helmholtz equation, SIAM J. Appl. Math., 56 (1996), 1664–1693.
[5] D. Colton and R. Kress, Inverse Acoustic and Electromagnetic Scatterig Theory, Springer, Berlin, 1998, 2nd edition.
[6] F. Cakoni, D. Colton and P. Monk, The determination of the surface conductivity of a partially coated dielectric, SIAM J. Appl. Math., 65 (2005), 767–789.
[7] K. C. Gupta, I. J. Bahl, P. Bhartia and R. Garg, Microstrip Lines and Slotlines, Artech House, London, 1996, 2nd edition.
[8] P. Hähner, On the uniqueness of the shape of a penetrable anisotropic obstacle, J. Comput. Appl. Math., 116 (2000), 167–180.
[9] J. R. James and P. S. Hall (Eds.), Handbook of Microstrip Antennas, Peter Peregrinus, London, 1988.
[10] S. Moskow, F. Santosa and J. Zhang, An approximate method for scattering by thin structures, SIAM J. Appl. Math., 66 (2005), 187–205.
[11] D. M. Pozar and D. H. Schaubert (Eds.), Microstrip Antennas: The Analysis and Design of Microstrip Antennas and Arrays, John Wiley & Sons, New York, 1995.
[12] S. Soussi, Second-harmonic generation in the undepleted pump approximation, Mult. Scal. Model Simul., 4 (2005), 115–148.
[13] W. L. Wendland and E. P. Stephan, A hyersingular boundary integral method for two-dimensional screen and crack problems, Arch. Ration. Mech. Anal., 112 (1990), 363–390.
[14] K. L. Wong, Design of Nonplanar Microstrip Antennas and Transmission Lines, Wiley Ser. Microwave Optical Engrg., Wiley-Interscience, New York, 1999.