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Bilinear Forms for the Recovery-Based Discontinuous Galerkin Method for Diffusion

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Abstract. The present paper introduces bilinear forms that are equivalent to the recovery-based discontinuous Galerkin formulation introduced by Van Leer in 2005. The recovery method approximates the solution of the diffusion equation in a discontinuous function space, while inter-element coupling is achieved by a local L_2 projection that recovers a smooth continuous function underlying the discontinuous approximation. Here we introduce the concept of a local "recovery polynomial basis" – smooth polynomials that are in the weak sense indistinguishable from the discontinuous basis polynomials – and show it allows us to eliminate the recovery procedure. The recovery method reproduces the symmetric discontinuous Galerkin formulation with additional penalty-like terms depending on the targeted accuracy of the method. We present the unique link between the recovery method and discontinuous Galerkin bilinear forms.

AMS subject classifications: 35K05, 65M60

Key words: Diffusion equation, discontinuous Galerkin methods, recovery basis, L_2 projection.

1 Introduction

Highly accurate schemes for solving the advection equation are preferably obtained by means of the discontinuous Galerkin (DG) method. This method approximates the solution in an element-wise continuous function space that is globally discontinuous. Because of the physics of advection, the method acquires a natural upwind character that renders the discontinuities at the cell interfaces harmless and stabilizes the scheme.

When combining advection with diffusion, though, we run into a problem: the discontinuous function space that works so well for advection does not combine naturally

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with the diffusion operator. In the course of 30 years, various bilinear DG forms have been introduced for approximation of the diffusion operator; the symmetric DG form, stabilized either with the penalty term of Baker [1] (usually attributed to Arnold [2]) or the penalty term of Bassi, Rebay et al. [3], also credited to Brezzi [4], are most widely used.

In constructing bilinear forms for diffusion, the essence lies in choosing the numerical fluxes that are responsible for the coupling of the discontinuous solution approximation across the cell interfaces. Traditionally, these numerical fluxes are defined such that the bilinear form satisfies a number of mathematical conditions (the more the better) such as symmetry, coercivity, boundedness, consistency and adjoint consistency. Until recently, however, an analysis in which bilinear forms for diffusion, with desirable mathematical properties, appear simply as a result of a physical argument, was lacking.

In 2005 Van Leer et al. presented the “recovery method.” Here the coupling through the numerical fluxes is obtained by arguing that, for diffusion, the discontinuous solution approximation should locally be regarded as an L_2 projection of a higher-order continuous function. This acknowledges the physical datum that diffusion produces a smooth solution at any $t>0$ even from discontinuous initial values. The local “recovered” function couples neighboring cells and provides the information for computing the diffusive fluxes.

In the present paper we consider the 1-D diffusion equation in order to present the link between the recovery method and traditional discontinuous Galerkin bilinear formulations. Key to the systematic derivation of the diffusive fluxes that appear in the bilinear forms is the discovery of the “recovery polynomial basis:” to each piecewise continuous polynomial basis of degree k defined on two adjacent cells corresponds a *unique* continuous polynomial space of degree $2k+1$. In consequence, to the approximation of the solution as an expansion in the discontinuous basis functions locally corresponds an *identical* expansion in the smooth recovery basis; the latter permits computing of the numerical fluxes across the cell interfaces. And, because of the duality of the polynomial spaces, the numerical fluxes in terms of the discontinuous basis functions follow immediately.

Thus, for any polynomial space of degree k the recovery method is equivalent to a unique, basis-independent bilinear discontinuous Galerkin formulation.

The outline of the paper is as follows. In the next section the recovery method is reviewed, in Section 3 the recovery basis is introduced, and in Section 4 the numerical fluxes are computed. These lead to the bilinear forms presented in Section 5. The final section lists the paper’s conclusions.

2 The recovery method

Let us consider the diffusion equation $u_t = Du_{xx}$ which for convenience we discretize on the regular infinite grid

$$\mathbb{Z}_h = \{ jh \mid j \in \mathbb{Z}, h>0 \}. \tag{2.1}$$

On the partitioning of open cells $\Omega_j =]jh,(j+1)h[$ we introduce the function space associated with discontinuous Galerkin methods

$$\mathcal{P}^k(\mathbb{Z}_h) = \Big\{ w : w \in \mathcal{P}^k(\Omega_j), j \in \mathbb{Z} \Big\}, \tag{2.2}$$

with $\mathcal{P}^k(\Omega_j)$ a polynomial space of degree k, and we introduce the jump and average operators that are useful for discontinuous Galerkin formulations. For a function w we define

$$\begin{aligned} [w]|_{(j+1)h} &= w|_{\Omega_j} \mathrm{n}_{\Omega_j} + w|_{\Omega_{j+1}} \mathrm{n}_{\Omega_{j+1}}, \\ \langle w \rangle|_{(j+1)h} &= \tfrac{1}{2}\Big(w|_{\Omega_j} + w|_{\Omega_{j+1}} \Big). \end{aligned} \tag{2.3}$$

Here, n_{Ω_j} is the one-dimensional outward normal of cell Ω_j at point $(j+1)h$.

The recovery method for the 1-D diffusion equation is based on the following weak formulation: find $u \in \mathcal{P}^k(\mathbb{Z}_h)$ such that

$$\begin{aligned} \sum_{\Omega_j \in \mathbb{Z}} \int_{\Omega_j} u_t v dx = & D \sum_{\Omega_j \in \mathbb{Z}} \int_{\Omega_j} u v_{xx} dx \\ & + D\Big([v] f_x|_{\Gamma_{\text{int}}} - [v_x] f|_{\Gamma_{\text{int}}} \Big), \quad \forall v \in \mathcal{P}^k(\mathbb{Z}_h), \end{aligned} \tag{2.4}$$

where $\Gamma_{\text{int}} = \{jh\}_{j\in\mathbb{Z}}$ is the set of cell-interfaces and f is the locally recovered smooth function whose value and derivative we use at each cell interface $x = eh$, $e \in \mathbb{Z}$. The recovery requirement is that f is indistinguishable from u in the weak sense on the cells Ω_{e-1} and Ω_e, meaning that we find $f \in \mathcal{P}^{2k+1}(\Omega_{e-1} \cup \Omega_e)$ such that

$$\sum_{j=e-1,e} \int_{\Omega_j} f v dx = \sum_{j=e-1,e} \int_{\Omega_j} u v dx, \quad \forall v \in \mathcal{P}^k(\Omega_{e-1}) \cup \mathcal{P}^k(\Omega_e). \tag{2.5}$$

In Van Leer et al. [5, 6] we have presented linear systems from which the polynomial coefficients of f follow. In the next section we introduce a new concept, the recovery basis, which is at the heart of the recovery procedure; it greatly simplifies both computation and further analysis of the diffusive fluxes.

3 The recovery basis

As the basis for (2.2) we consider polynomials $\phi_i(\xi)$ in the hierarchical space $\mathcal{P}^k(]0,1[)$ of degree k that satisfy

$$\int_0^1 \phi_i(\xi)\phi_l(\xi) d\xi = \begin{cases} 0 & \text{if } i \neq l \\ 1 & \text{if } i = l \end{cases} \qquad \text{with } i \text{ and } l = 0, \cdots, k. \tag{3.1}$$

For the sake of determinacy (it is not essential), we choose the orthonormal Legendre polynomials defined on the open interval $\xi =]0,1[$ (see [7], page 775):

$$\phi_n(\xi) = \frac{1}{2^n} \sqrt{2n+1} \sum_{m=0}^{[n/2]} (-1)^m \begin{pmatrix} n \\ m \end{pmatrix} \begin{pmatrix} 2n-2m \\ n \end{pmatrix} (-1+2\xi)^{n-2m}. \tag{3.2}$$

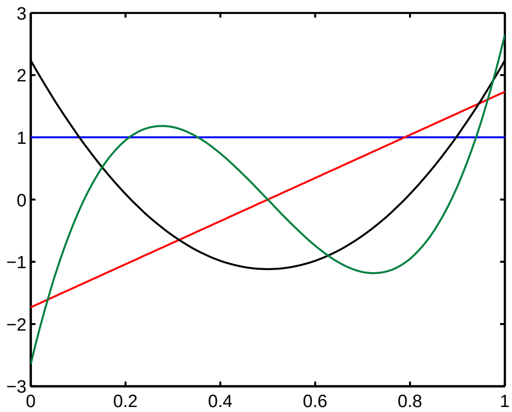


Figure 1: The first four orthonormal Legendre polynomials $\phi_0(\xi)=1$, $\phi_1(\xi)=\sqrt{3}(-1+2\xi)$, $\phi_2(\xi)=\sqrt{5}(1-6\xi+6\xi^2)$ and $\phi_3(\xi)=\sqrt{7}(-1+12\xi-30\xi^2+20\xi^3)$ defined on the interval $\xi=]0,1[$.

Fig. 1 shows the first four polynomials of this basis. On the grid (2.1), the function space (2.2) is spanned by the piecewise continuous polynomials

$$\phi_{i,j}(x)=\begin{cases}\phi_i(\frac{x-jh}{h}) & x\in\Omega_j,\\ 0 & \text{otherwise}.\end{cases} \tag{3.3}$$

To obtain a L_2 approximation of a given smooth function $U(x)$ (for instance an initial-value distribution) we solve the bilinear form: find $u\in\mathcal{P}^k(\mathbb{Z}_h)$ such that

$$\sum_{j\in\mathbb{Z}}\int_{\Omega_j}u\phi_{i,j}dx=\sum_{j\in\mathbb{Z}}\int_{\Omega_j}U\phi_{i,j}dx,\quad \forall\phi_{i,j}\in\mathcal{P}^k(\mathbb{Z}_h). \tag{3.4}$$

With the approximation

$$u=\sum_{j\in\mathbb{Z}}\sum_{i=0}^{k}c_{i,j}\phi_{i,j}(x), \tag{3.5}$$

and using (3.1), the solution of (3.4) reads

$$c_{i,j}=\int_0^1 U(h(\xi+j))\phi_i(\xi)d\xi,\quad i=0,\cdots,k,\quad j\in\mathbb{Z}. \tag{3.6}$$

The L_2 approximation (3.5) is in general *discontinuous* at the cell-interfaces. The function values and derivatives of $u(x)$ are formally not defined at the nodal points jh, whereas they were for $U(x)$. This motivates recovery.

We shall now go to the root of the recovery procedure. We consider two adjacent cells Ω_e and Ω_{e+1} and describe the local recovery procedure (2.5) for the piecewise continuous approximation (3.5). On the union of the cells we introduce the basis polynomials (to be specified later)

$$\psi_{i,j}(x)=\begin{cases}\psi_i(\frac{x-jh}{h}) & x\in\Omega_j\cup\Omega_{j+1},\quad i=0,\cdots,2k+1,\\ 0 & \text{otherwise}.\end{cases} \tag{3.7}$$

with $\psi_i(\xi)\in\mathcal{P}^{2(k+1)}(]0,2[)$, and we construct the polynomial space

$$\mathcal{P}^{2k+1}(\Omega_e\cup\Omega_{e+1})=\text{Span}\{\psi_{i,j}(x)\},\quad j=e,\quad i=0,\cdots,2k+1. \tag{3.8}$$

Writing the recovered smooth approximation as

$$f=\sum_{i=0}^{2k+1}a_i\psi_{i,e}(x),\quad x\in\Omega_e\cup\Omega_{e+1}, \tag{3.9}$$

our task is solving the bilinear form: find $f\in\mathcal{P}^{2(k+1)}(\Omega_e\cup\Omega_{e+1})$ such that

$$\sum_{j\in\mathbb{Z}}\int_{\Omega_j}f\phi_{i,j}dx=\sum_{j\in\mathbb{Z}}\int_{\Omega_j}u\phi_{i,j}dx=hc_{i,j},\quad j=e,e+1,\quad i=0,\cdots,k. \tag{3.10}$$

This implies solving the linear system

$$\begin{bmatrix} \int_0^1\psi_0(\xi)\phi_0(\xi)d\xi & \cdots & \int_0^1\psi_{2k+1}(\xi)\phi_0(\xi)d\xi \\ \vdots & \cdots & \vdots \\ \int_0^1\psi_0(\xi)\phi_k(\xi)d\xi & \cdots & \int_0^1\psi_{2k+1}(\xi)\phi_k(\xi)d\xi \\ \int_0^1\psi_0(\xi+1)\phi_0(\xi)d\xi & \cdots & \int_0^1\psi_{2k+1}(\xi+1)\phi_0(\xi)d\xi \\ \vdots & \cdots & \vdots \\ \int_0^1\psi_0(\xi+1)\phi_k(\xi)d\xi & \cdots & \int_0^1\psi_{2k+1}(\xi+1)\phi_k(\xi)d\xi \end{bmatrix}\begin{bmatrix} a_0\\ \vdots\\ a_k\\ a_{k+1}\\ \vdots\\ a_{2k+1}\end{bmatrix}=\begin{bmatrix} c_{0,e}\\ \vdots\\ c_{k,e}\\ c_{0,e+1}\\ \vdots\\ c_{k,e+1}\end{bmatrix}. \tag{3.11}$$

We now make the key observation: if we choose the $\psi_{i,e}(x)$ such that the matrix in (3.11) renders an *identity* matrix, the expansion of f in terms of the smooth basis is *identical* to the expansion of u in terms of the discontinuous basis. Then we will have $a_i=c_{i,e}$ and $a_{k+1+i}=c_{i,e+1}$ for $i=0,\cdots,k$, and

$$\begin{aligned} u|_{\Omega_e\cup\Omega_{e+1}}&=\sum_{i=0}^{k}c_{i,e}\phi_{i,e}(x)+\sum_{i=0}^{k}c_{i,e+1}\phi_{i,e+1}(x),\\ f&=\sum_{i=0}^{k}c_{i,e}\psi_{i,e}(x)+\sum_{i=0}^{k}c_{i,e+1}\psi_{k+1+i,e}(x),\quad x\in\Omega_e\cup\Omega_{e+1}. \end{aligned} \tag{3.12}$$

To achieve this simplicity we must require

$$\begin{cases} \int_0^1\psi_i(\xi)\phi_l(\xi)d\xi=\begin{cases}1 & i=l,\\ 0 & i\neq l,\end{cases}\\ \int_0^1\psi_i(\xi+1)\phi_l(\xi)d\xi=0;\end{cases}\qquad \begin{cases} \int_0^1\psi_{k+1+i}(\xi)\phi_l(\xi)d\xi=0,\\ \int_0^1\psi_{k+1+i}(\xi+1)\phi_l(\xi)d\xi=\begin{cases}1 & i=l,\\ 0 & i\neq l;\end{cases}\end{cases} \tag{3.13}$$

for i and $l=0,\cdots,k$.

This set of equations represents a weak interpolation problem and, just as for strong collocation interpolation, has a unique solution for each basis $\{\phi_i\}$. In consequence, each

Table 1: The orthonormal Legendre basis and the recovery basis for degrees $k=0,1$ and 2. The recovery basis satisfies the orthonormality conditions (3.13) and makes the matrix in (3.11) the identity matrix.

k	$\phi_i(\xi)$ $0<\xi<1,\ i=0,\cdots,k$	$\psi_i(\xi)$ $0<\xi<2,\ i=0,\cdots,2k+1$
0	1	$\frac{3}{2}-\xi$ $-\frac{1}{2}+\xi$
1	1 $(-1+2\xi)\sqrt{3}$	$\frac{1}{4}+\frac{21}{4}\xi-\frac{15}{2}\xi^2+\frac{5}{2}\xi^3$ $(-\frac{23}{12}+\frac{33}{4}\xi-\frac{17}{2}\xi^2+\frac{5}{2}\xi^3)\sqrt{3}$ $\frac{3}{4}-\frac{21}{4}\xi+\frac{15}{2}\xi^2-\frac{5}{2}\xi^3$ $(-\frac{7}{12}+\frac{17}{4}\xi-\frac{13}{2}\xi^2+\frac{5}{2}\xi^3)\sqrt{3}$
2	1 $(-1+2\xi)\sqrt{3}$ $(1-6\xi+6\xi^2)\sqrt{5}$	$\frac{9}{4}-\frac{75}{4}\xi+\frac{135}{2}\xi^2-\frac{185}{2}\xi^3+\frac{105}{2}\xi^4-\frac{21}{2}\xi^5$ $(\frac{3}{8}-\frac{147}{8}\xi+\frac{1155}{16}\xi^2-\frac{775}{8}\xi^3+\frac{1715}{32}\xi^4-\frac{21}{2}\xi^5)\sqrt{3}$ $(\frac{99}{40}-\frac{1083}{40}\xi+\frac{1233}{16}\xi^2-\frac{713}{8}\xi^3+\frac{1449}{32}\xi^4-\frac{42}{5}\xi^5)\sqrt{5}$ $-\frac{5}{4}+\frac{75}{4}\xi-\frac{135}{2}\xi^2+\frac{185}{2}\xi^3-\frac{105}{2}\xi^4+\frac{21}{2}\xi^5$ $(\frac{9}{8}-\frac{137}{8}\xi+\frac{1005}{16}\xi^2-\frac{705}{8}\xi^3+\frac{1645}{32}\xi^4-\frac{21}{2}\xi^5)\sqrt{3}$ $(-\frac{29}{40}+\frac{453}{40}\xi-\frac{687}{16}\xi^2+\frac{503}{8}\xi^3-\frac{1239}{32}\xi^4+\frac{42}{5}\xi^5)\sqrt{5}$

basis in the discontinuous polynomial space $\mathcal{P}^k(\Omega_e)\cup\mathcal{P}^k(\Omega_{e+1})$ has a unique recovery counterpart in $\mathcal{P}^{2k+1}(\Omega_e\cup\Omega_{e+1})$.

The discontinuous and smooth recovery polynomial bases satisfying (3.13) are given in Table 1 for some values of k. Fig. 2 shows the discontinuous orthonormal Legendre polynomials and the corresponding smooth recovery polynomials for $k=1$. Fig. 3 shows the piecewise constant function and its recovery counterpart for $k=0$ to 3. The smooth basis function tends weakly to the underlying discontinuous one as k is increased. Fig. 4 shows the L_2 approximation according to (3.12) of $f=\cos(\frac{9\pi}{4h}x)$ both in the discontinuous basis (left) and in the recovery basis (right). In the latter representation we can compute unique values of the function and its derivative at the cell interface.

4 Numerical fluxes

We are now ready to express the diffusive fluxes in (2.4), recovered according to (2.5), in terms of the discontinuous polynomials in the function space (2.2) for any degree k; we shall restrict ourselves to lower values of k.

We first consider the case k=1. Using the expansion (3.12) of f and the relevant entries

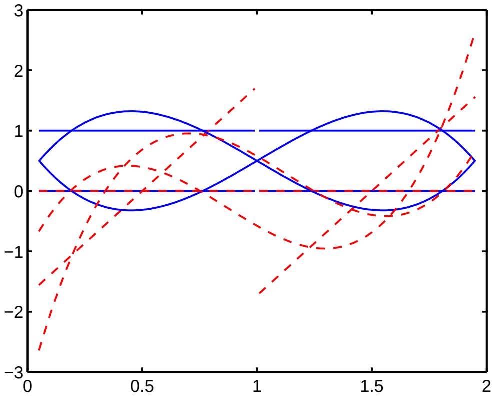


Figure 2: The discontinuous and recovery basis for $k=1$ and cells Ω_0 and Ω_1 with $h=1$.

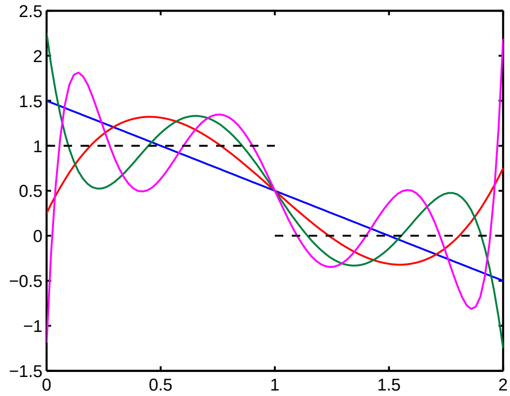
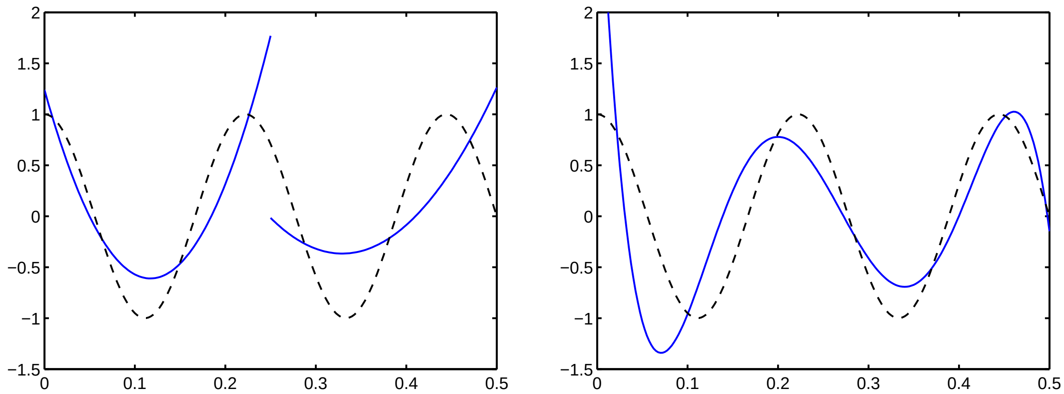


Figure 3: The piecewise constant function and its counterpart in the recovery basis for $k=0$ to 3.



(a) (b)

Figure 4: L_2 approximation of $f=\cos(\frac{9\pi}{4h}x)$ on cells Ω_0 and Ω_1 (h=1/4). (a) the approximation in the Legendre basis with $k=2$, (b) the corresponding recovered function in the recovery basis.

for ψ_i from Table 1 we compute the trace at $x=(e+1)h$:

$$\begin{aligned} f((e+1)h) &= \sum_{i=0,1} c_{i,e}\psi_i(1)+\sum_{i=0,1} c_{i,e+1}\psi_{2+i}(1), \\ &= \frac{1}{2}c_{0,e}+\frac{1}{3}c_{1,e}\sqrt{3}+\frac{1}{2}c_{0,e+1}-\frac{1}{3}c_{1,e+1}\sqrt{3}. \end{aligned} \tag{4.1}$$

Comparing this result with the expansion of u we see, with $\varepsilon\to 0$, that

$$\begin{aligned} u((j+1)h-\varepsilon) &= \sum_{i=0,1} c_{i,e}\phi_i(1) = c_{0,e}+c_{1,e}\sqrt{3}, \\ u((j+1)h+\varepsilon) &= \sum_{i=0,1} c_{i,e+1}\phi_i(0) = c_{0,e+1}-c_{1,e+1}\sqrt{3}. \end{aligned} \tag{4.2}$$

Table 2: The expansion of (4.3) and (4.4) for discontinuous polynomial space of degree $k=0,1$ and 2.

k	$f(jh)$	$f_x(jh)$
0	$\langle u\rangle$	$-\frac{1}{h}[u]$
1	$\langle u\rangle-\frac{1}{12}h[u_x]$	$-\frac{9}{4h}[u]+\langle u_x\rangle$
2	$\langle u\rangle-\frac{3}{64}h[u_x]$	$-\frac{15}{4h}[u]+\langle u_x\rangle-\frac{9}{240}h[u_{xx}]$

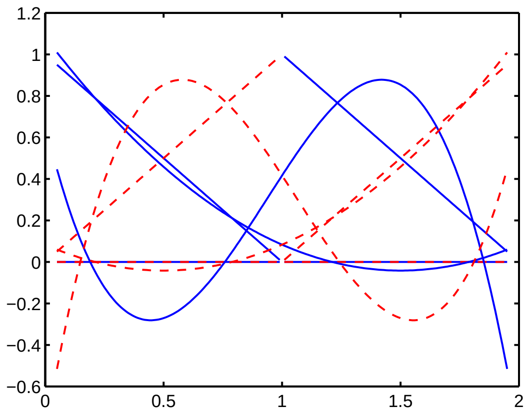


Figure 5: The typical finite-element hat-polynomials and their recovery counterpart. In these bases, the expansions (3.12) hold.

Applying the jump and average operators and using the fact that the Legendre basis is hierarchical we find

$$f((e+1)h)=\langle u\rangle|_{(e+1)h}-\frac{1}{12}h\,[u_x]|_{(e+1)h}, \tag{4.3}$$

and analogously

$$f_x((e+1)h)=-\frac{9}{4h}\,[u]|_{(e+1)h}+\langle u_x\rangle|_{(e+1)h}. \tag{4.4}$$

The derivation for other polynomial degrees is similar to the above procedure; Table 2 shows the results for $k=0$ to 2. As indicated before, and verifiable by construction, for a given degree k, to each Legendre polynomial (3.2) in the function space $\mathcal{P}^k(\Omega_e)\cup\mathcal{P}^k(\Omega_{e+1})\subset\mathcal{P}^k(\mathbb{Z}_h)$ corresponds a *unique* recovery polynomial in the function space $\mathcal{P}^{2k+1}(\Omega_e\cup\Omega_{e+1})$. In consequence, basis transformations apply on both polynomial spaces simultaneously, with the result that any polynomial in the space $\mathcal{P}^k(\Omega_e)\cup\mathcal{P}^k(\Omega_{e+1})\subset\mathcal{P}^k(\mathbb{Z}_h)$ has a unique recovery polynomial in the space $\mathcal{P}^{2k+1}(\Omega_e\cup\Omega_{e+1})$. As an example we show in Fig. 5 the typical finite-element hat-polynomials and their recovery counterparts satisfying the expansions (3.12). Two of the four hat polynomials are continuous at $x=1$, with the result that the corresponding recovery polynomial becomes a good approximation.

Because of the existence of the recovery polynomials, which span $\mathcal{P}^{2k+1}(\Omega_e \cup \Omega_{e+1})$, the recovered approximation $f \in \mathcal{P}^{2k+1}(\Omega_e \cup \Omega_{e+1})$ is independent of the basis on which it was computed and, likewise, the resulting diffusive fluxes are basis independent. Hence, starting from the hat-polynomials (k=1), one finds the same interface values of f and f_x as given in (4.3) and (4.4).

5 Bilinear forms for the recovery method

Given the unique expansions for several k in Table 2 we may finally reformulate the recovery method (2.4) with its recovery procedure (2.5) for these degrees as: find $u \in \mathcal{P}^k(\mathbb{Z}_h)$ such that

$$B_k(u,v) = \frac{1}{D} \sum_{\Omega_j \in \mathbb{Z}_h} \int_{\Omega_j} u_t v dx, \qquad \forall v \in \mathcal{P}^k(\mathbb{Z}_h), \tag{5.1}$$

where*

$$k=0:\ B_0(u,v) = -\frac{1}{h}[v][u]|_{\Gamma_{\text{int}}}, \tag{5.2}$$

$$\begin{aligned} k=1:\ B_1(u,v) = &- \sum_{\Omega_j \in \mathbb{Z}_h} \int_{\Omega_j} u_x v_x dx + [v]\langle u_x \rangle|_{\Gamma_{\text{int}}} \\ &+ \langle v_x \rangle [u]|_{\Gamma_{\text{int}}} - \frac{9}{4h}[v][u]|_{\Gamma_{\text{int}}} + \frac{1}{12}h[v_x][u_x]|_{\Gamma_{\text{int}}}, \end{aligned} \tag{5.3}$$

$$\begin{aligned} k=2:\ B_2(u,v) = &- \sum_{\Omega_j \in \mathbb{Z}_h} \int_{\Omega_j} u_x v_x dx + [v]\langle u_x \rangle|_{\Gamma_{\text{int}}} + \langle v_x \rangle [u]|_{\Gamma_{\text{int}}} \\ &- \frac{15}{4h}[v][u]|_{\Gamma_{\text{int}}} + \frac{3}{64}h[v_x][u_x]|_{\Gamma_{\text{int}}} - \frac{9}{240}h[v][u_{xx}]|_{\Gamma_{\text{int}}}. \end{aligned} \tag{5.4}$$

Clearly, the recovery-based discontinuous Galerkin method for $k \geq 1$ reproduces the symmetric weak formulation, with a sequence of penalty-like terms; The number of terms and their coefficients depend on the targeted accuracy of the method. For any k Baker's [1] interior penalty term appears. Note that for $k=2$ the last term is not symmetric. Higher-degree bilinear forms are obtainable by straightforward construction; so far we have gone up to $B_6(u,v)$. The high-order terms are always asymmetric, being proportional to $[\partial^k u/\partial x^k]$ and either $[v]$ or $[v_x]$.

Though we have numerical evidence of exponential order of convergence [6], we lack rigorous proofs of stability, coercivity and boundedness. We wish to point out, though, that the 1-D bilinear forms we found all are consistent with the weak form of the diffusion

*Here we use that

$$\sum_{\Omega_j \in \mathbb{Z}_h} \int_{\Omega_j} u v_{xx} dx = - \sum_{\Omega_j \in \mathbb{Z}_h} \int_{\Omega_j} u_x v_x dx + [u v_x]|_{\Gamma_{\text{int}}}$$

and $[v_x]\langle u \rangle = -\langle v_x \rangle [u]|_{\Gamma_{\text{int}}} + [u v_x]|_{\Gamma_{\text{int}}}$.

operator. To show this, we start from the interior-penalty method, which is known to be consistent. The present bilinear forms for $k \geq 1$ contain additional penalty terms; these, however, are proportional to $[\partial^k u/\partial x^k]$ and therefore vanish when u is replaced by the smooth exact solution.

Moreover, although not symmetric for $k \geq 2$, these bilinear forms are also adjoint consistent. Again, the internal-penalty method has the desired property; the added higher-order penalty terms do not affect this since they are proportional to either $[v]$ or $[v_x]$, which vanishes when v is replaced by the smooth exact solution.

6 Conclusion

In this article we have derived bilinear forms for the recovery-based discontinuous Galerkin method for the discretization of the 1-D diffusion equation. This method approximates the solution of the diffusion equation in a discontinuous function space, while inter-element coupling is achieved by locally recovering a higher-order continuous function from the continuous function.

At the heart of this recovery procedure is the existence of the recovery basis polynomials: polynomials, defined on two cells, that are in L_2-sense indistinguishable from the discontinuous basis functions.

An important property is that for a fixed degree to each non-smooth basis function in the discontinuous function space corresponds a unique locally smooth polynomial in the recovery function space, and visa-versa. We have shown that the diffusive flux (function value or derivative) computed from the recovered smooth approximation, expressed in terms of the interface jumps and averages of the discontinuous approximation, is unique; in particular it is independent of the bases in which the discontinuous and smooth approximations were expressed.

The recovery method reproduces the symmetric discontinuous Galerkin formulation, augmented with a number of not necessarily symmetric penalty terms; the number increases with the degree of the approximation space.

We submit this paper provides the tools to further analyze the stability, accuracy, and other properties of the recovery-based discontinuous Galerkin method.

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