

## An Adjoint State Method for Numerical Approximation of Continuous Traffic Congestion Equilibria

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**Abstract.** The equilibrium metric for minimizing a continuous congested traffic model is the solution of a variational problem involving geodesic distances. The continuous equilibrium metric and its associated variational problem are closely related to the classical discrete Wardrop's equilibrium. We propose an adjoint state method to numerically approximate continuous traffic congestion equilibria through the continuous formulation. The method formally derives an adjoint state equation to compute the gradient descent direction so as to minimize a nonlinear functional involving the equilibrium metric and the resulting geodesic distances. The geodesic distance needed for the state equation is computed by solving a factored eikonal equation, and the adjoint state equation is solved by a fast sweeping method. Numerical examples demonstrate that the proposed adjoint state method produces desired equilibrium metrics and outperforms the subgradient marching method for computing such equilibrium metrics.

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## 1 Introduction

In traffic flow for transportation and communication, network equilibrium models are commonly used for prediction of traffic patterns in transportation and communication networks that are subject to congestion. The idea of traffic equilibrium originated as early as 1924 in the work by Knight [12]. In 1952, Wardrop introduced two principles that

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formalize the notion of equilibrium [24]. Wardrop's first principle states that no driver may lower his/her transportation cost through unilateral action, which leads to the user-optimized equilibrium. Wardrop's second principle states that drivers behave cooperatively to minimize the total system travel time, which leads to the system-optimized equilibrium. These two principles have been put into firm foundation by treating the network equilibrium problem as a discrete convex programming problem in Chapter 3 of Beckmann et al. [2]. In a recent work [7] Carlier et al. introduced a continuous version of Wardrop's equilibria, proved the existence of continuous traffic congestion equilibrium by introducing a variational problem analogous to the discrete convex programming in Chapter 3 of [2], and related it to the optimal transportation problem with congestion. It turns out that such an equilibrium is linked to a certain metric, and all actually used paths (the continuous version of routes) must be geodesics for this metric. Based on the work in [7], Benmansour et al. [4] have shown that an equilibrium metric is the solution of a variational problem involving geodesic distances. Furthermore, to solve this particular variational problem, they have designed a subgradient marching method [3] to approximate continuous traffic congestion equilibria. This method requires intensive memory and is computationally inefficient.

In this paper, as an alternative approach, we propose a new adjoint state method which is efficient in both memory and computation to solve this variational problem. By using this adjoint state method, we first derive the gradient descent direction for a certain nonlinear functional in a continuous setting, and we then discretize the resulting gradient accordingly. This is different from the viewpoint of Benmansour et al. [4], where they discretized the nonlinear functional first and computed the derivatives of the discrete functional with respect to metrics in a discrete setting. In designing an efficient adjoint state method for continuous traffic congestion equilibria, there are two challenging computational issues: one is how to compute geodesic distances efficiently and accurately from a source location to many destination locations, as the distance function is not differentiable at the source location; the other is how to solve the adjoint state equation efficiently. To deal with the first difficulty we use the factored eikonal equation [11] to discretize the eikonal equation so that the source singularity can be treated with high accuracy. To deal with the second difficulty, we adopt a fast sweeping method as designed in [13]. Numerical examples demonstrate that the proposed adjoint state method produces desired equilibrium metrics. In a discrete setting with  $N$  grid points, in terms of computational memory, the subgradient marching method in [3, 4] requires  $\mathcal{O}(N^2)$  memory, while our new approach requires  $\mathcal{O}(N)$  memory; in terms of computational complexity, the subgradient marching method in [3, 4] is of  $\mathcal{O}(N^2 \log N)$ , while our approach is of  $\mathcal{O}(N)$ .

The outline of this work is as follows. In Section 2, we present the continuous traffic congestion model and its dual formulation as described in [4, 7]. In Section 3, we derive our adjoint state method to compute the equilibrium metric. In Section 4, we present numerical examples to illustrate the performance of our method and make comparison with the subgradient marching method. We conclude the paper with some remarks.

## 2 The model and the optimization formulation

In this section, we briefly recall the continuous traffic congestion model and its dual formulation introduced in [4, 7]. Interested researchers may refer to [4, 7] and references therein for details and proofs.

An equilibrium is related to the distribution of vehicles on all possible paths such that all actually used paths have the same cost and the total cost is the minimum. The distribution of vehicles is subject to the organization and communication of the city, for example, residents and services. Mathematically, the city is modeled by the closure of a connected open bounded set  $\Omega \subset \mathbb{R}^2$ . All possible paths are represented by curves  $C \triangleq W^{1,\infty}([0,1],\bar{\Omega})$  (with the usual topology of  $C^0([0,1],\mathbb{R}^2)$ ). A Borel probability measure  $Q$  on the sets of paths is used to model the distribution of vehicles, which is subject to a transportation plan  $\gamma$  that models movements on source-destination pairs.  $\gamma$  is a Borel probability measure on  $\bar{\Omega} \times \bar{\Omega}$ , and its marginals model the distribution of residents and services. See [4, 7] for more details.

The total congestion cost needs to be defined such that its minimum is related to an equilibrium. For a certain distribution  $Q$  of vehicles, the congestion effects depend on the traffic intensity  $i_Q$  associated to  $Q$ .  $i_Q$  is a Borel measure on  $\bar{\Omega}$ , such that

$$\int_{\bar{\Omega}} \phi(x) di_Q(x) \triangleq \int_C L_\phi(\sigma) dQ(\sigma), \quad (2.1)$$

for all  $\phi \in C^0(\bar{\Omega}, \mathbb{R}_+)$ , where

$$L_\phi(\sigma) \triangleq \int_0^1 \phi(\sigma(t)) |\dot{\sigma}(t)| dt,$$

and  $\sigma$  is a possible path in the set  $C$ ; see [4, 7].

A continuous non-negative function  $g(x, i(x))$  is used to model the congestion effects.  $g(\cdot, i)$  is the cost per unit length for a path when the traffic intensity is  $i$ .  $g(x, \cdot)$  is strictly increasing on  $\mathbb{R}_+$  for every  $x$  and  $g$  also satisfies the following: There exist  $a > 0$ ,  $b > 0$  and  $\alpha \in (0, 1)$ , such that

$$ai^\alpha \leq g(x, i) \leq b(i^\alpha + 1), \quad \forall i \in \mathbb{R}_+, \quad \forall x \in \Omega. \quad (2.2)$$

The total congestion cost is defined as  $\int_{\Omega} H(x, i(x)) dx$  with  $H$  linked to  $g$ ,

$$H(x, i) \triangleq \int_0^i g(x, s) ds, \quad \forall x \in \Omega, \quad \forall i \in \mathbb{R}_+. \quad (2.3)$$

A Wardrop equilibrium is a measure  $Q$  which is a minimizer of a convex optimization problem for the total congestion cost,

$$(\mathcal{P}) \quad \inf \left\{ \int_{\Omega} H(x, i_Q(x)) dx : Q \text{ with } i_Q \in L^q \right\}, \quad (2.4)$$

where  $q = 1 + \alpha$ . At an equilibrium  $Q$ , the drivers only use paths that are geodesics with respect to the congested metric  $\xi_Q$ ,

$$\xi_Q(x) \triangleq \frac{\partial H(x, i_Q)}{\partial i} = g(x, i_Q(x)), \quad x \in \Omega. \tag{2.5}$$

$\xi_Q$  is the equilibrium metric. Since  $H(x, \cdot)$  is strictly convex, by using the Legendre transform, a dual formulation can be derived to find the equilibrium metric  $\xi_Q$ .

For a continuous non-negative metric  $\xi \in C^0(\bar{\Omega}, \mathbb{R}_+)$ , the geodesic distance  $T_\xi$  is given as,

$$T_\xi(x, y) \triangleq \inf \{ L_\xi(\sigma) : \sigma \in C, \sigma(0) = x, \sigma(1) = y \}. \tag{2.6}$$

$T_\xi$  and  $L_\xi$  can be extended by sequential approximations to the cases of  $\xi$  in some  $L^p$  space with  $p > 2$ . The extensions are still denoted as  $T_\xi$  and  $L_\xi$  without notational confusion.

By using the dual formulation, we may consider the following optimization problem as suggested in [4, 7],

$$(\mathcal{P}^*) \quad \inf \{ \mathcal{J}(\xi) : \xi \in L^{q^*}, \xi \geq \xi_0 \}, \tag{2.7}$$

with

$$\mathcal{J}(\xi) \triangleq \int_\Omega H^*(x, \xi(x)) dx - \int_{\bar{\Omega} \times \bar{\Omega}} T_\xi(x, y) d\gamma(x, y), \tag{2.8}$$

where  $q^*$  is the conjugate exponent of  $q$  with  $1/q + 1/q^* = 1$ ,  $\xi_0 \triangleq g(x, 0)$ , and the Legendre transform of  $H$ ,

$$H^*(x, \xi) \triangleq \sup \{ \xi i - H(x, i) : i \geq 0 \}. \tag{2.9}$$

One can prove that under appropriate assumptions [4],

$$\begin{aligned} \min(\mathcal{P}) &= \min(\mathcal{P}^*) \\ \text{and an appropriate } \xi &\text{ solves } (\mathcal{P}^*) \text{ if and only if } \xi = \xi_Q \text{ for some} \\ \text{appropriate Borel probability measure } Q &\text{ solving } (\mathcal{P}). \end{aligned} \tag{2.10}$$

Once an equilibrium metric  $\xi_Q$  is obtained, the corresponding equilibrium intensity  $i_Q$  can be recovered by inverting  $\xi_Q(x) = g(x, i_Q(x))$ .

### 3 Adjoint state method

For the functional (2.8), we assume that  $\gamma$  is absolutely continuous with respect to the Lebesgue measure. Then according to the Radon-Nikodym Theorem, we can write  $d\gamma(x, y) = R(x, y) dx dy$  with  $R(x, y) \geq 0$  the Radon-Nikodym derivative [21]. Thus the functional is rewritten as

$$\mathcal{J}(\xi) = \int_\Omega H^*(x, \xi(x)) dx - \int_{\bar{\Omega} \times \bar{\Omega}} T_\xi(x, y) R(x, y) dx dy. \tag{3.1}$$

For each source  $x \in \bar{\Omega}$ , we define

$$\mathcal{J}^x(\xi) \triangleq - \int_{\bar{\Omega}} T_{\xi}(x, y) R(x, y) dy. \tag{3.2}$$

By the Fubini Theorem [21], we have

$$\mathcal{J}(\xi) = \int_{\Omega} H^*(x, \xi(x)) dx + \int_{\bar{\Omega}} \mathcal{J}^x(\xi) dx. \tag{3.3}$$

This is the functional that we want to minimize. An adjoint state approach will be designed to solve the related minimization problem.

### 3.1 Adjoint state equation and factored eikonal equation

The evaluation of  $\mathcal{J}$  or  $\mathcal{J}^x$  for a given source  $x$  requires geodesic distances  $T_{\xi}$ , which can be obtained as the viscosity solution of the following eikonal equation [9] (by denoting  $T(y) = T_{\xi}(x, y)$ ),

$$\begin{cases} |\nabla T(y)| = \xi(y), & y \in \bar{\Omega} \setminus \{x\}, \\ T(x) = 0. \end{cases} \tag{3.4}$$

**Remark 3.1.** By using the Hopf formula for the viscosity solution of (3.4) in [14], one can easily see that

- For a fixed source  $x$ , if two metrics  $\xi$  and  $\eta$  satisfy  $\xi \leq \eta$ , then  $T_{\xi}(y) \leq T_{\eta}(y)$  for all  $y \in \bar{\Omega}$ . This is also a simple implication of the comparison principle for viscosity solutions [9, 14].
- $T_{\xi}$  is concave in  $\xi$ , and since  $H^*$  is convex in  $\xi$ , the functional  $\mathcal{J}$  is convex in  $\xi$ .

In this work, to solve the eikonal equation we will adopt a factorization formulation of the eikonal equation, so-called the factored eikonal equation in [11]. This factorization formulation is specially designed to resolve the source singularity so that the accuracy of the numerical solution can be improved.

Let us consider a factored decomposition,

$$\xi(y) = \xi_1(y)\beta(y), \quad T(y) = T_1(y)\tau(y), \tag{3.5}$$

where  $T_1$  and  $\xi_1$  are assumed to be known and satisfy the eikonal equation

$$|\nabla T_1(y)| = \xi_1(y) \quad \text{for } y \in \bar{\Omega} \setminus \{x\}, \quad T_1(x) = 0.$$

Substituting decomposition (3.5) into the eikonal equation (3.4), we get the factored eikonal equation

$$T_1^2(y) |\nabla \tau(y)|^2 + 2T_1(y)\tau(y) \nabla T_1(y) \cdot \nabla \tau(y) + [\tau^2(y) - \beta^2(y)] \xi_1^2(y) = 0. \tag{3.6}$$

By choosing appropriate  $T_1$  such that it captures the source singularity of  $T$ , the underlying function  $\tau$  is smooth in the neighborhood of the source. In this work, we choose  $\xi_1 = 1$ , so  $T_1$  is the distance function to the source  $x$ , which has a singularity at the source  $x$ .

### 3.1.1 Minimization of $\mathcal{J}^x$

To minimize  $\mathcal{J}^x$ , we use the method of gradient descent. We first perturb the metric  $\xi$  by  $\epsilon\tilde{\xi}$ , which causes a corresponding change in  $T$  by  $\epsilon\tilde{T}$ . The change in  $\mathcal{J}^x$  is given by

$$\delta\mathcal{J}^x = -\epsilon \int_{\Omega} \tilde{T}(y)R(x,y)dy. \tag{3.7}$$

From the factored eikonal equation (3.6), the perturbations in  $\xi$  and  $T$  are related by (denote  $y = (y_1, y_2) \in \bar{\Omega} \subset \mathbb{R}^2$ )

$$(\tau + T_1 \nabla T_1 \cdot \nabla \tau)\tilde{\tau} + T_1(\nabla T_1 \tau + T_1 \nabla \tau) \cdot \nabla \tilde{\tau} = \xi \tilde{\xi} + \frac{\epsilon}{2}(\tilde{\xi}^2 - |\nabla(T_1 \tilde{\tau})|^2), \tag{3.8}$$

where  $\tilde{\tau}$  is defined through  $\tilde{T} = T_1 \tilde{\tau}$ .

Dropping the higher order terms, multiplying (3.8) by  $\epsilon\lambda$ , integrating it over  $\bar{\Omega}$ , applying integration by parts, and adding the resulting expression to (3.7), we get

$$\begin{aligned} \frac{\delta\mathcal{J}^x}{\epsilon} = & - \int_{\bar{\Omega}} T_1 \tilde{\tau} R(x,y) dy + \int_{\partial\Omega} \lambda \tilde{\tau} (T_1 \nabla T_1 \tau + T_1^2 \nabla \tau) \cdot \vec{n} dl \\ & + \int_{\bar{\Omega}} \{ \lambda (\tau + T_1 \nabla T_1 \cdot \nabla \tau) + \nabla \cdot [ -T_1 (\nabla T_1 \tau + T_1 \nabla \tau) \lambda ] \} \tilde{\tau} dy - \int_{\bar{\Omega}} \lambda \xi \tilde{\xi} dy, \end{aligned} \tag{3.9}$$

where  $\vec{n}$  is the unit outer normal to  $\bar{\Omega}$ . We choose  $\lambda$  satisfying

$$\lambda (\tau + T_1 \nabla T_1 \cdot \nabla \tau) + \nabla \cdot [ -T_1 (\nabla T_1 \tau + T_1 \nabla \tau) \lambda ] = T_1 R(x,y), \quad y = (y_1, y_2) \in \Omega, \tag{3.10a}$$

$$\lambda = 0, \quad y = (y_1, y_2) \in \partial\Omega. \tag{3.10b}$$

By introducing this adjoint state function  $\lambda$ , we eliminate the dependence of  $\tilde{\tau}$  and we get the perturbation in  $\mathcal{J}^x$  as,

$$\frac{\delta\mathcal{J}^x}{\epsilon} = - \int_{\bar{\Omega}} \lambda \xi \tilde{\xi} dy. \tag{3.11}$$

To minimize  $\mathcal{J}^x$  by the method of gradient descent, we could choose the perturbation  $\tilde{\xi} = \lambda \xi$  in  $\Omega$  and  $\tilde{\xi} = 0$  on  $\partial\Omega$ , which implies

$$\delta\mathcal{J}^x = -\epsilon \int_{\bar{\Omega}} \tilde{\xi}^2 dy \leq 0, \tag{3.12}$$

and the equality holds when  $\|\tilde{\xi}\|_{L^2(\Omega)} = 0$ .

**Remark 3.2.** Note that the characteristic direction for the adjoint state equation (3.10) is

$$\frac{dy}{dt} = -T_1(\nabla T_1 \tau + T_1 \nabla \tau) = -T_1 \nabla T,$$

which is the opposite to the characteristic direction for the eikonal equation (3.4) or the factored eikonal equation (3.6). If we assume that the metric outside the domain  $\Omega$  is

large (or infinity) and constant, then the characteristics of the eikonal equation (3.4) or the factored eikonal equation (3.6) are straight lines outside the domain and thus outgoing, that is, if a characteristic leaves the domain, it will not re-enter the domain again. As a result, the characteristics for the adjoint state equation (3.10) are incoming. Moreover, given a source location  $x$ , all the characteristics linking the source  $x$  and any other point  $y$  trace back to the source  $x$  and these characteristics will not intersect with each other except at the source according to the uniqueness of viscosity solution (e.g., [1, 9, 10]). Namely, these characteristics trace back to the source  $x$  and intersect at the source location only.

For the adjoint state equation (3.10), the term

$$\int_{\partial\Omega} \lambda \tilde{\tau} (T_1 \nabla T_1 \tau + T_1^2 \nabla \tau) \cdot \vec{n} dl$$

is eliminated by imposing  $\lambda = 0$  on  $\partial\Omega$ . Technically, we want this term to be non-positive. Since  $\tilde{\tau}$  is involved and one can not have control on  $\tilde{\tau}$ , it is impossible to control the sign of  $\int_{\partial\Omega} \lambda \tilde{\tau} (T_1 \nabla T_1 \tau + T_1^2 \nabla \tau) \cdot \vec{n} dl$  or  $\lambda \tilde{\tau} (T_1 \nabla T_1 \tau + T_1^2 \nabla \tau) \cdot \vec{n}$ . And we want to avoid the involvement of  $\tilde{\tau}$ , so we let this term be zero by setting  $\lambda = 0$  on  $\partial\Omega$ .

### 3.1.2 Minimization of $\mathcal{J}$

Utilizing the approach above, we have the following perturbation in  $\mathcal{J}$ ,

$$\frac{\delta \mathcal{J}}{\epsilon} = \int_{\Omega} \frac{\partial H^*(y, \zeta(y))}{\partial \zeta} \tilde{\zeta}(y) dy - \int_{\bar{\Omega}} \left( \int_{\bar{\Omega}} \lambda^x(y) dx \right) \tilde{\zeta}(y) \zeta(y) dy, \tag{3.13}$$

where  $\lambda^x$  is the solution of the adjoint state equation (3.10) with source  $x$ .

Consequently, we can choose the following perturbation to minimize  $\mathcal{J}$ ,

$$\tilde{\zeta}(y) = - \left( \frac{\partial H^*(y, \zeta(y))}{\partial \zeta} - \zeta(y) \int_{\bar{\Omega}} \lambda^x(y) dx \right), \quad y \in \Omega, \tag{3.14a}$$

$$\tilde{\zeta}(y) = 0, \quad y \in \partial\Omega. \tag{3.14b}$$

Then we have

$$\delta \mathcal{J} = -\epsilon \int_{\Omega} \tilde{\zeta}^2 dy \leq 0, \tag{3.15}$$

where the equality holds when  $\|\tilde{\zeta}\|_{L^2(\Omega)} = 0$ .

**Lemma 3.1.** *Let a non-negative  $\zeta \in L^p$  be fixed. Considering any  $h \in L^p$  such that  $\zeta + h$  is non-negative, we have*

$$\mathcal{J}(\zeta + h) - \mathcal{J}(\zeta) \geq (\nabla_{\zeta} \mathcal{J}, h), \tag{3.16}$$

where

$$(\nabla_{\zeta} \mathcal{J}, h) = \int_{\Omega} \frac{\partial H^*(y, \zeta(y))}{\partial \zeta} h(y) dy - \int_{\bar{\Omega}} \left( \int_{\bar{\Omega}} \lambda^x(y) dx \right) \zeta(y) h(y) dy. \tag{3.17}$$

*Proof.* For a fixed source point  $x$ , letting  $\epsilon = 1$  and  $\tilde{\xi} = h$  in Eq. (3.8), multiplying this equation with  $\lambda^x$ , and integrating it over  $\tilde{\Omega}$ , we have

$$\int_{\partial\Omega} \lambda^x \tilde{\tau} (T_1 \nabla T_1 \tau + T_1^2 \nabla \tau) \cdot \tilde{n} dl + \int_{\tilde{\Omega}} \{ \lambda^x (\tau + T_1 \nabla T_1 \cdot \nabla \tau) + \nabla \cdot [ -T_1 (\nabla T_1 \tau + T_1 \nabla \tau) \lambda^x ] \} \tilde{\tau} dy - \int_{\tilde{\Omega}} \lambda^x \xi h dy - \frac{1}{2} \int_{\tilde{\Omega}} \lambda^x (h^2 - |\nabla \tilde{T}|^2) dy = 0. \tag{3.18}$$

Since  $\lambda^x$  satisfies the adjoint state equation (3.10), we have

$$\int_{\tilde{\Omega}} T_1 \tilde{\tau} R(x, y) dy - \int_{\tilde{\Omega}} \lambda^x \xi h dy - \frac{1}{2} \int_{\tilde{\Omega}} \lambda^x (h^2 - |\nabla \tilde{T}|^2) dy = 0; \tag{3.19}$$

using Eq. (3.7), the above relation can be rewritten as

$$\mathcal{J}^x(\xi + h) - \mathcal{J}^x(\xi) = (\nabla_{\xi} \mathcal{J}^x, h) - \frac{1}{2} \int_{\tilde{\Omega}} \lambda^x (h^2 - |\nabla \tilde{T}|^2) dy, \tag{3.20}$$

where  $(\nabla_{\xi} \mathcal{J}^x, h) = - \int_{\tilde{\Omega}} \lambda^x \xi h dy$ .

Using the method of characteristics to solve Eq. (3.10), we have

$$\frac{d\lambda^x}{dt} = T_1 R(x, y) + \{ \nabla \cdot [ T_1 (\nabla T_1 \tau + T_1 \nabla \tau) ] - (\tau + T_1 \nabla T_1 \cdot \nabla \tau) \} \lambda^x. \tag{3.21}$$

Since  $\lambda^x = 0$  on  $\partial\Omega$  (at  $t = 0$ ) and  $T_1 R(x, y) \geq 0$ , by the integrating factor method we have  $\lambda^x \geq 0$ . By Eq. (3.8), we have (without factorization),

$$\begin{aligned} 2\nabla T \cdot \nabla \tilde{T} &= 2\xi h + (h^2 - |\nabla \tilde{T}|^2) \\ \Rightarrow \frac{1}{2}(h^2 - |\nabla \tilde{T}|^2) &= \nabla T \cdot \nabla \tilde{T} - \xi h \\ \Rightarrow \frac{1}{2}(h^2 - |\nabla \tilde{T}|^2) &\leq |\nabla T| |\nabla \tilde{T}| - \xi h \\ \Rightarrow \frac{1}{2}(h^2 - |\nabla \tilde{T}|^2) &\leq \xi |\nabla \tilde{T}| - \xi h \\ \Rightarrow \frac{1}{2}(h + |\nabla \tilde{T}|)(h - |\nabla \tilde{T}|) &\leq \xi (|\nabla \tilde{T}| - h) \\ \Rightarrow h &\leq |\nabla \tilde{T}| \text{ a.e. in } \Omega, \end{aligned} \tag{3.22a}$$

$$\begin{aligned} 2\nabla T \cdot \nabla \tilde{T} &= 2\xi h + (h^2 - |\nabla \tilde{T}|^2) \\ \Rightarrow \frac{1}{2}(|\nabla \tilde{T}|^2 - h^2) &= \xi h - \nabla T \cdot \nabla \tilde{T} \\ \Rightarrow \frac{1}{2}(|\nabla \tilde{T}|^2 - h^2) &\leq \xi h + |\nabla T| |\nabla \tilde{T}| \\ \Rightarrow \frac{1}{2}(|\nabla \tilde{T}|^2 - h^2) &\leq \xi h + \xi |\nabla \tilde{T}| \\ \Rightarrow \frac{1}{2}(|\nabla \tilde{T}| + h)(|\nabla \tilde{T}| - h) &\leq \xi (h + |\nabla \tilde{T}|) \\ \Rightarrow h &\geq -|\nabla \tilde{T}| \text{ a.e. in } \Omega. \end{aligned} \tag{3.22b}$$

Therefore, we have

$$\mathcal{J}^x(\xi+h) - \mathcal{J}^x \geq (\nabla_{\xi} \mathcal{J}^x, h). \quad (3.23)$$

With similar arguments for all sources  $x \in \Omega$  and the assumption that  $H^*$  is convex, we prove the lemma.  $\square$

Based on this lemma, according to optimization theory in [5, 8, 19, 23] we may design a convergent gradient descent algorithm to minimize the functional  $\mathcal{J}$  in Eq. (3.1).

## 3.2 Algorithm and numerical implementation

### 3.2.1 Optimization algorithm

The algorithm for minimizing the functional (3.1) consists of solving the factored eikonal equation and the adjoint state equation.

#### Optimization Algorithm:

1. Initialize  $\zeta^k$  for  $k=0$  with  $\zeta = \infty$  on  $\partial\Omega$ .
2. Obtain  $\tau$  by solving (3.6) using  $\zeta = \zeta^k$ .
3. Obtain  $\lambda$  by solving (3.10).
4. Obtain  $\tilde{\zeta}^k$  using (3.14).
5. Determine the gradient step  $\epsilon^k$  using, for example, the Armijo-Goldstein-Wolfe condition, or simply  $\epsilon^k = \epsilon$ .
6. Update
 
$$\zeta^{k+1} = \zeta^k + \epsilon^k \tilde{\zeta}^k. \quad (3.24)$$
7. Go back to Step 2 until  $\|\tilde{\zeta}^k\|_2 \leq tol$  or  $k \geq k_{\max}$ , where  $tol$  and  $k_{\max}$  are given convergence parameters.

Since the functional to be minimized is convex in  $\zeta$  (Remark 3.1), with Lemma 3.1, if we choose appropriate gradient steps  $\epsilon^k$  for iterations, the sequence  $\{\zeta^k\}$  will converge to the minimizer of the problem through the convergence of subgradient algorithms or  $\epsilon$ -subgradient algorithms (e.g., [5, 8, 19, 23]).

### 3.2.2 Fast sweeping method for the factored eikonal equation (3.6)

The fast sweeping method [6, 17, 18, 20, 25] has been designed as an efficient method for solving static convex Hamilton-Jacobi equations. The error of the numerical solution obtained by the fast sweeping method depends on the source singularity [15, 17, 18, 25]. In order to improve the accuracy, a factorization idea was introduced in [11]. By factoring the original function  $T$  as a product of a known function  $T_1$  and an underlying correction function  $\tau$ , we get a factored eikonal equation (3.6). This known function  $T_1$  captures the source singularity of  $T$  so that the underlying correction function  $\tau$  is smooth in the neighborhood of the source. By solving the factored eikonal equation (3.6) on  $\tau$ , the

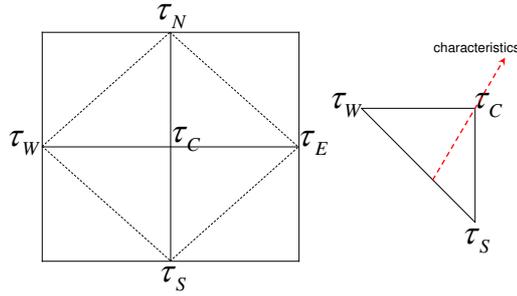


Figure 1: Rectangular mesh.

accuracy of the numerical solution for  $T$  can be improved in comparison to solving the original eikonal equation (3.4) directly. In [11], a fast sweeping scheme has also been designed to solve (3.6), which follows the causality of  $T$ . The new fast sweeping method for (3.6) is as efficient as the original fast sweeping method.

Here, we give a brief summary of the fast sweeping method for (3.6). Interested researchers may refer to [11] for more details.

First we discretize the domain  $\bar{\Omega}$  by a rectangular mesh  $\Omega^h$  with grid size  $h$ . For a fixed point  $C$  with its adjacent four triangles, we discretize (3.6) on each triangle, for example on triangle  $\triangle CWS$  as in Fig. 1, and we get a discretized equation:

$$T_1^2(C) \left| \left( \frac{\tau_C - \tau_W}{h}, \frac{\tau_C - \tau_S}{h} \right) \right|^2 + 2T_1(C)\tau_C \nabla T_1(C) \cdot \left( \frac{\tau_C - \tau_W}{h}, \frac{\tau_C - \tau_S}{h} \right) + (\tau_C^2 - \xi^2(C)) = 0. \quad (3.25)$$

When solving this equation, we enforce the following **causality condition**. Assuming that  $\tau_C^h$  is an appropriate root of Eq. (3.25), we require that the characteristic  $T_1 \nabla \tau_C^h + \tau_C^h \nabla T_1$  is in between the triangle  $\triangle CWS$ , that is,

$$T_1(C) \nabla \tau_C^h + \tau_C^h \nabla T_1(C) \geq 0, \quad (3.26)$$

with

$$\nabla \tau_C^h \approx \left( \frac{\tau_C^h - \tau_W}{h}, \frac{\tau_C^h - \tau_S}{h} \right)$$

(see Fig. 1).

Then we choose the minimum one  $\tau^h T_1$  from all four triangles, and we update  $\tau$  with the solution  $\tau^h$  of (3.25) corresponding to minimum  $\tau^h T_1$ .

**Fast Sweeping Algorithm for (3.6):**

1. Initialize the point source condition  $\tau(x)$  for a fixed source  $x$ , if  $T_1(x) \neq 0$ ,  $\tau(x) = T(x)/T_1(x)$ ; else  $\tau(x) = \xi(x)/\xi_1(x)$  from Eq. (3.6) or L'Hospital's rule.
2. Update the solution by Gauss-Seidel iterations with alternate sweeping. At each point, updating  $\tau$  according to the procedure above. For a rectangle mesh, four natural alternate orderings are:  $i=1:I, j=1:J, i=1:I, j=J:1, i=I:1, j=1:J, i=I:1$  and  $j=J:1$ .
3. Test convergence: for given convergence criterion  $tol > 0$ , repeat Step 2 until  $\|T_\xi^{k+1} - T_\xi^k\|_\infty \leq tol$ .

**Remark 3.3.** Denoting the left-hand side of (3.25) as  $H(\tau_C, \tau_W, \tau_S)$ , it has been proved that  $H(\tau_C, \tau_W, \tau_S)$  is consistent and monotone under the causality condition (3.26) (e.g., see [17, 18, 25]); therefore the numerical solution of the fast sweeping method will converge to the viscosity solution of the factored eikonal equation (3.6) as  $h \rightarrow 0$  by the convergence theorem of Barles-Souganidis [1] (see also [10, 17, 18, 25]).

**3.2.3 Fast sweeping method for adjoint state equation (3.10)**

For solving (3.10), we adopt the fast sweeping scheme in [13]. The adjoint state equation can be written in the following form,

$$c\lambda + (a\lambda)_x + (b\lambda)_y = f, \tag{3.27}$$

where  $c, a, b$  and  $f$  are given functions of  $(x, y) \in \bar{\Omega} \subset \mathbb{R}^2$  (without notational confusion, we use  $(x, y)$  to represent a point in  $\bar{\Omega} \subset \mathbb{R}^2$ ).

Considering a computational cell centered at  $(x_i, y_j)$  and discretizing the equation in conservation form, we get

$$c_{i,j}\lambda_{i,j} + \frac{1}{\Delta x} (a_{i+\frac{1}{2},j}\lambda_{i+\frac{1}{2},j} - a_{i-\frac{1}{2},j}\lambda_{i-\frac{1}{2},j}) + \frac{1}{\Delta y} (b_{i,j+\frac{1}{2}}\lambda_{i,j+\frac{1}{2}} - b_{i,j-\frac{1}{2}}\lambda_{i,j-\frac{1}{2}}) = f_{i,j}. \tag{3.28}$$

The values of  $\lambda$  on the interfaces,  $\lambda_{i\pm 1/2,j}$  and  $\lambda_{i,j\pm 1/2}$ , are determined according to propagation of the characteristics. In the case that  $a_{i+1/2,j} > 0$ , the characteristic for determining  $\lambda$  blows from left to right, and this suggests that we choose  $\lambda_{i,j}$  to define  $\lambda_{i+1/2,j}$ . Other cases are determined in a similar way.

With the following notations,

$$a_{i+\frac{1}{2},j}^\pm = \frac{a_{i+\frac{1}{2},j} \pm |a_{i+\frac{1}{2},j}|}{2}, \quad a_{i-\frac{1}{2},j}^\pm = \frac{a_{i-\frac{1}{2},j} \pm |a_{i-\frac{1}{2},j}|}{2}, \tag{3.29a}$$

$$b_{i,j+\frac{1}{2}}^\pm = \frac{b_{i,j+\frac{1}{2}} \pm |b_{i,j+\frac{1}{2}}|}{2}, \quad b_{i,j-\frac{1}{2}}^\pm = \frac{b_{i,j-\frac{1}{2}} \pm |b_{i,j-\frac{1}{2}}|}{2}, \tag{3.29b}$$

we get

$$c_{i,j}\lambda_{i,j} + \frac{1}{\Delta x} \left( (a_{i+\frac{1}{2},j}^+ \lambda_{i,j} + a_{i+\frac{1}{2},j}^- \lambda_{i+1,j}) - (a_{i-\frac{1}{2},j}^+ \lambda_{i-1,j} + a_{i-\frac{1}{2},j}^- \lambda_{i,j}) \right) + \frac{1}{\Delta y} \left( (b_{i,j+\frac{1}{2}}^+ \lambda_{i,j} + b_{i,j+\frac{1}{2}}^- \lambda_{i,j+1}) - (b_{i,j-\frac{1}{2}}^+ \lambda_{i,j-1} + b_{i,j-\frac{1}{2}}^- \lambda_{i,j}) \right) = f_{i,j}. \tag{3.30}$$

**Fast Sweeping Algorithm for Eqs. (3.27) and (3.30):**

1. Assign  $\lambda_{i,j} = 0$  at grid points on  $\partial\Omega$ .
2. Update  $\lambda_{i,j}$  at grid points in  $\Omega$  according to (3.30). As in the fast sweeping method for the factored eikonal equation, we sweep the whole domain with four alternate orderings.
3. Test convergence: for given convergence criterion  $tol > 0$ , repeat Step 2 until  $\|\lambda^{k+1} - \lambda^k\|_\infty \leq tol$ .

Denote the left-hand side of (3.30) as  $F(\lambda_{i,j}, \lambda_{i-1,j}, \lambda_{i+1,j}, \lambda_{i,j-1}, \lambda_{i,j+1})$ . For the adjoint state equation (3.10),

$$\begin{aligned} c &= \tau + T_1 \nabla T_1 \cdot \nabla \tau = \tau |\nabla T_1|^2 + T_1 \nabla T_1 \cdot \nabla \tau \\ &= \nabla T_1 \cdot (\tau \nabla T_1 + T_1 \nabla \tau) = \nabla T_1 \cdot \nabla T, \\ (a, b) &= -T_1 (\tau \nabla T_1 + T_1 \nabla \tau) = -T_1 \nabla T. \end{aligned}$$

We prove that the numerical scheme is consistent and monotone.

**Lemma 3.2.** *The numerical scheme (3.30) is consistent and monotone if  $\nabla T$  is approximated with linear interpolations.*

*Proof.* The consistency is obvious. We prove the monotonicity. Note that

$$\begin{aligned} \frac{\partial F}{\partial \lambda_{i-1,j}} &= \frac{-a_{i-\frac{1}{2},j}^+}{\Delta x} \leq 0, & \frac{\partial F}{\partial \lambda_{i+1,j}} &= \frac{a_{i+\frac{1}{2},j}^-}{\Delta x} \leq 0, & \frac{\partial F}{\partial \lambda_{i,j-1}} &= \frac{-b_{i,j-\frac{1}{2}}^+}{\Delta y} \leq 0, \\ \frac{\partial F}{\partial \lambda_{i,j+1}} &= \frac{b_{i,j+\frac{1}{2}}^-}{\Delta y} \leq 0, & \frac{\partial F}{\partial \lambda_{i,j}} &= c_{i,j} + \frac{a_{i+\frac{1}{2},j}^+ - a_{i-\frac{1}{2},j}^-}{\Delta x} + \frac{b_{i,j+\frac{1}{2}}^+ - b_{i,j-\frac{1}{2}}^-}{\Delta y}. \end{aligned}$$

The first four inequalities are obtained from the definitions (3.29). We want to prove that

$$\frac{\partial F}{\partial \lambda_{i,j}} = c_{i,j} + \frac{a_{i+\frac{1}{2},j}^+ - a_{i-\frac{1}{2},j}^-}{\Delta x} + \frac{b_{i,j+\frac{1}{2}}^+ - b_{i,j-\frac{1}{2}}^-}{\Delta y} \geq 0.$$

Without loss of generality, we assume that the source point is  $(0,0)$ , so we have  $T_1(x,y) = \sqrt{x^2 + y^2}$ . We also assume that  $a_{i+1/2,j} \geq 0$  and  $b_{i,j+1/2} \geq 0$ , so

$$a_{i+\frac{1}{2},j}^+ = a_{i+\frac{1}{2},j} = -T_1 T_x|_{(i+\frac{1}{2},j)} \quad \text{and} \quad b_{i,j+\frac{1}{2}}^+ = b_{i,j+\frac{1}{2}} = -T_1 T_y|_{(i,j+\frac{1}{2})}.$$

By definition,  $c_{i,j} = T_{1x} T_x + T_{1y} T_y|_{(i,j)}$ . Since we approximate  $T_x, T_y$  with linear approximations, we have  $T_x|_{(i,j)} = T_x|_{(i+1/2,j)}$  and  $T_y|_{(i,j)} = T_y|_{(i,j+1/2)}$ . Then  $T_x|_{(i,j)} \leq 0$  and  $T_y|_{(i,j)} \leq 0$ .

Thus we have

$$\frac{\partial F}{\partial \lambda_{i,j}} \geq \left( T_{1x}|_{(i,j)} - \frac{T_1|_{(i+\frac{1}{2},j)}}{\Delta x} \right) T_x|_{(i,j)} + \left( T_{1y}|_{(i,j)} - \frac{T_1|_{(i,j+\frac{1}{2})}}{\Delta y} \right) T_y|_{(i,j)} \geq 0.$$

The last inequality is obtained by

$$\left( T_{1x}|_{(i,j)} - \frac{T_1|_{(i+\frac{1}{2},j)}}{\Delta x} \right) \leq 0 \quad \text{and} \quad \left( T_{1y}|_{(i,j)} - \frac{T_1|_{(i,j+\frac{1}{2})}}{\Delta y} \right) \leq 0,$$

which can be verified by simple calculation.

In conclusion, we prove that  $F(\lambda_{i,j}, \lambda_{i-1,j}, \lambda_{i+1,j}, \lambda_{i,j-1}, \lambda_{i,j+1})$  is consistent and monotone. □

**Remark 3.4.** The consistency and monotonicity of (3.30) guarantee that the numerical solution of the fast sweeping method will converge to the viscosity solution of the adjoint state equation (3.10) as  $\Delta x \rightarrow 0$ ,  $\Delta y \rightarrow 0$  by the convergence theorem of Barles-Souganidis [1] (see also [10, 17, 18, 25]).

### 3.3 Relation to the subgradient marching method, discrete formulation

In [3,4], a subgradient marching algorithm was proposed to solve the optimization problem (2.7) based on a discrete functional on a fixed rectangular discretization  $\Omega^h$  of  $\Omega$ . The functional to be minimized is the following

$$\mathcal{J}^h(\xi) \triangleq h^2 \sum_{l,k} H^*(x_{l,k}, \xi_{l,k}) - \sum_{i,j} T_\xi^h(S_i, D_j) \gamma_{i,j}, \quad (3.31)$$

where  $\xi_{l,k} = \xi(x_{l,k})$ ,  $T_\xi^h(S_i, D_j)$  is the numerical solution to the eikonal equation (3.4) by the fast marching method [22], the weights  $\gamma_{i,j}$  represent the coupling on the sources  $\{S_i\}$  and destinations  $\{D_j\}$ , and  $\sum_{i,j} \gamma_{i,j} = 1$ .

The discrete functional  $\mathcal{J}^h$  is viewed as a function on variables  $\{\xi_{l,k}\}$ . In order to minimize this functional by the method of gradient descent, the direction of gradient descent is chosen to be the opposite to the derivative of  $\mathcal{J}^h$  at  $\{\xi_{l,k}\}$ ,

$$-\frac{\partial \mathcal{J}^h}{\partial \xi_{l,k}} = -\left( h^2 \sum_{l,k} \frac{\partial H^*(x_{l,k}, \xi_{l,k})}{\partial \xi_{l,k}} - \sum_{i,j} \gamma(S_i, D_j) \frac{\partial T_\xi^h(S_i, D_j)}{\partial \xi_{l,k}} \right). \quad (3.32)$$

On the right hand side,  $\partial H^*(x_{l,k}, \xi_{l,k}) / \partial \xi_{l,k}$  in the first term is just the derivative of a function, while  $\partial T_\xi^h(S_i, D_j) / \partial \xi_{l,k}$  in the second term is a sub-differential which is calculated by a subgradient marching algorithm [4].

The convergence of the minima and minimizers of the discretized formulation (3.31) to those of the continuous formulation (2.8) has been shown in [4, 16] through a  $\Gamma$ -convergence proof.

**Remark 3.5.** On a given mesh with  $N$  grid points, we briefly analyze computational complexities and memory requirements of the two methods.

For the adjoint state method, the fast sweeping method for both the factored eikonal equation and the adjoint state equation is of computational complexity  $\mathcal{O}(N)$ ; therefore the total computational complexity is  $\mathcal{O}(N)$  [11, 13, 25]. The adjoint state and the updated metric need to be stored to carry out each iteration; therefore the memory requirement is  $\mathcal{O}(N)$ .

On the other hand, for the subgradient marching method, the complexity of the fast marching method for solving the eikonal equation is  $\mathcal{O}(N \log N)$  with the factor  $\log N$  as a result of the heap-sorting process [22]; thus the computational cost for updating the subgradient at each grid point is  $\mathcal{O}(N \log N)$ . Since one needs to update the subgradient at  $N$  grid points, the total complexity is  $\mathcal{O}(N^2 \log N)$  [3]. The updated metric and the

subgradient at each grid point need to be stored to carry out each iteration; the former requires  $\mathcal{O}(N)$  memory space and the latter requires  $\mathcal{O}(N^2)$  memory space. Therefore, the total memory requirement is  $\mathcal{O}(N^2)$  (see [3]).

## 4 Numerical examples

A few examples are presented to illustrate the method. For all the examples, we choose  $H^*(x, \xi) = \xi^3/3$ , which implies that  $H(x, i) = 2i^{2/3}/3$  and  $\xi_0 = 0$ . Similar examples have been used in [4] for the subgradient marching method. Here all our computations are based on C-codes and carried out on Michigan State University Mathematics Department Research Server.

The city is modeled with  $\Omega = (0, 1) \times (0, 1)$ . We use a  $101 \times 101$  mesh to discretize  $\bar{\Omega}$ . For the optimization algorithm, we use the Armijo-Goldstein-Wolfe condition to determine gradient steps. For a single pair of source and destination (Example 4.1), we also compare the results of our method with those by the subgradient marching method.

**Example 4.1.** First we consider a single source-destination pair  $(S, D)$ . We choose the traffic strength  $\gamma(S, D) = 1$  to connect the source and the destination, and the results are shown in Fig. 2. We see that the equilibrium metric is symmetric about the source and the destination.

With the computed equilibrium metric  $\xi$ , we integrate the characteristic ODE

$$\frac{dx}{dt} = -\frac{\nabla T_{\xi}(S, x)}{\xi}$$

to find geodesics. Fig. 3 shows the distance by solving the factored eikonal equation (3.6) with equilibrium metric  $\xi$  (contour plots), and some geodesics (black curves). The figure reflects the concept of the Wardrop equilibrium; namely, each path is a geodesic.

We compare our results with those computed by the subgradient marching method [4]. Fig. 4 shows the decreasing of functional values versus the number of iterations and the contours of computed metrics by both methods. The Armijo-Goldstein-Wolfe condition is also used to determine the gradient step for the subgradient marching method. The  $L^2$ -difference between the two computed metrics is 0.0704. In terms of computational time on a  $101 \times 101$  mesh, the subgradient marching method uses 64,088 seconds ( $\approx 17.8$  hours) of CPU time, while our adjoint state method only uses 2,382 seconds ( $\approx 0.667$  hours) of CPU time.

In addition, we also try to compare two methods on a refined mesh ( $201 \times 201$ ). Our adjoint state method uses 14,158 seconds ( $\approx 3.93$  hours), while the subgradient marching method is too restrictive to implement in terms of memory and computational time.

**Example 4.2.** Next we consider the case of one source  $S$  and two destinations  $D_1, D_2$ , with (a)  $\gamma(S, D_1) = 0.5, \gamma(S, D_2) = 0.5$  and (b)  $\gamma(S, D_1) = 0.1, \gamma(S, D_2) = 0.9$ . Figs. 5 and 6 show the results.

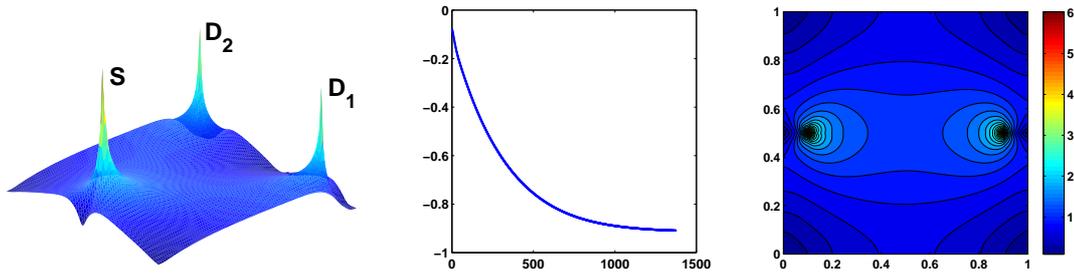


Figure 2: Example 4.1. One source-destination pair. Left: surf plot of  $\zeta$ ; Middle and right: the decrease of functional value and contour plot of  $\zeta$ .

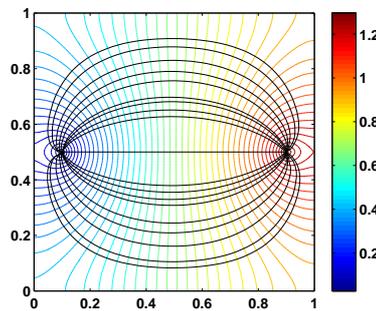


Figure 3: Example 4.1. Distance with equilibrium metric  $\zeta$  (contour plots) and geodesics (black curves).

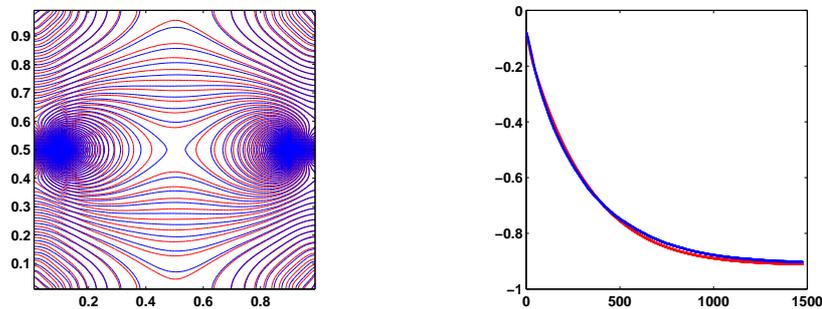


Figure 4: Example 4.1. Left: metrics computed by two methods, the  $L^2$ -difference is 0.0704; Right: functional value. Blue: subgradient marching method, Red: adjoint state method.

When the traffic strengths to each destination are equal, the equilibrium metric is symmetric about the source and destinations; when the traffic strength to one destination is dominant to the other one, the equilibrium metric exactly reflects the situation.

With the computed equilibrium metric  $\zeta$ , we also integrate the characteristic ODE  $dx/dt = -\nabla T_{\zeta}(S, x)/\zeta$  to find geodesics. Fig. 7 shows some geodesics. The figure also reflects the concept of the Wardrop equilibrium; namely, each path is a geodesic.

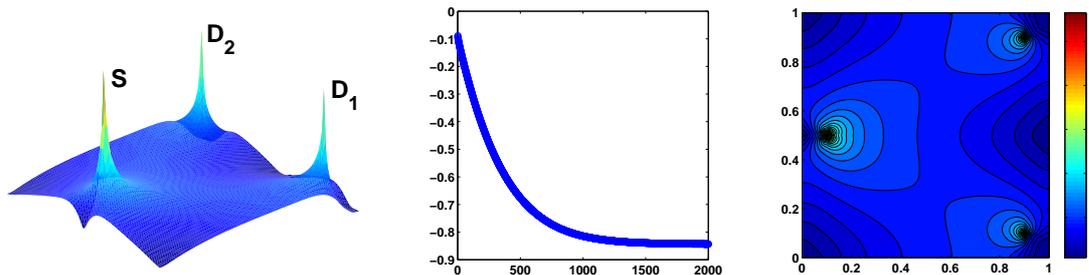


Figure 5: Example 4.2 (a). One source, two destinations. Left: surf plot of  $\zeta$ ; Middle and right: the decrease of functional value and contour plot of  $\zeta$ .

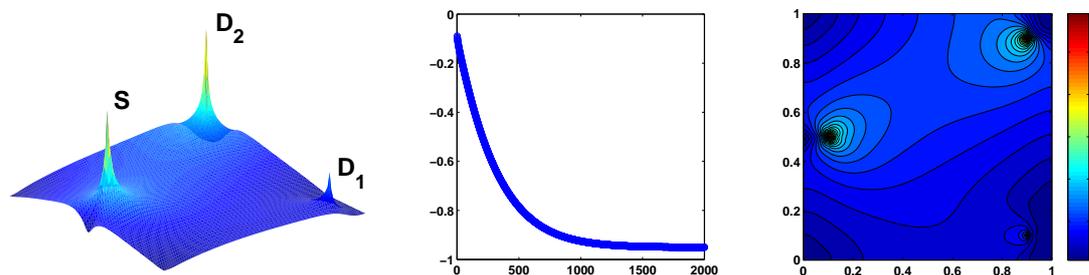


Figure 6: Example 4.2 (b). One source, two destinations. Left: surf plot of  $\zeta$ ; Middle and right: the decrease of functional value and contour plot of  $\zeta$ .

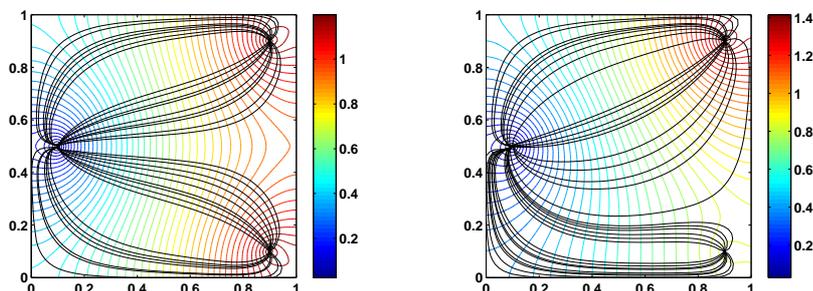


Figure 7: Example 4.2. Distance with equilibrium metric  $\zeta$  (contour plot) and geodesics (black curves). Left: (a); Right: (b).

**Example 4.3.** We consider the case of two sources  $S_1, S_2$  and two destinations  $D_1, D_2$  with  $\gamma(S_1, D_1) = 1.0/2.25$ ,  $\gamma(S_1, D_2) = 0.5/2.25$ , and  $\gamma(S_2, D_1) = 0.25/2.25$ ,  $\gamma(S_2, D_2) = 0.5/2.25$ . Fig. 8 shows the results. We see that if the traffic strength starting from source one is twice of that from source two, then the equilibrium metric reflects this situation.

With the computed equilibrium metric  $\zeta$ , we also integrate the characteristic ODE  $dx/dt = -\nabla T_\zeta(S, x)/\zeta$  to find geodesics. Fig. 9 shows some geodesics from the source  $S_1$  to the two destinations  $D_1$  and  $D_2$ . The figure also reflects the concept of the Wardrop equilibrium; namely, each path is a geodesic.

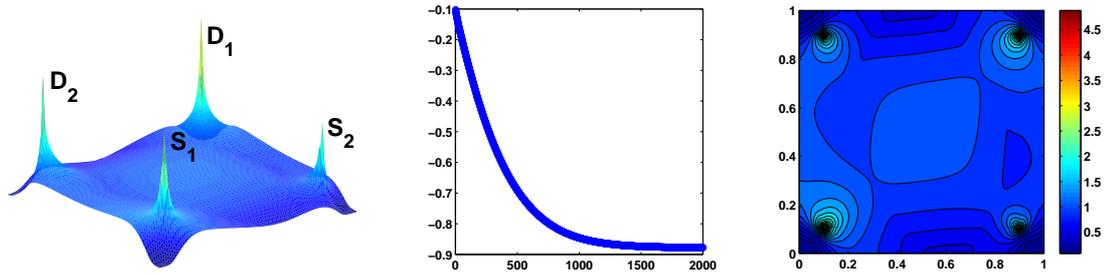


Figure 8: Example 4.3. Two sources, two destinations. Left: surf plot of  $\zeta$ ; Middle and right: the decay of functional value and contour plot of  $\zeta$ .

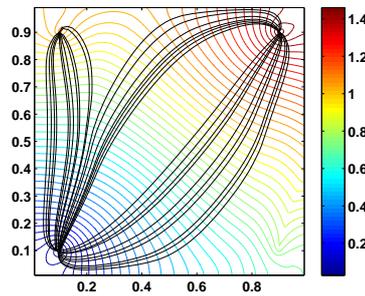


Figure 9: Example 4.3. Distance with equilibrium metric  $\zeta$  (contour plots) and geodesics (black curves).

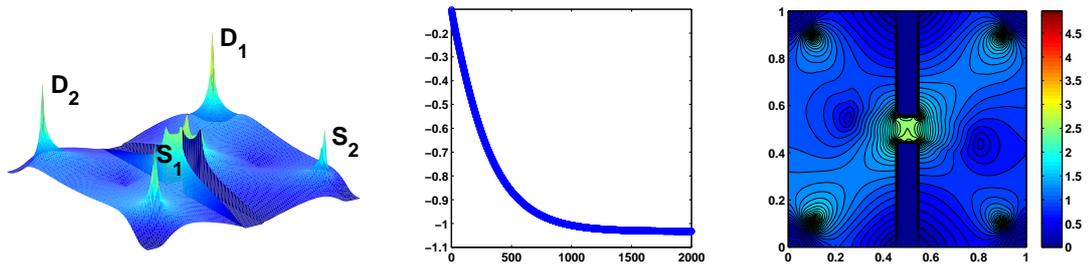


Figure 10: Example 4.4. Two sources and two destinations with river. Left: surf plot of  $\zeta$ ; Middle and right: the decrease of functional value and contour plot of  $\zeta$ .

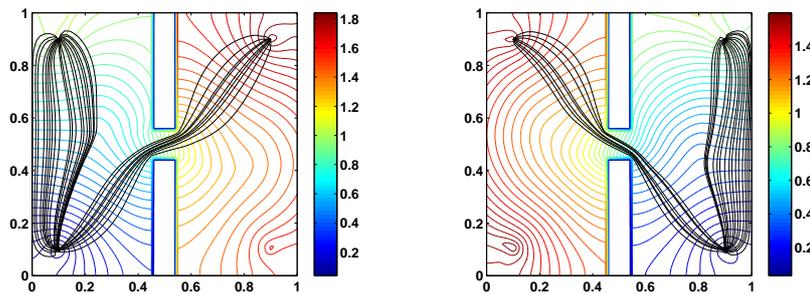


Figure 11: Example 4.4. Distance with equilibrium metric  $\zeta$  (contour plots) and geodesics (black curves). Left: Source  $S_1$ ; Right: Source  $S_2$ .

**Example 4.4.** In this example, we add a river running through the city with a bridge connecting two sides. Two sources, two destinations, and the traffic strengths are chosen as in Example 4.3. This example has also been studied in [4]. Fig. 10 shows the results.

With the computed equilibrium metric  $\xi$ , for each source, we solve the factored eikonal equation (3.6), and also integrate the characteristic ODE  $dx/dt = -\nabla T_\xi(S, x)/\xi$  to find geodesics. Fig. 11 shows some geodesics. The figure shows some paths, namely, geodesics, connecting each source to the destinations. For a source and a destination on different sides of the river, they are connected by geodesics through the bridge.

## 5 Conclusions

We have proposed an adjoint state method to compute the equilibrium metric for a dual formulation of a continuous traffic congestion model introduced in [4, 7]. The optimization problem is solved with the method of gradient descent, and the adjoint state method is designed to provide the gradient descent direction. The adjoint state equation (3.10) and the factored eikonal equation (3.6) can be solved efficiently by the fast sweeping method designed in [13] and [11], respectively. One key point is to impose appropriate boundary conditions for the derived adjoint state equation so that the problem is well-posed.

On a given mesh with  $N$  grid points, for each iteration, the memory requirement of our approach is  $\mathcal{O}(N)$  and the computational complexity is  $\mathcal{O}(N)$ . For the subgradient marching method [4], the memory requirement is  $\mathcal{O}(N^2)$  and the computational complexity is  $\mathcal{O}(N^2 \log N)$ .

The methodology proposed here, by designing an adjoint state method with appropriate boundary conditions, can be applied to other optimization problems involving geodesic distances and related metrics.

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