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Operator Factorization for Multiple-Scattering Problems and an Application to Periodic Media

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Abstract. This work concerns multiple-scattering problems for time-harmonic equations in a reference generic media. We consider scatterers that can be sources, obstacles or compact perturbations of the reference media. Our aim is to restrict the computational domain to small compact domains containing the scatterers. We use Robin-to-Robin (RtR) operators (in the most general case) to express boundary conditions for the interior problem. We show that one can always factorize the RtR map using only operators defined using single-scatterer problems. This factorization is based on a decomposition of the diffracted field, on the whole domain where it is defined. Assuming that there exists a good method for solving single-scatterer problems, it then gives a convenient way to compute RtR maps for a random number of scatterers.

AMS subject classifications: 35C15, 35J05, 35Q60, 74J20

Key words: Multiple-scattering, harmonic wave equation, exact boundary conditions, periodic media.

1 Introduction

The present study has been motivated by the computation of wave propagation in locally perturbed periodic media. The typical application is numerical modeling of photonic crystals (see e.g. [7, 10]). The starting point is the method developed in [5] and [4] for the treatment of one small local defect (typically localized in one or a few periodicity cells). Our objective in this paper is to treat the case of several well separated defects of this nature by exploiting the existing method for one single defect. This problem enters the more general framework of multiple-scattering (see for instance [8]). The outline of the article is the following: in Section 2 we present our model problem in the more general case of a

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propagation medium which is a perturbation of a given reference medium (which will be the periodic medium in the application) and we present our main objective: determine a transparent "Robin-to-Robin boundary condition" to reduce the effective computation to a small neighbourhood of the local defect. In Section 3 we present a method of decomposition of the solution of the multiple scattering problem into a sum of solution of single-scattering problems. This is the basis of the factorization of the transparent operator as a product of two operators that can be determined by solving only single-scattering problems (Section 4). Finally, in Section 5 we present numerical results obtained by applying this method to the case of a periodic reference medium.

2 Model problem and objectives

2.1 Setting of the problem

Let (Ω_j), $1 \le j \le N$, be a family of bounded, connected, open sets of $\Omega = \mathbb{R}^n$, with at least a Lipschitz boundary. The domain $\Omega_{int} := \bigcup_j \Omega_j$ will play the role of the desired computational domain (with the subscript "int" standing for interior). They are supposed to contain the support of the sources and the regions where the true propagation domain differs from a reference media which is supposed to have a simpler structure. When $N=1$ or equivalently when only one of the Ω_j's exists, one can speak of a single-scattering problem. To be more precise, we wish to solve the following Helmholtz equation in Ω:

Find u in $H^1(\Omega)$ such that:

$$-\Delta u - n^2(\omega^2 + i\varepsilon\omega)u = f, \quad \text{in } \Omega, \tag{2.1}$$

where $\varepsilon > 0$ represents the absorption of the medium (possibly arbitrary small).

We suppose that the functions $f \in L^2(\Omega)$ $n \in L^\infty(\Omega)$ are such that:

- $supp\ f \subset \Omega_{int}$, so we will write $f_j \in L^2(\Omega) = \chi_{\Omega_j} f$ such that $supp\ f_j \subset \Omega_j$.
- There exists a reference function $n_{ref} \in L^\infty(\Omega)$, such that $supp\ (n^2 - n_{ref}^2) \subset \Omega_{int}$.
- For almost every $x \in \Omega$, $0 < n_- \le n(x) \le n_+$, $0 < n_- \le n_{ref}(x) \le n_+$.

With these technical hypothesis, the problem is well posed by Lax-Milgram's theorem.

Remark 2.1. In order to avoid lengthy notations, we have omitted on purpose the case where the domain Ω contains obstacles where the solution is not defined. This case can of course be treated using the method we will present here, provided that the Ω_j's are such that they contain all these obstacles.

It is interesting to remark that the model problem (2.1) takes into account "real" scattering problems, i.e. problems of the form: Find the total field u_{tot} such that $u_{tot} = u_{inc} + u_{diff}$, where u_{inc} is the "incident" field, u_{diff} the "diffracted" field and

$$-\Delta u_{tot} - n^2(\omega^2 + i\varepsilon\omega)u_{tot} = 0, \quad \text{in } \Omega,$$

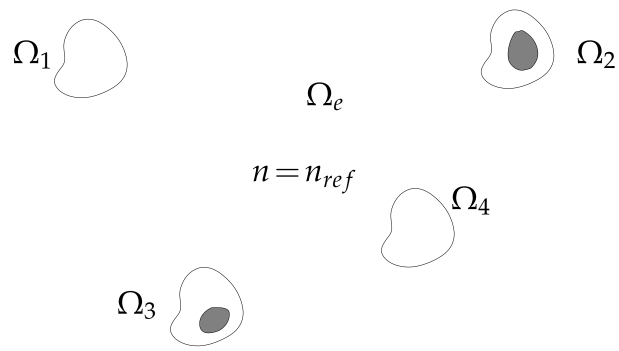


Figure 1: Example of geometry, for N=4 (obstacles are figured in grey).

with $u_{diff} \in H^1(\Omega)$ and $\Delta u_{inc} + n_{ref}^2(\omega^2 + i\varepsilon\omega) u_{inc} = 0$, as the problem posed in terms of diffracted field rewrites:

$$-\Delta u_{diff} - n^2(\omega^2 + i\varepsilon\omega) u_{diff} = (n^2 - n_{ref}^2)(\omega^2 + i\varepsilon\omega) u_{inc}, \quad \text{in } \Omega,$$

which is of the same kind than (2.1) since $supp\ (n^2 - n_{ref}^2) \subset \Omega_{int}$. Moreover one could add non-compactly supported sources to the incident field, provided one can effectively compute the incident field in the unperturbed medium (see for instance [3] for the treatment of non-compactly supported sources in homogeneous media).

We are speaking of multiple scattering problem because of the existence of several disconnected domains Ω_j's. The simplest reference medium is the case of homogeneous medium (n_{ref} =constant) for which there exists many methods for solving both single-scattering or multiple-scattering problems. The present study has been motivated by the case where n_{ref} is a periodic function: $n_{ref}(x_1 + L, \cdots, x_n + L) = n_{ref}(x_1, \cdots, x_n)$. In such a situation, an efficient method for solving single-scattering problems has been designed in [5] and [4]. In this case, the scatterer Ω_1 is a square union of periodicity cells, and one constructs a transparent boundary condition on $\partial\Omega_1$ to reduce the effective computation to the domain Ω_1. What is important to emphasize is that the cost of the resolution of this local problem as well as the construction of the transparent boundary condition increases rapidly with the size of the domain Ω_1, that is why it is not interesting to treat a multiple scattering problem in a single interior domain that would contain all the domain Ω_j introduced before (the scatterers). This is particularly clear when one wishes to treat the case of small but distant scatterers. We aim at finding a method, as it is the case for many methods for solving multiple-scattering problem, that relies on existing methods for single-scattering problems. In this paper, this project will be achieved in the context of the so-called Robin-to-Robin boundary conditions (see Section 2.2) on the boundary of: $\Omega_e = \Omega \setminus \bigcup_j \overline{\Omega_j}$. This construction requires the study of Helmholtz problems defined on Ω_e, with Robin data on $\partial\Omega_e$.

2.2 The exterior problem

As announced before, we will use Robin-boundary data's on our artificial boundaries. One should note that of course it is possible to use the classical Neumann or Dirichlet datas (which we will use in our numerical examples for simplicity), and that all the following results still hold with these datas, provided that one uses the correct trace spaces. We must now introduce a few notations. Let us define the trace operator:

$$\mathcal{E}_e : u \in H^1(\Delta,\Omega_e \text{ or } \Omega\setminus\overline{\Omega_e}) \quad\longrightarrow\quad (\partial_{n_e}u-\beta u)|_{\partial\Omega_e} \in H^{-1/2}(\partial\Omega_e), \tag{2.2}$$

where n_e is the exterior normal to $\Omega\setminus\Omega_e$, $\beta\neq 0$. We then consider problems of the form: For any $\varphi\in H^{-1/2}(\partial\Omega_e)$, find $u_e(\varphi)\in H^1(\Omega_e)$ such that:

$$-\Delta u_e(\varphi)-n_{ref}^2(\omega^2+i\varepsilon\omega)u_e(\varphi)=0, \quad \text{in } \Omega_e, \tag{2.3}$$

$$\mathcal{E}_e(u_e(\varphi))=\varphi, \qquad \text{on } \partial\Omega_e, \tag{2.4}$$

and we assume for the rest of the article that β is chosen such that the problem is coercive (in the complex sense). It is the case for instance as soon as $\Im(\beta)>0$. We will call this problem an exterior multiple-scattering problem.

We also need to define the "conjugated" (with respect to $\mathcal{E}_e$) trace operator:

$$\mathcal{S}_e : u \in H^1(\Delta,\Omega_e \text{ or } \Omega\setminus\overline{\Omega_e}) \quad\longrightarrow\quad (\partial_{n_e}u+\beta u)|_{\partial\Omega_e} \in H^{-1/2}(\partial\Omega_e). \tag{2.5}$$

This operator allows us to define the RtR operator for the exterior problem (2.3)-(2.4), as:

$$\Lambda_e : \varphi\in H^{-1/2}(\partial\Omega_e) \quad\longrightarrow\quad \mathcal{S}_e(u_e(\varphi))\in H^{-1/2}(\partial\Omega_e), \tag{2.6}$$

which leads to a reformulation of (2.1) on a bounded domain.

2.3 Reformulation on a bounded domain

Let us introduce the solution u_{int} of the following boundary value problem: Find $u_{int}\in H^1(\Omega\setminus\Omega_e)$ such that:

$$-\Delta u_{int}-n^2(\omega^2+i\varepsilon\omega)u_{int}=f, \quad \text{in } \Omega_{int}, \tag{2.7}$$

$$-\mathcal{S}_e(u_{int})+\Lambda_e(\mathcal{E}_e(u_{int}))=0, \quad \text{on } \partial\Omega_e. \tag{2.8}$$

It is well known that (2.7)-(2.8) is equivalent to (2.1) in the sense that:

- If u is the solution of (2.1) $u_{|\Omega_{int}}=u_{int}$.
- Starting from u_{int} the solution of (2.7)-(2.8), the solution u of is given by:

$$u=\begin{cases} u=u_{int}, & \text{in } \Omega_{int}, \\ u_{|\Omega_e}=u_e(\mathcal{E}_e(u_{int})), & \text{in } \Omega_e. \end{cases}$$

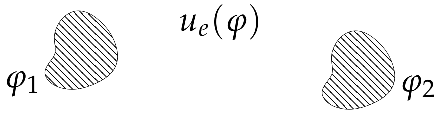


Figure 2: Example of exterior problem with two scatterers.

That is why we call this boundary condition a transparent RtR boundary condition. What follows has been directly inspired by two articles [6, 9] dedicated respectively to direct and inverse multiple-scattering problems in homogeneous media. Our contribution has been essentially to reformulate, formalize and generalize the key-idea of the wave splitting developed in these two articles. This consists in decomposing the exterior field in N single-scatterer-"diffracted" fields. This will allow us to factorize Λ_e using only single-scatterer operators.

3 Decomposition of the solution of the exterior problem

Let us define precisely what we mean by decomposition of solution of the exterior problem. For any exterior multiple-scattering problem, such as the simple example with two scatterers on Fig. 2, it is convenient to identify a data on $\partial\Omega_e$ with the vector of its restrictions on the $\partial\Omega_i$. In other words, if φ is a data for the exterior problem, we will still call $\varphi=(\varphi_i)_{1\le i\le N}$ where $\varphi_i=\varphi_{|\partial\Omega_i}$. Of course, to get one component of the complete data on $\partial\Omega_e$, we use the following trace operators:

$$\mathcal{E}_j : u\in H^1(\Delta,\Omega_j \text{ or } \Omega\setminus\overline{\Omega_j}) \quad \longrightarrow \quad (\partial_{n_j}u-\beta u)|_{\partial\Omega_j}\in H^{-1/2}(\partial\Omega_j), \tag{3.1}$$

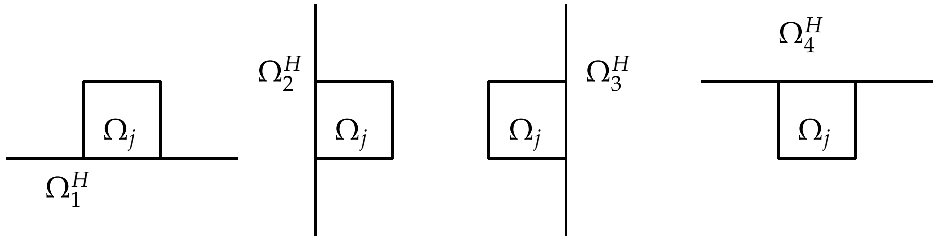
where we set $n_j=n_e$ on $\partial\Omega_e\cap\partial\Omega_j$. Naturally, we associate with each Ω_j a single-scatterer exterior problem which uses the data given by $\mathcal{E}_j$: For $\widetilde{\varphi}\in H^{-1/2}(\partial\Omega_j)$, find $u_{e,j}(\varphi)\in H^1(\Omega\setminus\overline{\Omega_j})$ such that:

$$-\Delta u_{e,j}(\widetilde{\varphi})-n_{ref}^2(\omega^2+i\varepsilon\omega)u_{e,j}(\widetilde{\varphi})=0, \quad \text{in } \mathbb{R}^n\setminus\overline{\Omega_j}, \tag{3.2}$$

$$\mathcal{E}_j(u_{e,j}(\widetilde{\varphi}))=\widetilde{\varphi}, \qquad \text{on } \partial\Omega_j. \tag{3.3}$$

Let us now explain what we mean by being able to solve a single-scattering problem. By this, we mean that one is able to compute the solution of (3.2)-(3.3) in all $\mathbb{R}^n\setminus\overline{\Omega_j}$. In the case of a homogeneous media and of spherical Ω_j's, one can expand the solution in terms of Bessel's function in all $\mathbb{R}^n\setminus\overline{\Omega_j}$, and thus proceed to an analytic resolution of the exterior single-scattering problem (see [6] for instance).

Here we are interested by more general media, and particularly in periodic media. If one chooses Ω_j's that are squared union of periodicity cells, one can resort to the semi-analytic method of [5]. The idea is to decompose the space in four half-planes $(\Omega_k^H)_{1\le k\le 4}$ surrounding Ω_j:



In each Ω_k^H, one applies the Bloch-Floquet transform in the direction of the boundary of Ω_k^H, and gets a family of half wave-guide problems that can be solved thanks to local cell-problems (see [4]), which can be solved numerically. In practice, this gives a way to reconstruct sequentially the solution in all $\mathbb{R}^n \backslash \overline{\Omega_j}$ periodicity cell per periodicity cell, combining the local cell-problems and the inverse Bloch-Floquet transform. Moreover, a system of integral equations on the boundary of each Ω_k^H can be derived, and its solution combined with the RtR operator of each half-plane gives the RtR operator for the single-scattering problem, which will be of use in the following.

The meaning of the decomposition can now be explained. We want to know whether for each $\varphi \in H^{-1/2}(\partial\Omega_e)$ there exists $(\widetilde{\varphi}_i)_i$ such that, in Ω_e:

$$u_e(\varphi) = \sum_i u_{e,i}(\widetilde{\varphi}_i), \tag{3.4}$$

which is illustrated on Fig. 3. Of course we also would like such a decomposition to be unique. This will be the object of the two following subsections. The numerical use of formula (3.4) is clear, provided that:

- One is able to construct the $\widetilde{\varphi}_i$ from φ_i: as we shall see, the proof of the existence of the φ_i is constructive.
- Compute $u_{e,i}(\widetilde{\varphi}_i)$ from $\widetilde{\varphi}_i$ which corresponds to solving a single-scattering problem.

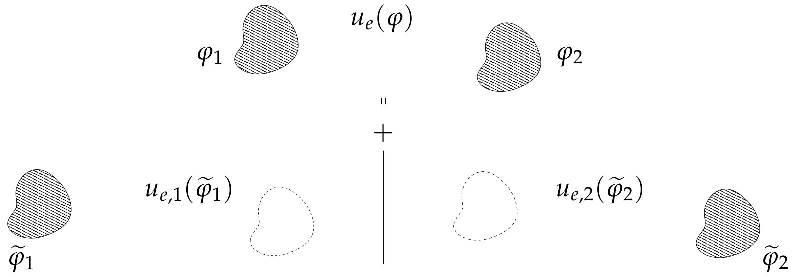


Figure 3: Example of decomposition with two scatterers.

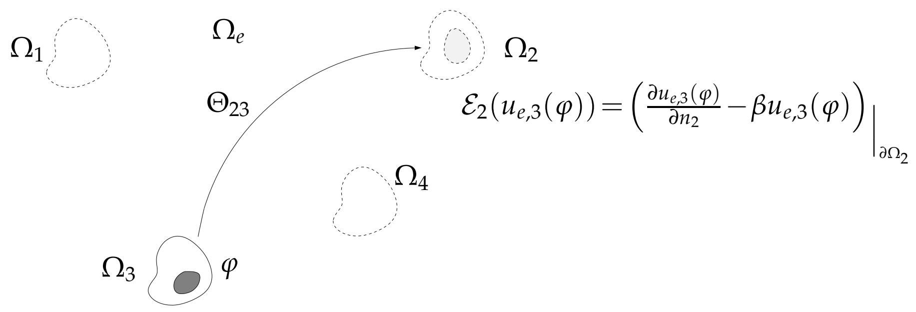


Figure 4: The operator Θ_{ij}.

3.1 Existence of the exterior decomposition

In the case of a homogeneous media, existence is obtained through explicit integral representation formulas (see [6] or [9]). We shall adopt here a more abstract and general approach. As suggested by Fig. 3, constructing the decomposition will require to look carefully at the traces of the solution of an exterior single-scattering problem on the boundaries of the other scatterers. In other words, we see on Fig. 3 that we need to know $\mathcal{E}_1(u_{e,2}(\widetilde{\varphi}_2)$ to find $\widetilde{\varphi}_1$ and reciprocally. This leads us to introduce new operators which will allow us to express this link between traces. Using the single-scatterer exterior problems (3.2)-(3.3), we define the propagation operators Θ_{ij}:

$$\Theta_{ij} : \varphi \in H^{-1/2}(\partial\Omega_j) \quad \longrightarrow \quad \mathcal{E}_i(u_{e,j}(\varphi)) \in H^{-1/2}(\partial\Omega_i), \tag{3.5}$$

as described on Fig. 4. One remarks that $\Theta_{ii} = I_d(H^{-1/2}(\partial\Omega_i))$.

Remark 3.1. As explained in the previous section, being able to solve a single-scattering problem as explain in section means in particular that one is able to compute each of the operators Θ_{ij}.

We use the identification of an element of $H^{-1/2}(\partial\Omega_e)$ with the vector of its restrictions on each $H^{-1/2}(\partial\Omega_j)$ to define the following operator:

$$\Theta : \varphi \in \prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j) \quad \longrightarrow \quad \Theta\varphi = \left(\sum_{j=1}^{N} \Theta_{ij}\varphi_j\right)_{1\le i\le N} \in \prod_{i=1}^{N} H^{-1/2}(\partial\Omega_j). \tag{3.6}$$

We can rewrite Θ as a pseudo-matrix: $\Theta = \left(\Theta_{ij}\right)_{1\le i,j\le N}$. This new operator will be useful to characterize our decomposition. Indeed, first assume that there exists a decomposition of the field:

$$u_e = \sum_{j=1}^{N} u_{e,j}(\widetilde{\varphi}_j).$$

We will then get, applying the trace operator $\mathcal{E}_i$:

$$\mathcal{E}_i(u_e) = \sum_{j=1}^{N} \mathcal{E}_i(u_{e,j}(\widetilde{\varphi}_j)),$$

which can be rewritten using the $\Theta_{i,j}$'s:

$$\mathcal{E}_i(u_e) = \sum_{j=1}^{N} \Theta_{ij}\left(\mathcal{E}_j(u_{e,j}(\widetilde{\varphi}_j))\right),$$

which is also, by definition of the exterior single-scatterer problem:

$$\mathcal{E}_i(u_e) = \sum_{j=1}^{N} \Theta_{ij}\widetilde{\varphi}_j.$$

If we call φ the vector of the boundary values of u_e, i.e. $\varphi_j = \mathcal{E}_j(u_e)$ we get the equation:

$$\varphi = \Theta\widetilde{\varphi}. \tag{3.7}$$

This is the key relation that will allow us to prove the existence of a decomposition, and explicit the fundamental link between the operator Θ and the decomposition:

Theorem 3.1. *The operator Θ is invertible from $\prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j)$ to $\prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j)$ if and only if the solution of the exterior problem has a unique decomposition in the sense of Eq.* (3.4) *for all data φ. More precisely:*

(i) Θ surjective $\Leftrightarrow$ existence of a decomposition.

(ii) Θ injective $\Leftrightarrow$ uniqueness of the decomposition.

Proof. (i) Θ surjective $\Rightarrow$ existence of a decomposition.

Let φ be the vector of boundary datas of the exterior solution. then as Θ is surjective, $\exists\ \widetilde{\varphi}$ such that $\varphi = \Theta\widetilde{\varphi}$. We consider then the $u_{e,i}(\widetilde{\varphi}_i)$, whose sum is by linearity of the problem solution in Ω_e of the exterior problem, with data

$$\mathcal{E}_e\left(\sum_{i=1}^{n} u_{e,j}(\widetilde{\varphi}_i)\right) = \Theta\widetilde{\varphi} = \varphi,$$

since (3.7). By uniqueness of the exterior solution, we get $u_e = \sum_i u_{e,i}(\widetilde{\varphi}_i)$ in all Ω_e, which proves existence of the decomposition.

Existence of a decomposition $\Rightarrow$ Θ surjective.

Let $\varphi = (\varphi_i)_{1\le i\le N}$ be a vector of $\prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j)$. We know that there exists a decomposition $u_{e,i}(\widetilde{\varphi}_i)$ of $u_e(\varphi)$, the solution of the exterior problem for the boundary data φ. By (3.7), we have $\varphi = \Theta\widetilde{\varphi}$ which proves surjectivity.

(ii) Θ injective $\Rightarrow$ uniqueness of the decomposition.

If we take two decompositions $u_{e,i}(\widetilde{\varphi}_i)$ and $u_{e,i}(\widetilde{\psi}_i)$, then we get $\Theta(\widetilde{\varphi}-\widetilde{\psi})=0$ since (3.7), from which we get $\widetilde{\varphi}=\widetilde{\psi}$ by injectivity of Θ, which proves uniqueness.

Uniqueness of the decomposition $\Rightarrow$ Θ injective.

Let $(\widetilde{\varphi}_i)_{1\leq i\leq N}$ be such that $\Theta\widetilde{\varphi}=0$. We notice that the only exterior solution with zero boundary data is $u_e(0)=0$. However, if consider the $u_{e,i}(\widetilde{\varphi}_i)$, their sum is solution of the same exterior problem (2.3)-(2.4) than $u_e(0)$, with, since (3.7), the boundary data

$$\mathcal{E}_e\left(\sum_{i=1}^{n} u_{e,j}(\widetilde{\varphi}_i)\right)=\Theta\widetilde{\varphi},$$

i.e. zero.By uniqueness, $\sum_i u_{e,i}(\widetilde{\varphi}_i)=u_e(0)$, so $u_{e,i}(\widetilde{\varphi}_i)$ is an exterior decomposition of $u_e(0)$. By uniqueness of this decomposition, $\widetilde{\varphi}=0$, and Θ is injective. $\square$

To conclude about the existence of an exterior decomposition, it consequently remains to prove that Θ is surjective. This will result from the following property:

Lemma 3.1. *For $i\neq j$, the operators Θ_{ij} are compact from $H^{-1/2}(\partial\Omega_j)$ to $H^{-1/2}(\partial\Omega_i)$, and the operator Θ is of Fredholm type.*

Proof. It is a direct consequence of the definition of Θ_{ij}. For φ in $H^{-1/2}(\partial\Omega_j)$, $u_{e,j}(\varphi)$ is in $H^2_{loc}(\mathbb{R}^n\backslash\Omega_j)$, so for $i\neq j$, its boundary data on $\partial\Omega_i$ is at least in $H^{1/2}(\partial\Omega_i)$. So the range space of Θ_{ij} is included in $H^{1/2}(\partial\Omega_i)$, which is compactly embedded in $H^{-1/2}(\partial\Omega_i)$, so Θ_{ij} is compact. Now we notice that Θ can be written $I_d+\mathcal{K}$ which is compact as we have just proved. So Θ is of Fredholm type. $\square$

According to Theorem 3.1 and Lemma 3.1, proof of the existence and uniqueness of the decomposition is reduced to the injectivity of Θ, that is to say the uniqueness property.

3.2 Uniqueness of the decomposition

The uniqueness property will be a direct consequence of the following uniqueness result:

Theorem 3.2. *Let u_e be the solution of the exterior problem* (2.3)-(2.4)*. If there exists a decomposition such that:*

$$u_e\equiv\sum_i \widetilde{u}_i, \quad in\ \Omega_e, \qquad \widetilde{u}_i\in H^1(\Omega\backslash\overline{\Omega_i}),$$
$$-\Delta\widetilde{u}_i-n_{ref}^2(\omega^2+i\varepsilon\omega)\widetilde{u}_i=0, \quad in\ \Omega\backslash\overline{\Omega_i},$$

then this decomposition is unique.

Figure 5: Domain of definition of w_1 and w_2.

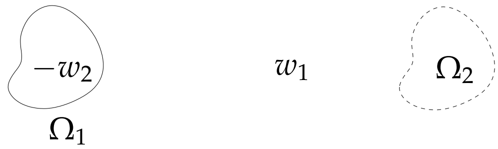


Figure 6: The function w.

Proof. The proof we derive here has been strongly inspired by [9] for the homogeneous media. However we use slightly different arguments to obtain the final conclusion, which allows us to state our result in our more general setting. We first consider the case of two scatterers (we will see that the case of N scatterers can be easily deduced from this one). Let v_1, v_2 be another decomposition, and:

$$w_i = \widetilde{u}_i - \widetilde{v}_i \ \text{ defined in } \ \Omega \backslash \overline{\Omega}_i,$$

then remark that in $\Omega \backslash \overline{\Omega}_1 \cup \overline{\Omega}_2$:

$$w_1 + w_2 = (\widetilde{u}_1 + \widetilde{u}_2) - (\widetilde{v}_1 + \widetilde{v}_2) = u - u = 0, \quad \text{in } H^1(\Omega \backslash \overline{\Omega}_1 \cup \overline{\Omega}_2).$$

In particular, we have: $w_{1|_{\partial\Omega_j}} = -w_{2|_{\partial\Omega_j}}$, $j=1,2$. As

$$-\Delta w_i - n_{ref}^2 (\omega^2 + i\varepsilon\omega) w_i = 0, \quad \text{in } \ \Omega \backslash \overline{\Omega}_i,$$

we moreover have $w_i \in H^2_{loc}(\Omega \backslash \overline{\Omega}_i)$, as shown on Fig. 5.

In other words, w_1 is regular in the neighbourhood of $\partial\Omega_2$, so one can take the trace of its normal derivative. The same thing will hold for w_2 in the neighbourhood of $\partial\Omega_1$. As one can also take the trace of the normal derivative of $w_1 + w_2$ as it is zero and consequently regular in the neighbourhood of $\partial\Omega_1$ and $\partial\Omega_2$, we get:

$$\partial_n w_1|_{\partial\Omega_j} = -\,\partial_n w_2|_{\partial\Omega_j}, \quad \text{in } H^{-1/2}(\partial\Omega_j), \quad j=1,2.$$

We consider the function w defined by: $w = w_1$ in $\overline{\Omega}_2$ and $w = -w_2$ in $\Omega \backslash \overline{\Omega}_2$ which solves the Helmholtz problem separately in Ω_2 and $\Omega \backslash \overline{\Omega}_2$, and is continuous in value and normal derivative on the boundary of Ω_2. Then, w solves the homogeneous Helmholtz equation for the reference medium, in all Ω. Thus $w = 0$, and consequently $w_2 = 0$ in $\Omega \backslash \overline{\Omega_2}$. We use the same arguments to prove that $w_1 = 0$.

For N scatterers, we take a particular w_j and we call $\breve{w}_1 = w_j$, $\breve{w}_2 = \sum_{i \neq j} w_i$, and use the same argument to prove that $w_j = 0$. We iterate the procedure to prove the result on all the w_i's left. $\square$

We can finally deduce the following result:

Theorem 3.3. *The operator Θ is invertible from $\prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j)$ to $\prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j)$.*

Proof. We know from Lemma 3.1 that the invertibility of Θ is ruled by Fredholm alternative. Theorem 3.1 says that the injectivity of Θ is equivalent to the uniqueness of the exterior decomposition. Moreover, Theorem 3.2 applied to two decompositions $u_{e,i}(\varphi_i)$ $u_{e,i}(\psi_i)$ shows that $\varphi_i = \psi_i$ and gives uniqueness of the decomposition. As a consequence, Θ is injective, so it is also surjective and even an isomorphism by virtue of Fredholm alternative. $\square$

From this last theorem, we immediately get the following existence result:

Theorem 3.4. *For every data $\varphi \in \prod_{j=1}^{N} H^{-1/2}(\partial\Omega_j) = H^{-1/2}(\partial\Omega_e)$, there exists a unique decomposition of the solution of the exterior problem* (2.3)-(2.4).

Our constructive proof of the existence of the decomposition allows us to know how to solve (2.3)-(2.4) for the data φ, easily starting from exterior problem of the kind (3.2)-(3.3), i.e. for a single-scatterer, and applying the algorithm:

- Solve $\widetilde{\varphi} = (\Theta)^{-1}\varphi$.
- Solve N problems of the kind (3.2)-(3.3) with boundary data $\widetilde{\varphi}_i$ on $\partial\Omega_i$.
- Sum in Ω_e of the solutions of these problems, and we get the solution of the scattering-problem with N scatterers.

An interesting remark is that if we choose a Gauss-Seidel algorithm to invert Θ, then we get an algorithm proposed in [1] for solving multi-scattering exterior problems.

4 Factorization of the operator Λ_e

In this section, we explain how to exploit the decomposition to construct a factorization of the operator Λ_e with the help of operators that can be constructed by solving single-scattering problems. We define the trace operators "conjugated" of the $\mathcal{E}_j$:

$$\mathcal{S}_j : u \in H^1(\Delta, \Omega_j \text{ or } \Omega\backslash\overline{\Omega_j}) \quad \longrightarrow \quad (\partial_{n_j} u + \beta u)|_{\partial\Omega_j} \in H^{-1/2}(\partial\Omega_j), \tag{4.1}$$

which are analogous to $\mathcal{S}_e$, but on a single boundary. For the same reason that we have introduced the Θ_{ij}'s, we introduce the operators $\widetilde{\Lambda}_{ij}$, which are Robin-to-Robin operators:

$$\widetilde{\Lambda}_{ij} : \varphi \in H^{-1/2}(\partial\Omega_j) \quad \longrightarrow \quad \mathcal{S}_i(u_{e,j}(\varphi)) \in H^{-1/2}(\partial\Omega_i). \tag{4.2}$$

Note that the Λ_{ij}'s are constructed as the Θ_{ij}'s from the solution of single-scattering problems, the difference being that we take the "conjugated" trace operator.

Lets get back to the solution of the original exterior problem (2.3)-(2.4). We use the decomposition (3.4) and apply the operators $\mathcal{E}_i$ and $\mathcal{S}_i$:

$$\mathcal{E}_i(u_e(\varphi))=\sum_{j=1}^{N}\mathcal{E}_i(u_{e,j}(\widetilde{\varphi}_j)) \quad \text{and} \quad \mathcal{S}_i(u_e(\varphi))=\sum_{j=1}^{N}\mathcal{S}_i(u_{e,j}(\widetilde{\varphi}_j)),$$

which gives using the operators Θ_{ij} and $\widetilde{\Lambda}_{ij}$:

$$\mathcal{E}_i(u_e(\varphi))=\sum_{j=1}^{N}\Theta_{ij}\left(\mathcal{E}_j(u_{e,j}(\widetilde{\varphi}_j))\right) \quad \text{and} \quad \mathcal{S}_i(u_e(\varphi))=\sum_{j=1}^{N}\widetilde{\Lambda}_{ij}\left(\mathcal{E}_j(u_{e,j}(\widetilde{\varphi}_j))\right).$$

Using the definition of Λ_e, (3.3) and the fact that $\varphi_{|\partial\Omega_i}=\varphi_i$, we obtain:

$$\varphi_i=\mathcal{E}_i(u_e)=\sum_{j=1}^{N}\Theta_{ij}\widetilde{\varphi}_j \quad \text{and} \quad (\Lambda_e\varphi)_i=(\mathcal{S}_e(u_e(\varphi)))_i=\mathcal{S}_i(u_e(\varphi))=\sum_{j=1}^{N}\widetilde{\Lambda}_{ij}\widetilde{\varphi}_j.$$

If we define $\widetilde{\Lambda}:\prod_{j=1}^{N}H^{-1/2}(\partial\Omega_j)\longrightarrow\prod_{j=1}^{N}H^{-1/2}(\partial\Omega_j)$ the same way as Θ, then the last two equalities become, as Θ is invertible since Theorems 3.2 and 3.1:

$$\Lambda_e\varphi=\widetilde{\Lambda}(\Theta)^{-1}\varphi. \tag{4.3}$$

Then it is obvious since (4.3) that Λ_e can be factorized the following way:

$$\Lambda_e=\widetilde{\Lambda}(\Theta)^{-1}.$$

We have of course the following result, as a direct consequence of (2.7)-(2.8):

Theorem 4.1. *Problem* (2.1) *is equivalent to:*

$$-\Delta u_{int}-n^2(\omega^2+i\varepsilon\omega)u_{int}=f_i, \qquad \text{in } \Omega_i=1,\cdots,N, \tag{4.4}$$
$$-\mathcal{S}_e(u_{int})+\widetilde{\Lambda}((\Theta)^{-1}(\mathcal{E}_e(u_{int})))=0, \quad \text{on } \partial\Omega_e \tag{4.5}$$

in the sense that if u_{int} is the solution of this new problem, $u_{|\Omega_{int}}=u_{int}$ and $u_{|\Omega_e}=u_e(\mathcal{E}_e(u_{int}))$.

This shows how we should choose boundary datas for the interior problem in order to get the restriction of the solution in all Ω, and how to describe these boundary datas using only the ones corresponding to single scatterer problems.

5 Numerical results

In this section, our reference media will be periodic, and we will use the method of [5] to solve single-scattering problems, and use the same first order Raviart-Thomas finite elements for the discretization (we omit the details).

We will know present some numerical results. As we have seen in the previous sections, to get the solution everywhere in Ω_e it is enough to know the $\widetilde{\varphi}_j$'s and then to apply equation (3.4) to get the solution of the multiple-scattering problem. Of course, one can always solve the problem (4.4)-(4.5), and then apply the inverse of Θ to the Robin-trace of its solution to get the $\widetilde{\varphi}_j$'s. However, from a practical point of view, we want to avoid the inversion of the full matrix Θ. That is why we have introduced the $\widetilde{\varphi}_j$'s as new unknowns in our problem, and we solve the augmented system:

$$
\begin{aligned}
&-\Delta u_{int} - n^2(\omega^2 + i\varepsilon\omega)u_{int} = f_i, && \text{in } \Omega_i, \quad i=1,\cdots,N, && (5.1)\\
&-\mathcal{S}_e(u_{int}) + \widetilde{\Lambda}(\widetilde{\varphi}) = 0, && \text{on } \partial\Omega_e, && (5.2)\\
&\Theta\widetilde{\varphi} = \mathcal{E}_e(u_{int}), && \text{on } \partial\Omega_e, && (5.3)
\end{aligned}
$$

which is of course equivalent to (4.4)-(4.5). We will now present several numerical illustrations. The solution will be represented on 13 periodicity cells in each direction (one defect will be at the center), that contains all the obstacles, with the following parameters:

- We use the frequency $\omega = 5$ and the absorption parameter $\varepsilon = 0.1$. The media n_{ref}^2 we will use will be of magnitude 1 to 5 (1 for the homogeneous media, 1 and 5 for the piecewise constant case), so the wavelength and the size of the periodicity cell will be of the same order, as well as the scatterers.
- The periodicity cell will be a square of length 1, on which we use a uniform structured quadrilateral grid. We use $Nx = Ny = 40$ squares in each direction x and y, thus resulting to a mesh precision $h = 0.025$.
- Eqs. (5.2)-(5.3) are handled as described in [5] through the Floquet-Bloch transform. We use piecewise constants to handle the Floquet-variable, with $Nk = 60$ basis functions on $[0,2\pi]$.

5.1 Validation in the case of a homogeneous media

A homogeneous media being only a particular case of periodic media, it is natural to use it to validate our method as we can compute a reference solution by other means. We will consider a source problem with several obstacles, and apply our method to get the solution. We use the media and source presented on Fig. 7. We next show on Fig. 8 the solution of each of the single-scattering problem with data $\widetilde{\varphi}_j$, which is given when we solve the interior problem, and then we combine this to get the global solution (on Fig. 9). It is clear that each of the single-scatterer diffracted field corresponds to what we would have got if we had only solved the single-scatterer problem. In particular, the last single-scattered field corresponds to a source problem without obstacles, and the global solution is a perturbation of this solution. We have also tested our method by increasing the size of the Ω_j's, and we still get the same results. Finally, we have validated our method by comparing the results we obtained with those obtained by another computational code that puts all the obstacle inside the same computational domain, with suitable boundary

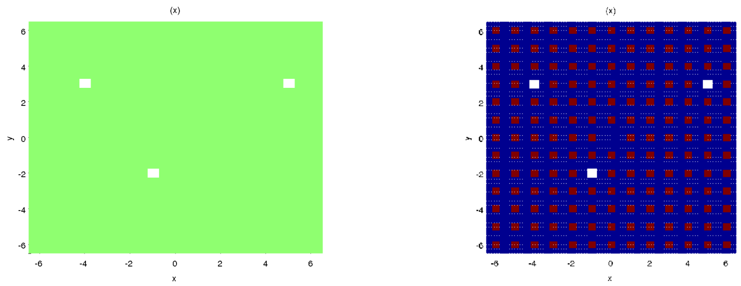


Figure 7: Media used in Subsection 5.1 (homogeneous) and Subsection 5.2 (piecewise constant).

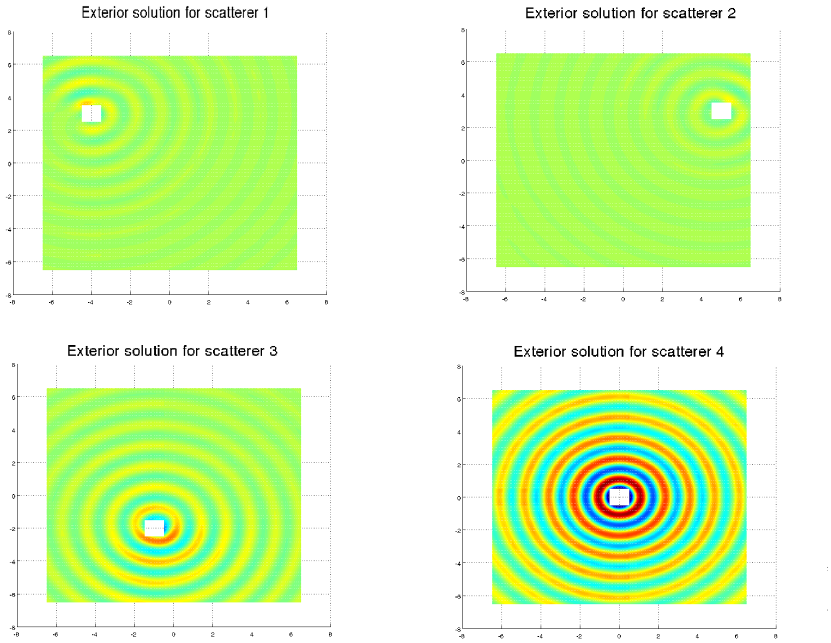


Figure 8: Solution of each of the single-scattering problem, interior and global solution for a homogeneous media.

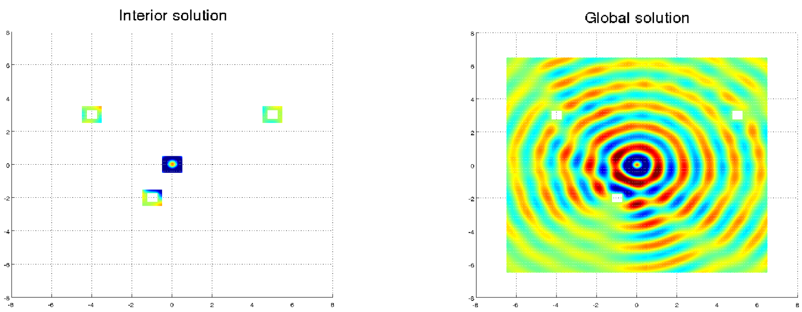


Figure 9: Interior and global solution for a homogeneous media.

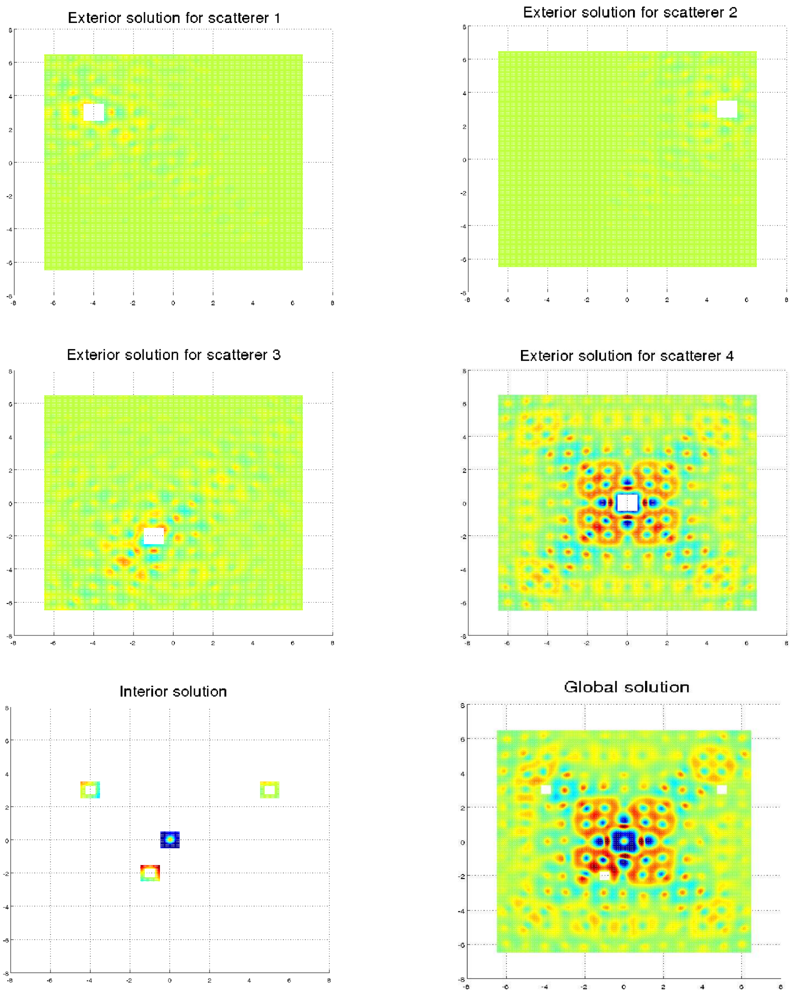


Figure 10: Solution of each of the single-scattering problem, interior and global solution for the piecewise constant periodic media of Fig. 7.

conditions. We have used Marc Durufle's code MONTJOIE (see [2]), which can handles perfectly matched layers, suitable for homogeneous media. We get the same results as those of Fig. 9, that is why we have not reproduced them here.

5.2 A more general periodic media

We will now present results in the case of a more complicated periodic media. We will again consider a source problem, in a periodic media where n_{ref} is piecewise constant, as shown on Fig.7. Again, we show on Fig. 10 the solution of each single-scattering problem, and the interior and global solutions. It is of course less obvious in this case to give a qualitative interpretation of the solutions, due to the very complicated structure of the solution. Nevertheless, we see that the three squared obstacles give diffracted fields of the same kind, and that the solution is clearly a source solution perturbed by the obstacles.

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