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Power Laws and Skew Distributions

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Abstract. Power-law distributions and other skew distributions, observed in various models and real systems, are considered. A model, describing evolving systems with increasing number of elements, is considered to study the distribution over element sizes. Stationary power-law distributions are found. Certain non-stationary skew distributions are obtained and analyzed, based on exact solutions and numerical simulations.

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1 Introduction

Power laws are observed in many systems. Particularly, one has to note the critical phenomena in interacting many-particle systems, which are associated with cooperative fluctuations of a large number of microscopic degrees of freedom. The singularities of various quantities in vicinity of the phase transition point are described by the critical exponents. It has been rigorously shown for a class of exactly solved models [1–3], which are mainly the two-dimensional lattice models. For three-dimensional systems, exact results are difficult to obtain and approximate methods are usually used. A review of numerical results, as well as of the applied here standard perturbative renormalization group (RG) methods can be found, e.g., in [4]. An alternative approach has been proposed in [5]. There are also many textbooks devoted to this topic, e.g., [6–9]. A general review of critical phenomena in various systems can be found, e.g., in [10]. Recently, the role of quantum fluctuations in critical phenomena has been reviewed and discussed in [11].

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Goldstone mode power-law singularities are observed also below the critical temperature in some systems, where the order parameter is an n-component vector with $n>1$ (see, e.g., [12–17]). These systems are spin models having $\mathcal{O}(n)$ rotational symmetry in zero external field. This is an interesting example of power law behavior, exhibited by the transverse and longitudinal correlation functions in the ordered phase. Moreover, according to the recent Monte Carlo (MC) simulation results [18–20], it is very plausible that this behavior is described by nontrivial exponents, as predicted in [17].

For a general review, one has to mention that phase transitions described by power laws and critical exponents are observed in variety of systems, such as social, economical, biological systems, as well as vehicular traffic flow, which are often referred in literature as non-physical systems. In particular, traffic flow is a driven one-dimensional system in which, unlike to one-dimensional equilibrium systems, phase transitions are observed. Formation of a car cluster on the road is analogous to aggregation phenomena in many physical systems [21]. The widely used approach in description of the vehicular traffic, as well as the traffic in biological systems such as ants, is the simulation by cellular automata models. One can mention here the famous Nagel-Schreckenberg model [22], which has numerous extensions, e.g., [23–29]. A good review about this topic can be found in [30]. Stochastic fluctuations play an important role here. A new approach to this problem, emphasizing the role of the stochasticity, has been introduced in [31]. The master equation is used here to describe the jam formation on a road as a stochastic one-step process, in which the size of a car cluster is a stochastic variable. The results of this approach have been summarized in the review paper [32], as well as in the recent textbook [33]. The critical behavior, found in a simple traffic flow model considered in [32], is described by the mean-field exponent $\beta=1/2$ for the order parameter (see p. 75 in [32]).

The power laws in critical phenomena have been discussed in [34] in a general context of many other examples, where the power-law distributions emerge. A distinguishing feature of the critical phenomena is the existence of certain length scale, which diverges at specially chosen parameters, i.e., at the critical point. It results in a scale-free or power-law distribution. In some cases, however, no fine tuning of parameters is necessary to observe the critical phenomena. It refers to systems exhibiting the self-organized criticality. Any such system adjusts itself to the critical point due to some dynamical process. The percolation on square lattice have been discussed in [34] as an example of critical phenomena and the forest fire model-as an example of the self-organized criticality. Spin systems with global rotational symmetry could be added here as a different example of the power-law behavior at a divergent length scale. Namely, the correlation length in such systems is divergent at vanishing external field not only at the critical temperature, but also below it. It results in the already mentioned here power-law Goldstone mode singularities.

Apart from the appearance of the divergent length scale, there are also other mechanisms how the power laws emerge. Many examples have been reviewed and discussed in [10, 34–37] pointing out the ubiquitous observation of power law distributions in nature. A tool for analyzing power law distributed empirical data in presented in [36]. A set of mechanisms for power laws can be found in [10] based on self-organized criticality like damage and fracture of materials as well as multiplicative recurrence with stochastic variables. Power laws are common patterns in nature [37] and economics (known as Pareto distributions [34, 35]). The underlying cause seems to be stochasticity. We have addressed this question in Section 2 by considering in detail certain model of evolving system.

2 Emergence of power-law distribution and other skew distributions in evolving systems

Power-law distribution can be considered as a particular case of the so called skew distributions. Examples of skew distributions are considered in various papers in literature, e.g., Zipf's law (power law) in rank-size distribution of cities [38] and log-normal distribution as a long-tailed duration distribution for disability in aged people [39]. A class of non-Gaussian distributions with power-law tails has been considered in [40]. In Chapter 6 of [10], the stretched exponential function family and its generation is reviewed as intermediate between thin (like Gaussian) and fat tail (like power law) distributions. Certain extreme deviation mechanism has been discussed in [41], which can explain the appearance of stretched-exponential distribution in a number of physical and other systems, exhibiting anomalous probability distribution functions and relaxation behaviors. Examples are anomalous relaxations in glasses and velocity distribution in turbulent flow.

A generalized family of distributions, including the Pareto power law distribution as well as the Weibull distribution, has been considered in [42, 43]. It has been shown here that the power law family is nested into the Weibull family as certain limit case (see Eqs. (1) to (8) in [43]).

In [44] (see also comment on this paper [45]) some interesting ideas are developed how skew distributions such as power law, log-normal and Weibull distributions emerge in general evolving systems and what makes the difference between them. According to [45], however, no correct answers to these fundamental questions have been found in [44]. Therefore, we have reconsidered this problem and have found an example, where the Weibull distribution really emerges.

We consider certain evolving system consisting of N elements, introduced already in [45]. Therefore, we will repeat some basic definitions and relations of [45], which are necessary for the actual extended study. Each element of the evolving system has certain size, which is a discrete stochastic variable taking one of the values x_n, where $n=1,2,3$, etc. The size of each element can increase from x_n to x_{n+1} with the transition rate $w(n)$. The number of elements $N=N(t)$ also increases with time, i.e., a new element of the minimal size x_1 is generated with certain rate $\mathcal{W}(N)$. Assuming $\mathcal{W}(N)=rN$, the probability $P(N,t)$ of having N elements at time t is given by the master equation

$$\frac{\partial P(N,t)}{\partial t}=r(N-1)P(N-1,t)-rNP(N,t). \tag{2.1}$$

We consider a system having $N(t=0)=N(0)$ elements at the beginning. Thus, the initial condition reads

$$P(N,t=0)=\delta_{N,N(0)}. \tag{2.2}$$

The equation for the mean number of elements $\langle N\rangle(t)=\sum_N NP(N,t)$ is obtained via multiplying both sides of (2.1) by N and summing up from $N(0)$ to infinity. It yields

$$\frac{d\langle N\rangle}{dt}=r\langle N\rangle, \tag{2.3}$$

which gives the solution

$$\langle N\rangle(t)=N(0)e^{rt}. \tag{2.4}$$

Let $\mathcal{P}(N_1,N_2,\cdots;t)$ be the probability of having N_1 elements of size x_1, N_2 elements of size x_2 and so on, at time t. The time evolution of our system can be described by the master equation for $\mathcal{P}(N_1,N_2,\cdots;t)$ in an infinitely-dimensional space of stochastic variables N_n. A quantity of interest is the probability $p(n,t)$ that a randomly chosen element has size x_n at time t.

In the thermodynamic limit $N(0)\to\infty$, considered further on, the relative fluctuations of N_n around their mean values $\langle N_n\rangle$ are vanishingly small and we have

$$p(n,t)=\frac{\langle N_n\rangle(t)}{\langle N\rangle(t)}. \tag{2.5}$$

In this case, the mean numbers of elements obey simple balance equations

$$\frac{d\langle N_n\rangle}{dt}=w(n-1)\langle N_{n-1}\rangle-w(n)\langle N_n\rangle : n\geq 2, \tag{2.6a}$$

$$\frac{d\langle N_1\rangle}{dt}=r\langle N\rangle-w(1)\langle N_1\rangle. \tag{2.6b}$$

From (2.3) and (2.5)-(2.6b) we obtain

$$\frac{\partial p(n,t)}{dt}=w(n-1)p(n-1,t)-[w(n)+r]p(n,t) : n\geq 2, \tag{2.7a}$$

$$\frac{\partial p(1,t)}{dt}=r-[w(1)+r]p(1,t). \tag{2.7b}$$

These are the basic relations, introduced already in [45].

In the following, we will consider two particular examples. For $w(n)=\lambda$, the stationary solution of (2.7a)-(2.7b) is an exponential function

$$p^{st}(n)=\frac{r}{\lambda}\Big(1+\frac{r}{\lambda}\Big)^{-n}=\frac{r}{\lambda}e^{-\gamma n} \quad \text{with } \gamma=\ln\Big(1+\frac{r}{\lambda}\Big). \tag{2.8}$$

If $x_n=(1+b)^{n-1}$ and $w(n)=\lambda$, then Eq. (2.7a) for $p(n,t)\equiv p(x_n,t)$ becomes

$$\frac{\partial p(x_n,t)}{\partial t}=-(r+\lambda)p(x_n,t)+\lambda p(x_n-\delta_n,t) \tag{2.9}$$

with $\delta_n = x_n b/(1+b)$. This equation is similar to that one obtained in [44] (cf. Eq. (7) in [44]) and discussed also in [45]. The stationary solution $p^{st}(x_n)$ is a power-law

$$p^{st}(x_n) \propto x_n^{-\alpha} \tag{2.10}$$

with $\alpha = \ln(1+r/\lambda)/\ln(1+b)$.

Another example is $x_n = n$ and $w(n) = \lambda n$. One should note the similarity of this growth mechanism with the auto-catalysis in chemical reactions in presence of birth of new entrants [46–48] as well as with the mechanism of fault growth considered in [49]. In this case we obtain

$$\frac{\partial p(x_n,t)}{\partial t} = -(r+\lambda x_n)p(x_n,t) + \lambda(x_n-1)p(x_n-1,t). \tag{2.11}$$

At large n or large x_n, it is expected that the probability $p(x_n,t)$ changes almost continuously with x_n, so that $p(x_n,t)$ can be approximated by a continuous function $p(x,t)$, which has the meaning of the probability density. For large x_n, the stationary solution of (2.11) is a power-law

$$p^{st}(x_n) \propto x_n^{-\alpha} \quad \text{at} \;\; x_n \to \infty \tag{2.12}$$

with $\alpha = 1 + r/\lambda$.

Similarly as in models of aggregation with injection [50, 51], the size distribution is cut-off at $x_n \sim \Lambda(t)$, where the upper cut-off parameter $\Lambda(t)$ diverges at $t \to \infty$. The non-stationary solution converges to the stationary (power-law) one at $r > 0$ and $t \to \infty$ in the sense that the probability distribution becomes time-independent for $x \ll \Lambda(t)$, whereas $r = 0$ is a special case where $\lim_{t\to\infty} p(x_n,t) = 0$ holds for any fixed n. Indeed, the size of each element can only increase with time and no new elements appear if $r = 0$.

The total number of elements N is conserved at $r = 0$ and these elements evolve independently of each other. Hence, $p(n,t)$ in this case can be interpreted as the probability to have certain size x_n at time t for a system consisting of one element.

Assuming the initial condition $p(n,t=0) = \delta_{n,1}$, the exact solution of (2.7a)-(2.7b) at $r = 0$ is

$$p(n,t) = \frac{(\lambda t)^{n-1}}{(n-1)!} e^{-\lambda t} \tag{2.13}$$

for $w(n) = \lambda$ and

$$p(n,t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{n-1} \tag{2.14}$$

for $w(n) = \lambda n$, as it can be easily verified by a direct substitution. Inserting $n-1 = \ln x_n/\ln(1+b)$ in (2.13), we obtain the solution for the case $x_n = (1+b)^{n-1}$ in terms of the element sizes x_n

$$p(x_n,t) = \frac{(\lambda t)^{\ln x_n/\ln(1+b)}}{\Gamma(1+\ln x_n/\ln(1+b))} e^{-\lambda t}. \tag{2.15}$$

Continuum description $x_n \to x$ is valid here for $b \to 0$ and large n around the distribution maximum, i.e., for $n \approx \lambda t \to \infty$. Besides, the probability density is $\propto p(x,t)/x$ in this case, since the density of points is varied as $1/x$.

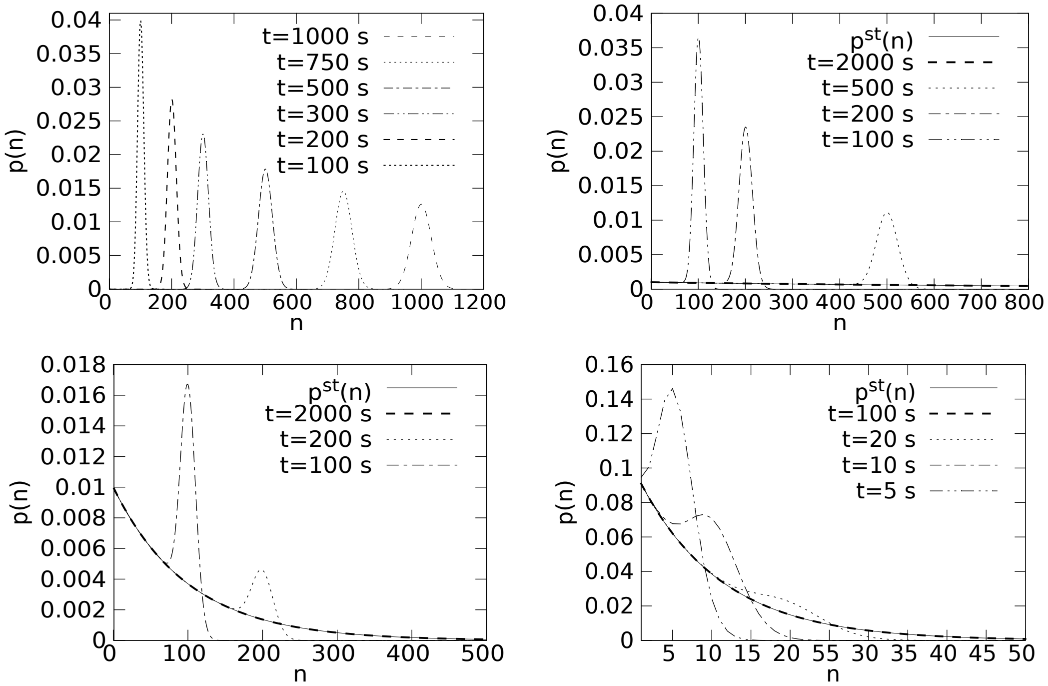

Figure 1: Probability distributions for $w(n)=\lambda=1\mathrm{s}^{-1}$ at four different values of r (in s^{-1}): $r=0$, $r=0.001$, $r=0.01$, $r=0.1$ (r value increases form left to right and top to bottom). Dashed lines represent calculated results by solving Eqs. (2.7a)-(2.7b) numerically at different time moments t. Solid line shows stationary solution given by Eq. (2.8).

The probability distribution can be obtained numerically by simulating stochastic trajectories, corresponding to the master equations (2.7a)-(2.7b). The results for $w(n)=\lambda=1$ at four different values of r are shown and compared with the stationary distribution (2.8) in Fig. 1.

At $r=0$ (top left), the probability distribution is represented by a moving maximum at $n\approx\lambda t$, as consistent with (2.13). This maximum is also well seen at small positive values of r ($r=0.001;0.01$). However, it becomes less distinct with increasing of r. Generally, the distribution is cut-off at n values, which are somewhat larger than λt. At long times, remarkably smaller than λt values of n and positive r, the time-dependent distribution is well consistent with the exponential stationary distribution (2.8), which is power-law distribution (2.10) depending on size x_n.

In the other example, where $x_n=n$ and $w(n)=\lambda n$, Eq. (2.14) yields the asymptotic solution at $r=0$

$$p(x_n,t)=e^{-\lambda t}e^{-\frac{x_n}{e^{\lambda t}}}=\frac{1}{x_n}\frac{x_n}{e^{\lambda t}}e^{-\frac{x_n}{e^{\lambda t}}} \tag{2.16}$$

valid for $x_n\sim e^{\lambda t}\to\infty$. It is obtained using the identity $\lim_{z\to\infty}(1+a/z)^z=e^a$. According to the last equality in (2.16), this exponential distribution is a special case of the Weibull

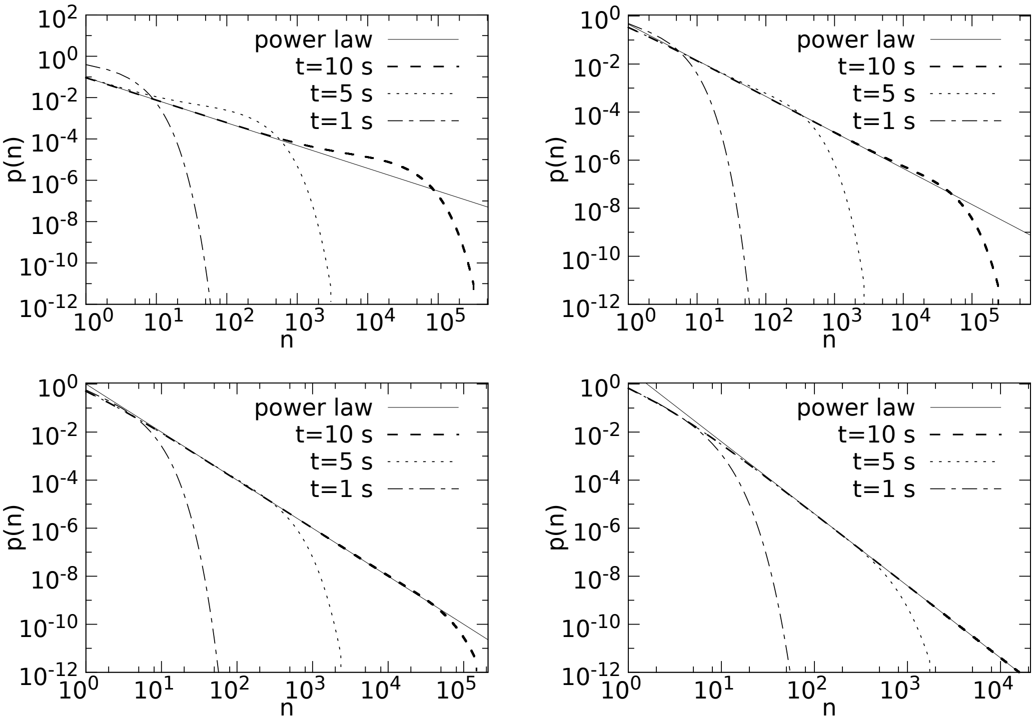

Figure 2: Probability distributions for $w(n)=n\lambda$ with $\lambda=1$ s^{-1} at four different values of r (in s^{-1}): $r=0.1$, $r=0.5$, $r=1$, $r=2$ (r value increases form left to right, top to bottom). Dashed lines represent calculated results by solving Eq. (2.11) numerically at different time moments t. Solid line represents power law solution from Eq. (2.12) and proportionality constant was determined by fitting numerical results of Eq. (2.11) at large n for maximal calculated time, because power law solution should be valid at large value of n and long time t.

distribution

$$p_{\text{Weibull}}(x,t) \equiv f(x;\eta,\gamma) = \frac{\gamma}{x}\left(\frac{x}{\eta}\right)^{\gamma} e^{-(\frac{x}{\eta})^{\gamma}} \tag{2.17}$$

with the scale parameter $\eta = e^{\lambda t}$ and the shape parameter $\gamma = 1$. The simulation results for four positive values of r are shown in Fig. 2.

The simulation results confirm the expected convergence to the stationary power-law distribution (2.12) for large $x_n = n$ and long times t.

The last term in (2.11) can be evaluated approximately by using the Taylor expansion of $yp(y,t)$ around $y = x$. In the linear approximation at $r = 0$, it leads to a continuous equation

$$\frac{\partial p(x,t)}{\partial t} = -\lambda p(x,t) - \lambda x \frac{\partial p(x,t)}{\partial x} \tag{2.18}$$

with $p(x,t)$ being the probability density. This equation is valid for large x in certain cases, where the higher order expansion terms are small. It satisfies the conservation law of total probability, as it can be easily verified performing the integration over x by parts.

It has a solution

$$p(x,t)=\frac{1}{x}\mathcal{F}\Big(\frac{x}{\eta(t)}\Big) \tag{2.19}$$

with the shape function $\mathcal{F}(z)$ and the scale parameter $\eta(t)=\eta_0 e^{\lambda t}$. Here $\mathcal{F}(z)$ is an arbitrary function, which has continuous first derivative. The Weibull distribution (2.17) is a particular case of (2.19). Hence, if we choose the initial condition corresponding to (2.19) at $t=0$, then this equation represents the solution also at $t>0$. This solution has a simple interpretation: it corresponds to the growth of the size x of each element according to the deterministic approximation $dx/dt=\lambda x$. The second-order derivative neglected in (2.18) is responsible for the diffusion effect, which can change the shape of the distribution. Eq. (2.18) is a good approximation within a finite time interval for the initial distribution in the form of (2.19) with large η_0. In this case, the second- and higher-order derivatives, neglected in (2.18) are small for any $x\sim\eta_0$, i.e., the diffusion effect is small within not too long time interval. Consequently, any skew distribution of the general form (2.19) can be observed as a transient behavior at appropriate initial distribution of element sizes, provided that no new elements are generated, i.e., $r=0$.

3 Conclusions

Below we summarize the main points of this paper:

1. The emergence of stationary power-law distributions of element sizes, as well as of non-stationary skew distributions, such as the Weibull distribution, has been considered in certain evolving systems, where the mean total number of elements $\langle N\rangle$ grows exponentially with time t as $\langle N\rangle(t)=N(0)e^{rt}$ and the size of each element also grows with time (Section 2).
2. Our exact results for the model with no particle injection ($r=0$) show that in this special case, where the total number of elements is fixed, the non-stationary long-time solutions can be different skew distributions given by Eqs. (2.15) and (2.16). The solution (2.16) is a particular case of the Weibull distribution. Our analysis shows that transient distributions of a general approximate form (2.19) can be also observed. The analytical solutions have been compared with the results of numerical simulations of the corresponding master equations, providing also the probability distributions for the general case of $r\geq 0$. A convergence of a time-dependent solution to the stationary (power-law) one is observed for large element sizes at $r>0$ and time $t\to\infty$.

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