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Stochastic Multi-Symplectic Integrator for Stochastic Nonlinear Schrödinger Equation

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Abstract. In this paper we propose stochastic multi-symplectic conservation law for stochastic Hamiltonian partial differential equations, and develop a stochastic multi-symplectic method for numerically solving a kind of stochastic nonlinear Schrödinger equations. It is shown that the stochastic multi-symplectic method preserves the multi-symplectic structure, the discrete charge conservation law, and deduces the recurrence relation of the discrete energy. Numerical experiments are performed to verify the good behaviors of the stochastic multi-symplectic method in cases of both solitary wave and collision.

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Key words: Stochastic nonlinear Schrödinger equations, stochastic multi-symplectic Hamiltonian systems, multi-symplectic integrators.

1 Introduction

As is well known, for deterministic Hamiltonian systems, the symplectic integrators for ODEs (see [7,11]), and multi-symplectic integrators for PDEs ([6,8,14]) have been investigated in the last decades, including lots of analysis on accuracy, efficiency, and long-time

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behavior, besides the character of preserving the symplectic or multi-symplectic geometric structure. For stochastic Hamiltonian systems, [15] established the theory about the stochastic symplectic methods which preserve the symplectic structure of the stochastic ODEs. To our knowledge, however, there is no reference about the multi-symplectic structure of stochastic PDEs till now. This motivates us to investigate stochastic PDEs with such structure, and find stochastic multi-symplectic integrators for this kind of stochastic PDEs. We take the stochastic nonlinear Schrödinger equations as the keystone mainly because they describe many physical phenomena and play an important role in fluid dynamics, nonlinear optics, plasma physics, etc [9, 17]. The noise sources usually represent the effect of the neglected terms yields to nonlinear Schrödinger equation in the modelison. A lot of qualitative characteristic for such small noises are presented in several chapters of references [13], including nonlinear-Schrödinger, Korteweg-de Vries, Sine-Gordon equations, etc. The stochastic nonlinear Schrödinger equation can be considered as a generalization of the deterministic nonlinear Schrödinger equation, or from another point of view a perturbation of them. In [5], a perturbed inverse scattering transform technique is used to study nonlinear Schrödinger equation with random terms. If the noise is a space independent case, a transformation can be used to convert the stochastic equation into corresponding deterministic case. Suppose some smooth conditions, the stochastic nonlinear Schrödinger equation has a unique global solution for some cases. For multiplicative noise, [1,16] studied and described some theoretical analysis. For more details about the theoretical aspects of stochastic nonlinear Schrödinger equations refer to [2] and references therein.

This paper is organized as follows. In the next section, we define the stochastic multi-symplectic PDEs, with proof of their preservation of the stochastic multi-symplectic conservation law, and finally give the definition of the stochastic multi-symplectic integrators which preserve the discrete stochastic multi-symplectic conservation law. The stochastic multi-symplectic form for the stochastic nonlinear Schrödinger equation with multiplicative noise is presented, which possesses the charge conservation law. The midpoint method is then used to construct the stochastic multi-symplectic integrator, the concrete form of which for the given Schrödinger equation is obtained by introducing the exact mathematical definition of the space-time noise. Section 3 is contributed to the theoretical analysis of the conservation properties of the obtained stochastic multi-symplectic scheme, including the discrete multi-symplectic conservation law, and the discrete charge conservation law. Furthermore, the recursion formula of a specific energy conservation law is presented. In section 4, numerical experiments are performed to testify the effectiveness of the stochastic multi-symplectic scheme listed in the previous section, for the case of both solitary wave and the collision of solitons. Section 5 is a conclusion.

2 Stochastic NLS equation and multi-symplectic integrator

In this paper, we consider a stochastic nonlinear Schrödinger equation with multiplicative noise:

$$\mathrm{i}\varphi_t=\varphi_{xx}+2|\varphi|^{2\sigma}\varphi+\varepsilon\varphi\circ\dot{\chi},\qquad t>0,\quad \varepsilon>0,\quad x\in\mathcal{R},\tag{2.1}$$

where $\varphi=\varphi(x,t)$ is a complex-valued function, and $\circ$ denotes Stratonovich product.

This equation is usually proposed as a model of energy transfer in a monolayer molecular aggregate in the presence of thermal fluctuations, and the noise here arises as a real valued potential, so that $\dot{\chi}$ is defined as a real-valued white noise which is delta correlated in time and either smooth or delta correlated in space.

If the last term at the right hand side of Eq. (2.1), i.e., the noise term, is eliminated, we reattain the famous deterministic nonlinear Schrödinger equation

$$\mathrm{i}\varphi_t=\varphi_{xx}+2|\varphi|^{2\sigma}\varphi,\tag{2.2}$$

to which plenty of literatures have been contributed [17]. It is well known the solitary wave exhibits stable for subcritical nonlinearity ($\sigma<2$), and unstable for critical ($\sigma=2$) or supercritical ($\sigma>2$) nonlinearity in this deterministic case. In fact, it is a multi-symplectic Hamiltonian system [9,12], which possesses the charge conservation law:

$$Q(t)=\int_{\mathcal{R}}|\varphi(x,t)|^2dx=\int_{\mathcal{R}}|\varphi(x,t_0)|^2dx=Q_0,\tag{2.3}$$

where $|\varphi|^2$ stands for spatial probability density.

Moreover, it preserves the energy conservation law:

$$H(t)=\int_{\mathcal{R}}|\varphi_x(x,t)|^2-\frac{1}{\sigma+1}|\varphi(x,t)|^{2(\sigma+1)}dx=H(t_0).\tag{2.4}$$

The equalities (2.3) and (2.4) have been two important criteria of measuring whether a numerical simulation is good or not.

It is noted that the stochastic nonlinear Schrödinger equation (2.1) can be considered as a white noise random perturbation of the deterministic equation (2.2), and the size of the noise is described by the real-value parameter $\varepsilon>0$. As $\varepsilon\to 0$, the solution is converges to the unique solution trajectory of the deterministic equation. Then, we can say that the stochastic model would be more realistic, and can be observed similar evolution phenomena about the solution as the deterministic case. We can prove that, under the Stratonovich product, the charge conservation law (2.3) is conserved by the solution of (2.1). This is stated in the following theorem.

Theorem 2.1. *The stochastic nonlinear Schrödinger equation* (2.1) *possesses the charge conservation law*

$$Q(t)=\int_{\mathcal{R}}|\varphi(x,t)|^2dx=\int_{\mathcal{R}}|\varphi(x,t_0)|^2dx=Q_0.$$

Based on the fact that $\dot{\chi}$ is real-valued, the assertion of the theorem can be proved easily by multiplying both sides of Eq. (2.1) by $\bar{\varphi}$, which is the conjugate of φ, and then taking the imaginary part and integrating it over the whole space domain. The proof is analogous to that of the deterministic case, so we ignore it here.

If we set $\varphi = p + iq$, where p, q are real-valued functions, we can separate (2.1) into the following form

$$\begin{cases} p_t = q_{xx} + 2(p^2+q^2)^\sigma q + \varepsilon q \circ \dot{\chi}, \\ -q_t = p_{xx} + 2(p^2+q^2)^\sigma p + \varepsilon p \circ \dot{\chi}. \end{cases}$$

By introducing two additional new variables, $v = p_x, w = q_x$, and defining a state variable $z = (p,q,v,w)^T$, the equation above can be transformed to the compact form

$$Mz_t + Kz_x = \nabla S_1(z) + \nabla S_2(z) \circ \dot{\chi}, \tag{2.5}$$

where

$$M = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

and

$$S_1(z) = \frac{1}{2}v^2 + \frac{1}{2}w^2 + \frac{1}{(\sigma+1)}(p^2+q^2)^{\sigma+1}, \qquad S_2(z) = \frac{1}{2}\varepsilon(p^2+q^2).$$

If we ignore the noise term, a corresponding deterministic PDE is obtained. According to [6] and references therein, a deterministic partial differential equation system is called a multi-symplectic Hamiltonian system if it can be written in the form

$$Mz_t + Kz_x = \nabla S(z), \quad z \in \mathcal{R}^d, \tag{2.6}$$

where M,K are skew-symmetric matrices, and S is a smooth function of the state variable z. Eq. (2.6) has the multi-symplectic conservation law

$$\partial_t \omega + \partial_x \kappa = 0, \tag{2.7}$$

with ω,κ being the two forms, i.e., $\omega = \frac{1}{2} dz \wedge M dz$, $\kappa = \frac{1}{2} dz \wedge K dz$.

The multi-symplectic conservation law (2.7) means that symplecticity changes locally and synchronously both in time and in space directions.

Then spontaneously, we have the problem of how to extend the multi-symplectic theory of the deterministic Hamiltonian partial differential equations to the stochastic systems?

Similarly, we define the stochastic multi-symplectic Hamiltonian system to have the form (2.5), where, M,K are skew-symmetric matrices, and $\dot{\chi}$ is a real-valued white noise which is delta correlated in time, and either smooth or delta correlated in space. The gradients of S_1 and S_2 are with respect to the inner product $< \cdot,\cdot >$ on R^d.

Now we give the precise mathematical definition of the noise $\dot{\chi}$. To this end, firstly, a probability space $(\Omega,\mathcal{F},\mathcal{P})$ is given together with a filtration $(\mathcal{F}_t)_{t>0}$. Then, we define the cylindrical Wiener process $W(t,x,\varpi)$ on $L^2(\mathcal{R},\mathcal{R})$, the space of square integrable functions of $\mathcal{R}$, associated to the stochastic basis $(\Omega,\mathcal{F},\mathcal{P},(\mathcal{F}_t))$, as

$$W(t,x,\varpi)=\sum_{i\in\mathrm{N}}\beta_i(t,\varpi)e_i(x), \qquad t\geq 0, \quad x\in\mathcal{R}, \quad \varpi\in\Omega. \tag{2.8}$$

Here, $e_i, i\in\mathrm{N}$ is any orthonormal basis of $L^2(\mathcal{R},\mathcal{R})$, and $(\beta_i)_{i\in\mathrm{N}}$ is a sequence of independent real Brownian motions on $(\Omega,\mathcal{F},\mathcal{P},(\mathcal{F}_t))$. Obviously, $\beta_i(t)=(W(t,x,\varpi),e_i)$, $i\in\mathrm{N}$, $t\geq 0$. Then, the space-time noise $\dot{\chi}$ is the time distributional derivative of cylindrical Wiener process, i.e., $\dot{\chi}dt=d_tW$.

According to the precise mathematical definition of the noise, the stochastic multi-symplectic Hamiltonian system (2.5) can be rewritten into the form:

$$Md_tz+Kz_xdt=\nabla S_1(z)dt+\nabla S_2(z)\circ d_tW, \qquad z\in\mathcal{R}^d. \tag{2.9}$$

We have the following theorem.

Theorem 2.2. *The stochastic multi-symplectic Hamiltonian system* (2.9) *preserves the stochastic multi-symplectic conservation law locally*

$$d_t\omega(t,x)+\partial_x\kappa(t,x)dt=0, \tag{2.10a}$$

i.e.,

$$\int_{x_0}^{x_1}\omega(t_1,x)dx+\int_{t_0}^{t_1}\kappa(t,x_1)dt=\int_{x_0}^{x_1}\omega(t_0,x)dx+\int_{t_0}^{t_1}\kappa(t,x_0)dt, \tag{2.10b}$$

where $\omega(t,x)=\frac{1}{2}dz\wedge Mdz$, $\kappa(t,x)=\frac{1}{2}dz\wedge Kdz$ *are the differential 2-forms associated with the two skew-symmetric matrices* M *and* K, *respectively, and* $(x_0,x_1)\times(t_0,t_1)$ *is the local definition domain of* $z(x,t)$.

Proof. Let $dz_{1,x,i}$, $dz_{0,x,i}$, $dz_{t,0,i}$, $dz_{0,0,i}$, $dz_{1,1,i}$, and $dz_{1,0,i}$, $i=1,\cdots,d$ be the i-th components of the differential forms $dz(t_1,x)$, $dz(t_0,x)$, $dz(t,x_1)$, $dz(t,x_0)$, $dz(t_0,x_0)$, $dz(t_1,x_1)$, and $dz(t_1,x_0)$, respectively, and M_{ij},K_{ij}, $i,j=1,\cdots,d$ be the elements of the matrices M,K.

We have

$$\begin{aligned}
&\int_{x_0}^{x_1}\omega(t_1,x)dx-\int_{x_0}^{x_1}\omega(t_0,x)dx\\
=&\frac{1}{2}\int_{x_0}^{x_1}\left[\sum_{i=1}^{d}dz_{1,x,i}\wedge\sum_{j=1}^{d}M_{ij}dz_{1,x,j}-\sum_{i=1}^{d}dz_{0,x,i}\wedge\sum_{j=1}^{d}M_{ij}dz_{0,x,j}\right]dx\\
=&\frac{1}{2}\int_{x_0}^{x_1}\sum_{i=1}^{d}\sum_{j=1}^{d}M_{ij}(dz_{1,x,i}\wedge dz_{1,x,j}-dz_{0,x,i}\wedge dz_{0,x,j})dx.
\end{aligned}$$

Using the formula of changing variables in differential forms, the above can be rewritten as

$$\begin{aligned}
&\int_{x_0}^{x_1}\sum_{i=1}^{d}\sum_{j=1}^{d}M_{ij}(dz_{1,x,i}\wedge dz_{1,x,j}-dz_{0,x,i}\wedge dz_{0,x,j})dx\\
=&\int_{x_0}^{x_1}\sum_{i=1}^{d}\sum_{j=1}^{d}M_{ij}\Big[\Big(\sum_{l=1}^{d}\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}dz_{0,0,l}\Big)\wedge\Big(\sum_{k=1}^{d}\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}dz_{0,0,k}\Big)\\
&\qquad-\Big(\sum_{l=1}^{d}\frac{\partial z_{0,x,i}}{\partial z_{0,0,l}}dz_{0,0,l}\Big)\wedge\Big(\sum_{k=1}^{d}\frac{\partial z_{0,x,j}}{\partial z_{0,0,k}}dz_{0,0,k}\Big)\Big]dx\\
=&\sum_{l=1}^{d}\sum_{k=1}^{d}\Big[\sum_{i=1}^{d}\sum_{j=1}^{d}M_{ij}\int_{x_0}^{x_1}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}-\frac{\partial z_{0,x,i}}{\partial z_{0,0,l}}\frac{\partial z_{0,x,j}}{\partial z_{0,0,k}}\Big)dx\Big]\,dz_{0,0,l}\wedge dz_{0,0,k}.
\end{aligned}\tag{2.11}$$

Similarly, we can obtain

$$\begin{aligned}
&\int_{t_0}^{t_1}\kappa(t,x_1)dt-\int_{t_0}^{t_1}\kappa(t,x_0)dt\\
=&\frac{1}{2}\int_{t_0}^{t_1}\sum_{i=1}^{d}\sum_{j=1}^{d}K_{ij}(dz_{t,1,i}\wedge dz_{t,1,j}-dz_{t,0,i}\wedge dz_{t,0,j})dt\\
=&\frac{1}{2}\sum_{l=1}^{d}\sum_{k=1}^{d}\Big[\sum_{i=1}^{d}\sum_{j=1}^{d}K_{ij}\int_{t_0}^{t_1}\Big(\frac{\partial z_{t,1,i}}{\partial z_{0,0,l}}\frac{\partial z_{t,1,j}}{\partial z_{0,0,k}}-\frac{\partial z_{t,0,i}}{\partial z_{0,0,l}}\frac{\partial z_{t,0,j}}{\partial z_{0,0,k}}\Big)dt\Big]\,dz_{0,0,l}\wedge dz_{0,0,k}.
\end{aligned}\tag{2.12}$$

Define

$$a_{l,k}(t_1,x_1)=\sum_{i=1}^{d}\sum_{j=1}^{d}M_{ij}\int_{x_0}^{x_1}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}-\frac{\partial z_{0,x,i}}{\partial z_{0,0,l}}\frac{\partial z_{0,x,j}}{\partial z_{0,0,k}}\Big)dx,\tag{2.13}$$

$$b_{l,k}(t_1,x_1)=\sum_{i=1}^{d}\sum_{j=1}^{d}K_{ij}\int_{t_0}^{t_1}\Big(\frac{\partial z_{t,1,i}}{\partial z_{0,0,l}}\frac{\partial z_{t,1,j}}{\partial z_{0,0,k}}-\frac{\partial z_{t,0,i}}{\partial z_{0,0,l}}\frac{\partial z_{t,0,j}}{\partial z_{0,0,k}}\Big)dt.\tag{2.14}$$

Combining Eqs. (2.11) and (2.12), we get

$$\begin{aligned}
&\int_{x_0}^{x_1}\omega(t_1,x)dx+\int_{t_0}^{t_1}\kappa(t,x_1)dt-\int_{x_0}^{x_1}\omega(t_0,x)dx-\int_{t_0}^{t_1}\kappa(t,x_0)dt\\
=&\frac{1}{2}\sum_{l=1}^{d}\sum_{k=1}^{d}[a_{l,k}(t_1,x_1)+b_{l,k}(t_1,x_1)]dz_{0,0,l}\wedge dz_{0,0,k}.
\end{aligned}$$

Then, the equality (2.10) is fulfilled if and only if

$$\sum_{l=1}^{d}\sum_{k=1}^{d}[a_{l,k}(t_1,x_1)+b_{l,k}(t_1,x_1)]dz_{0,0,l}\wedge dz_{0,0,k}=0.\tag{2.15}$$

Set t_0,x_0 fixed, and change the variables t_1,x_1. It's not difficult to check that, if t_1 is taken as the initial time t_0, we have $a_{l,k}(t_1,x_1)\equiv 0$, $l,k=1,\cdots,d$, and $b_{l,k}(t_1,x_1)\equiv 0$, $l,k=1,\cdots,d$, because the upper and lower integral limits become the same.

So the condition (2.15) holds, if the differential of $a_{l,k}(t_1,x_1)+b_{l,k}(t_1,x_1)$ with respect to t_1 can be proved to be zero, i.e.,

$$d_{t_1}a_{l,k}(t_1,x_1)+d_{t_1}b_{l,k}(t_1,x_1)=0,\qquad l,k=1,\cdots,d. \tag{2.16}$$

Consider the i-th component equation of the stochastic Hamiltonian system (2.9)

$$\sum_{j=1}^{d}M_{ij}d_{t_1}z_{1,x,j}+\sum_{j=1}^{d}K_{ij}\frac{\partial}{\partial_x}z_{1,x,j}dt=\frac{\partial S_1(z)}{\partial z_{1,x,i}}dt+\frac{\partial S_2(z)}{\partial z_{1,x,i}}\circ d_{t_1}W.$$

Taking partial derivatives with respect to $z_{0,0,k}$ and $z_{0,0,l}$ on both sides of the equation above yields

$$\begin{aligned}\sum_{j=1}^{d}M_{ij}d_{t_1}\Big(\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\Big)=&-\sum_{j=1}^{d}K_{ij}\frac{\partial}{\partial_x}\Big(\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\Big)dt+\sum_{j=1}^{d}\frac{\partial^2 S_1(z)}{\partial z_{1,x,i}\partial z_{1,x,j}}\Big(\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\Big)dt\\&+\sum_{j=1}^{d}\frac{\partial^2 S_2(z)}{\partial z_{1,x,i}\partial z_{1,x,j}}\Big(\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\Big)\circ d_{t_1}W,\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}&\sum_{i=1}^{d}M_{ij}d_{t_1}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\Big)=-\sum_{i=1}^{d}M_{ji}d_{t_1}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\Big)\\=&\sum_{i=1}^{d}K_{ji}\frac{\partial}{\partial_x}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\Big)dt-\sum_{i=1}^{d}\frac{\partial^2 S_1(z)}{\partial z_{1,x,j}\partial z_{1,x,i}}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\Big)dt-\sum_{i=1}^{d}\frac{\partial^2 S_2(z)}{\partial z_{1,x,j}\partial z_{1,x,i}}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\Big)\circ d_{t_1}W,\end{aligned} \tag{2.18}$$

respectively. Due to (2.13), we get

$$\begin{aligned}d_{t_1}a_{l,k}(t_1,x_1)=&\sum_{i=1}^{d}\int_{x_0}^{x_1}\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\sum_{j=1}^{d}M_{ij}d_{t_1}\Big(\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\Big)dx\\&+\sum_{j=1}^{d}\int_{x_0}^{x_1}\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\sum_{i=1}^{d}M_{ij}d_{t_1}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\Big)dx.\end{aligned} \tag{2.19}$$

Substituting (2.17), (2.18) into (2.19), and noticing that

$$\frac{\partial^2 S_\theta(z)}{\partial z_{1,x,j}\partial z_{1,x,i}}=\frac{\partial^2 S_\theta(z)}{\partial z_{1,x,i}\partial z_{1,x,j}},\qquad \theta=1,2,$$

we obtain

$$
\begin{aligned}
d_{t_1}a_{l,k}(t_1,x_1) &= -\sum_{i=1}^{d}\sum_{j=1}^{d}K_{ij}\left[\int_{x_0}^{x_1}\frac{\partial}{\partial_x}\Big(\frac{\partial z_{1,x,i}}{\partial z_{0,0,l}}\cdot\frac{\partial z_{1,x,j}}{\partial z_{0,0,k}}\Big)dx\right]dt_1\\
&= -\sum_{i=1}^{d}\sum_{j=1}^{d}K_{ij}\Big(\frac{\partial z_{1,1,i}}{\partial z_{0,0,l}}\cdot\frac{\partial z_{1,1,j}}{\partial z_{0,0,k}}-\frac{\partial z_{1,0,i}}{\partial z_{0,0,l}}\cdot\frac{\partial z_{1,0,j}}{\partial z_{0,0,k}}\Big)dt_1. \qquad (2.20)
\end{aligned}
$$

On the other hand, according to (2.14), we have

$$
d_{t_1}b_{l,k}(t_1,x_1)=\sum_{i=1}^{d}\sum_{j=1}^{d}K_{ij}\Big(\frac{\partial z_{1,1,i}}{\partial z_{0,0,l}}\cdot\frac{\partial z_{1,1,j}}{\partial z_{0,0,k}}-\frac{\partial z_{1,0,i}}{\partial z_{0,0,l}}\cdot\frac{\partial z_{1,0,j}}{\partial z_{0,0,k}}\Big)dt_1. \qquad (2.21)
$$

Then, the equality (2.16) results from adding (2.20) and (2.21).

This completes the proof. □

Then we get the conclusion that, stochastic nonlinear Schrödinger equation (2.1) possesses the stochastic multi-symplectic conservation law (2.10), and thus is the stochastic multi-symplectic Hamiltonian system. Now, the question is, what kind of numerical methods have the ability of preserving the discrete form of the stochastic multi-symplectic conservation law when they are applied to the stochastic multi-symplectic Hamiltonian system, i.e., stochastic multi-symplectic integrators?

To construct stochastic multi-symplectic integrators, we apply the midpoint rule to Eq. (2.5), both in temporal and spatial directions, and get the following full-discrete form

$$
M\left(\frac{z^{n+1}_{j+\frac{1}{2}}-z^{n}_{j+\frac{1}{2}}}{\Delta t}\right)+K\left(\frac{z^{n+\frac{1}{2}}_{j+1}-z^{n+\frac{1}{2}}_{j}}{\Delta x}\right)=\nabla S_1(z^{n+\frac{1}{2}}_{j+\frac{1}{2}})+\nabla S_2(z^{n+\frac{1}{2}}_{j+\frac{1}{2}})\dot{\chi}^{n+\frac{1}{2}}_{j+\frac{1}{2}}, \qquad (2.22)
$$

with $z^{n+\frac{1}{2}}_{j}=\frac{1}{2}(z^{n+1}_{j}+z^{n}_{j})$, $z^{n}_{j+\frac{1}{2}}=\frac{1}{2}(z^{n}_{j+1}+z^{n}_{j})$ and $z^{n+\frac{1}{2}}_{j+\frac{1}{2}}=\frac{1}{4}(z^{n}_{j}+z^{n}_{j+1}+z^{n+1}_{j}+z^{n+1}_{j+1})$.

We would like to mention that, for deterministic nonlinear Schrödinger equations, the midpoint rule is a representative method of constructing multi-symplectic integrator, see [8] and references therein. In the case of stochastic ODEs, midpoint rule is also an important symplectic method, see e.g. [10] and [15].

For the full-discrete method (2.22), we have the following result.

Theorem 2.3. *The discretization* (2.22) *is a stochastic multi-symplectic integrator, i.e., it satisfies the discrete stochastic multi-symplectic conservation law:*

$$
(\omega^{n+1}_{j+\frac{1}{2}}-\omega^{n}_{j+\frac{1}{2}})\Delta x+(\kappa^{n+\frac{1}{2}}_{j+1}-\kappa^{n+\frac{1}{2}}_{j})\Delta t=0, \qquad (2.23)
$$

where $\omega^{n}_{j+\frac{1}{2}}=\frac{1}{2}dz^{n}_{j+\frac{1}{2}}\wedge Mdz^{n}_{j+\frac{1}{2}}$, $\kappa^{n}_{j}=\frac{1}{2}dz^{n+\frac{1}{2}}_{j}\wedge Kdz^{n+\frac{1}{2}}_{j}$.

Proof. Taking differential in the phase space on both sides of (2.22), we obtain

$$\begin{aligned}&\Delta x M(dz^{n+1}_{j+\frac{1}{2}}-dz^{n}_{j+\frac{1}{2}})+\Delta t K(dz^{n+\frac{1}{2}}_{j+1}-dz^{n+\frac{1}{2}}_{j})\\=&\Delta t\Delta x\Big[\nabla^2 S_1(z^{n+\frac{1}{2}}_{j+\frac{1}{2}})dz^{n+\frac{1}{2}}_{j+\frac{1}{2}}+\nabla^2 S_2(z^{n+\frac{1}{2}}_{j+\frac{1}{2}})dz^{n+\frac{1}{2}}_{j+\frac{1}{2}}\dot{\chi}^{n+\frac{1}{2}}_{j+\frac{1}{2}}\Big],\end{aligned}$$

Then, using

$$dz^{n+\frac{1}{2}}_{j+\frac{1}{2}}:=\frac{1}{2}(dz^{n+1}_{j+\frac{1}{2}}+dz^{n}_{j+\frac{1}{2}})=\frac{1}{2}(dz^{n+\frac{1}{2}}_{j+1}+dz^{n+\frac{1}{2}}_{j})$$

to perform wedge product with the above equation yields

$$\Delta x\Big(dz^{n+1}_{j+\frac{1}{2}}\wedge Mdz^{n+1}_{j+\frac{1}{2}}-dz^{n}_{j+\frac{1}{2}}\wedge Mdz^{n}_{j+\frac{1}{2}}\Big)+\Delta t\Big(dz^{n+\frac{1}{2}}_{j+1}\wedge Kdz^{n+\frac{1}{2}}_{j+1}-dz^{n+\frac{1}{2}}_{j}\wedge Kdz^{n+\frac{1}{2}}_{j}\Big)=0,$$

where the equality is due to the symmetry of $\nabla^2 S_1(z)$ and $\nabla^2 S_2(z)$.

This completes the proof. □

In the analysis above, we just need that $\dot{\chi}$ is a real-valued function. Now we give its concrete discrete form.

For convenience, denote

$$\begin{aligned}&\delta_x^+ z_j:=\frac{z_{j+1}-z_j}{\Delta x},\qquad \delta_t^+ z^n:=\frac{z^{n+1}-z^n}{\Delta t},\qquad \delta_x^+\delta_x^- z_j:=\frac{z_{j+1}-2z_j+z_{j-1}}{\Delta x^2},\\&(u,v)=\Delta x\sum_j u_j\overline{v_j},\qquad \|z\|_2=\sqrt{(z,z)},\end{aligned}$$

and rewrite the numerical scheme (2.22), corresponding to the continuous equation (2.5), into

$$\begin{cases}\delta_t^+ p^n_{j+\frac{1}{2}}=\delta_x^+ w^{n+\frac{1}{2}}_j+2\Big((p^{n+\frac{1}{2}}_{j+\frac{1}{2}})^2+(q^{n+\frac{1}{2}}_{j+\frac{1}{2}})^2\Big)^\sigma q^{n+\frac{1}{2}}_{j+\frac{1}{2}}+\varepsilon q^{n+\frac{1}{2}}_{j+\frac{1}{2}}\dot{\chi}^{n+\frac{1}{2}}_{j+\frac{1}{2}},\\-\delta_t^+ q^n_{j+\frac{1}{2}}=\delta_x^+ v^{n+\frac{1}{2}}_j+2\Big((p^{n+\frac{1}{2}}_{j+\frac{1}{2}})^2+(q^{n+\frac{1}{2}}_{j+\frac{1}{2}})^2\Big)^\sigma p^{n+\frac{1}{2}}_{j+\frac{1}{2}}+\varepsilon p^{n+\frac{1}{2}}_{j+\frac{1}{2}}\dot{\chi}^{n+\frac{1}{2}}_{j+\frac{1}{2}},\\\delta_x^+ p^{n+\frac{1}{2}}_j=v^{n+\frac{1}{2}}_{j+\frac{1}{2}},\\\delta_x^+ q^{n+\frac{1}{2}}_j=w^{n+\frac{1}{2}}_{j+\frac{1}{2}}.\end{cases}$$

Recalling that $\varphi=p+\mathrm{i}q$, and eliminating the additionally introduced variables v and w, we get the equation of φ:

$$\begin{aligned}\mathrm{i}(\delta_t^+\varphi^n_{j+\frac{1}{2}}+\delta_t^+\varphi^n_{j-\frac{1}{2}})=&2\delta_x^+\delta_x^-\varphi^{n+\frac{1}{2}}_j+2|\varphi^{n+\frac{1}{2}}_{j+\frac{1}{2}}|^{2\sigma}\varphi^{n+\frac{1}{2}}_{j+\frac{1}{2}}+2|\varphi^{n+\frac{1}{2}}_{j-\frac{1}{2}}|^{2\sigma}\varphi^{n+\frac{1}{2}}_{j-\frac{1}{2}}\\&+\varepsilon\varphi^{n+\frac{1}{2}}_{j+\frac{1}{2}}\dot{\chi}^{n+\frac{1}{2}}_{j+\frac{1}{2}}+\varepsilon\varphi^{n+\frac{1}{2}}_{j-\frac{1}{2}}\dot{\chi}^{n+\frac{1}{2}}_{j-\frac{1}{2}}.\end{aligned}\tag{2.24}$$

Set the space domain to be $[x_L,x_R]$, and compute the numerical solution at the points $x_0,x_1,\cdots,x_J$, $J=(x_R-x_L)/\Delta x$. $\dot{\chi}_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ can be regarded as an approximation of integral

$$\frac{1}{\Delta x\Delta t}\int_{j\Delta x}^{(j+1)\Delta x}\int_{t_n}^{t_{n+1}}\dot{\chi}\,dsdx,\qquad j\in\mathcal{N}$$

in the local box $(x_L+j\Delta x,x_L+(j+1)\Delta x)\times(t_n,t_{n+1})$.

According to the precise mathematical definition of the cylindrical Wiener process (2.8) and the space-time noise, it follows

$$\begin{aligned}\dot{\chi}_{j+\frac{1}{2}}^{n+\frac{1}{2}}&=\frac{1}{\Delta x\Delta t}\int_{j\Delta x}^{(j+1)\Delta x}\int_{t_n}^{t_{n+1}}\sum_{i\in\mathrm{N}}e_i(x)d\beta_i(s)dx\\&=\frac{1}{\Delta x\Delta t}\sum_{i\in\mathrm{N}}\left(\int_{j\Delta x}^{(j+1)\Delta x}e_i(x)\,dx\right)(\beta_i(t_{n+1})-\beta_i(t_n)).\end{aligned}\tag{2.25}$$

As an orthonormal basis of $L^2(\mathcal{R})$, $e_i,i\in\mathrm{N}$ can be chosen as

$$\begin{aligned}&e_j=\frac{1}{\sqrt{\Delta x}}\mathbf{1}_{[x_L+(j-1/2)\Delta x,x_L+(j+1/2)\Delta x)},\qquad j=1,2,\cdots,J-1.\\&e_0=\frac{1}{\sqrt{\Delta x/2}}\mathbf{1}_{[x_L,x_L+1/2\Delta x)},\qquad e_J=\frac{1}{\sqrt{\Delta x/2}}\mathbf{1}_{[x_R-1/2\Delta x,x_R)}.\end{aligned}$$

It is not difficult to verify that $\int_{x_L+(j-1/2)\Delta x}^{x_L+(j+1/2)\Delta x}e_i(x)dx=0$, if $i\neq j$, $j=0,1,\cdots,J$, $i\in\mathrm{N}$.

Inserting this expression into (2.25), we have

$$\dot{\chi}_{j+\frac{1}{2}}^{n+\frac{1}{2}}=\frac{1}{2\Delta t\sqrt{\Delta x}}(\beta_j(t_{n+1})-\beta_j(t_n)+\beta_{j+1}(t_{n+1})-\beta_{j+1}(t_n)),\qquad j=0,1,\cdots,J-1.$$

Since $(\beta_j(t_{n+1})-\beta_j(t_n))/\sqrt{\Delta t}$ are independent random variables with $\mathcal{N}(0,1)$ distribution, we can select $(\chi_j^{n+\frac{1}{2}})_{n\geq 0}$, $j=0,1,\cdots,J$ as a sequence of independent random variables with normal law $\mathcal{N}(0,1)$, and produce a new vector $(\chi_0^{n+\frac{1}{2}},\cdots,\chi_J^{n+\frac{1}{2}})$ at each time increment. Furthermore, (2.24) can be written as

$$\begin{aligned}\mathrm{i}(\delta_t^+\varphi_{j+\frac{1}{2}}^n+\delta_t^+\varphi_{j-\frac{1}{2}}^n)=&2\delta_x^+\delta_x^-\varphi_j^{n+\frac{1}{2}}+2|\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}+2|\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}}\\&+\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}})+\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\chi_j^{n+\frac{1}{2}}+\chi_{j-1}^{n+\frac{1}{2}}),\end{aligned}\tag{2.26}$$

with

$$\begin{aligned}&\varphi_j^{n+\frac{1}{2}}=\frac{1}{2}(\varphi_j^{n+1}+\varphi_j^n),\qquad \varphi_{j+\frac{1}{2}}^n=\frac{1}{2}(\varphi_{j+1}^n+\varphi_j^n),\\&\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}=\frac{1}{4}(\varphi_j^n+\varphi_{j+1}^n+\varphi_j^{n+1}+\varphi_{j+1}^{n+1}).\end{aligned}$$

The following discussions are all based on the full-discretized stochastic multi-symplectic scheme (2.26). The implicit full-discreted scheme can be solved by use of the fixed point algorithm starting from initial values φ_j^0, $j=1,2,\cdots,J$.

3 Conservative properties of the stochastic multi-symplectic integrator

This section investigates the global conservative properties of the stochastic multi-symplectic scheme (2.26).

Theorem 3.1. *Under the periodic boundary conditions, the stochastic multi-symplectic scheme* (2.26) *satisfies the discrete charge conservation law, i.e.,*

$$\Delta x\sum_j|\varphi_{j+\frac{1}{2}}^{n+1}|^2=\Delta x\sum_j|\varphi_{j+\frac{1}{2}}^{n}|^2. \tag{3.1}$$

Proof. Multiply Eq. (2.26) by $\overline{\varphi_j^{n+\frac{1}{2}}}$, i.e., the conjugate of $\varphi_j^{n+\frac{1}{2}}$, sum over all spatial grid points j, and take the imaginary part. Then, under the periodic boundary conditions, the left-side becomes

$$\Delta x\sum_j \mathrm{i}(\delta_t^+\varphi_{j+\frac{1}{2}}^n+\delta_t^+\varphi_{j-\frac{1}{2}}^n)\overline{\varphi_j^{n+\frac{1}{2}}}=\frac{\mathrm{i}\Delta x}{\Delta t}\sum_j(\varphi_{j+\frac{1}{2}}^{n+1}\overline{\varphi_{j+\frac{1}{2}}^{n+1}}-\varphi_{j+\frac{1}{2}}^{n}\overline{\varphi_{j+\frac{1}{2}}^{n}}-\varphi_{j+\frac{1}{2}}^{n}\overline{\varphi_{j+\frac{1}{2}}^{n+1}}+\varphi_{j+\frac{1}{2}}^{n+1}\overline{\varphi_{j+\frac{1}{2}}^{n}}).$$

The first term of right-side is a real-valued function, so is the second term.

It follows from the last term

$$\begin{aligned}&\Delta x\sum_j\left[\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}})+\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\chi_j^{n+\frac{1}{2}}+\chi_{j-1}^{n+\frac{1}{2}})\right]\overline{\varphi_j^{n+\frac{1}{2}}}\\ =&\frac{\varepsilon\Delta x}{\sqrt{\Delta t\Delta x}}\sum_j|\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^2(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}}).\end{aligned}$$

The term on the right-side of the equality is real-valued, since $\chi_j^{n+\frac{1}{2}}$ is real-valued. Then, by taking the imaginary part, we obtain the discrete charge conservation law (3.1).

This completes the proof. □

The result of this theorem is evidently consistent with the charge conservation law (2.3), which means that the charge conservation law can be exactly preserved by the proposed stochastic multi-symplectic integrator. For deterministic multi-symplectic Hamiltonian PDEs, multi-symplectic methods have the stability in the sense of the charge conservation law (see [9]). We see that this is also the case for stochastic context.

The next two results concern the error estimation of the discrete global energy of the stochastic nonlinear Schrödinger equation. Due to the noise term in the equation, the energy of system is not constant any more. However, the transits of the discrete global energy can be derived by using the stochastic multi-symplectic integrators.

Theorem 3.2. *Under the periodic boundary conditions, the stochastic multi-symplectic scheme* (2.26) *satisfies the following recursion of discrete global energy conservation law, i.e.,*

$$\begin{aligned}\mathcal{H}^{n+1}-\mathcal{H}^{n}=&-2\Delta x\sum_j|\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}(|\varphi_{j+\frac{1}{2}}^{n+1}|^2-|\varphi_{j+\frac{1}{2}}^{n}|^2)+\frac{2}{\sigma+1}\Delta x\sum_j(|\varphi_{j+\frac{1}{2}}^{n+1}|^{2(\sigma+1)}-|\varphi_{j+\frac{1}{2}}^{n}|^{2(\sigma+1)})\\&-\Delta x\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\sum_j(|\varphi_{j+\frac{1}{2}}^{n+1}|^2-|\varphi_{j+\frac{1}{2}}^{n}|^2)(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}}).\end{aligned}\tag{3.2}$$

Here, the discrete global energy of the scheme (2.26) *at time t_n is defined as*

$$\mathcal{H}^n=-\Delta x\sum_j|\delta_x^+\varphi_j^n|^2+\frac{2}{\sigma+1}\Delta x\sum_j|\varphi_{j+\frac{1}{2}}^n|^{2(\sigma+1)}.$$

Proof. Multiplying Eq. (2.26) by $\Delta t\delta_t^+\overline{\varphi_j^n}$, summing up for j over the spatial domain, and taking the real part, we can obtain the results as follows.

The left-side reads

$$\Delta x\sum_j\mathrm{i}(\delta_t^+\varphi_{j+\frac{1}{2}}^n+\delta_t^+\varphi_{j-\frac{1}{2}}^n)(\Delta t\delta_t^+\overline{\varphi_j^n})=\frac{\mathrm{i}\Delta x}{2\Delta t}\sum_j(\varphi_{j+\frac{1}{2}}^{n+1}-\varphi_{j+\frac{1}{2}}^n)(\overline{\varphi_{j+\frac{1}{2}}^{n+1}}-\overline{\varphi_{j+\frac{1}{2}}^n}).$$

Thus the real part of the expression above is zero.

Similarly, for the first term of right-side of (2.26) it holds

$$\begin{aligned}&\Delta x\sum_j 2(\delta_x^+\delta_x^-\varphi_j^{n+\frac{1}{2}})(\Delta t\delta_t^+\overline{\varphi_j^n})\\=&\frac{1}{\Delta x}\sum_j(\varphi_{j+1}^{n+1}\overline{\varphi_{j+1}^{n+1}}-2\varphi_j^{n+1}\overline{\varphi_j^{n+1}}+\varphi_{j-1}^{n+1}\overline{\varphi_{j-1}^{n+1}})-(\varphi_{j+1}^{n}\overline{\varphi_{j+1}^{n}}-2\varphi_j^{n}\overline{\varphi_j^{n}}+\varphi_{j-1}^{n}\overline{\varphi_{j-1}^{n}})\\=&-\Delta x\sum_j(|\delta_x^+\varphi_j^{n+1}|^2-|\delta_x^+\varphi_j^n|^2).\end{aligned}$$

The last but one equality results from taking the real part, while the last one follows from the periodic boundary conditions.

For the second term,

$$\begin{aligned}&\Delta x\sum_j(2|\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}+2|\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}})(\Delta t\delta_t^+\overline{\varphi_j^n})\\=&2\Delta x\sum_j|\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}(\varphi_{j+\frac{1}{2}}^{n+1}+\varphi_{j+\frac{1}{2}}^n)(\overline{\varphi_{j+\frac{1}{2}}^{n+1}}-\overline{\varphi_{j+\frac{1}{2}}^n})=2\Delta x\sum_j|\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}|^{2\sigma}(|\varphi_{j+\frac{1}{2}}^{n+1}|^2-|\varphi_{j+\frac{1}{2}}^n|^2),\end{aligned}$$

as we take the real part in the last equality.

It follows from the last term that

$$\begin{aligned}
&\Delta x\sum_j\Big[\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}})+\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j-\frac{1}{2}}^{n+\frac{1}{2}}(\chi_j^{n+\frac{1}{2}}+\chi_{j-1}^{n+\frac{1}{2}})\Big](\Delta t\delta_t^+\overline{\varphi_j^n})\\
=&\Delta x\sum_j\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\varphi_{j+\frac{1}{2}}^{n+\frac{1}{2}}(\overline{\varphi_j^{n+1}}-\overline{\varphi_j^n}+\overline{\varphi_{j+1}^{n+1}}-\overline{\varphi_{j+1}^n})(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}})\\
=&\frac{\varepsilon}{2\sqrt{\Delta t\Delta x}}\Delta x\sum_j(|\varphi_{j+\frac{1}{2}}^{n+1}|^2-|\varphi_{j+\frac{1}{2}}^n|^2)(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}}).
\end{aligned}$$

The last equality is deduced for taking the real part.

Through the calculations above and the definition of discrete global energy, we then have the transformation of the global energy (3.2). ☐

For the integrable case $\sigma=1$, we have the following remark.

Remark 3.1. If $\sigma=1$, the discrete global energy conservation law of the stochastic multi-symplectic scheme (2.26) satisfies

$$\begin{aligned}
\mathcal{H}^{n+1}-\mathcal{H}^n=&\frac{\Delta x}{2}\sum_j(|\varphi_{j+\frac{1}{2}}^{n+1}|^2-|\varphi_{j+\frac{1}{2}}^n|^2)(|\varphi_{j+\frac{1}{2}}^{n+1}-\varphi_{j+\frac{1}{2}}^n|^2)\\
&-\frac{\varepsilon\Delta x}{2\sqrt{\Delta t\Delta x}}\sum_j(|\varphi_{j+\frac{1}{2}}^{n+1}|^2-|\varphi_{j+\frac{1}{2}}^n|^2)(\chi_j^{n+\frac{1}{2}}+\chi_{j+1}^{n+\frac{1}{2}}).
\end{aligned}$$

4 Numerical experiments

The main purpose of this section is to show the good numerical behavior of the stochastic multi-symplectic integrator. We take $\sigma=1$, which means that the deterministic equation is integrable as the subcritical physics case. It is possible to obtain qualitative information on the influence of the noise with small amplitude $\varepsilon\to 0$.

When we take no account of the noise term, we come back to the deterministic nonlinear Schrödinger equation. Choosing the initial value as

$$\varphi|_{t=0}=\frac{1}{\sqrt{2}}\sec\Big(\frac{1}{\sqrt{2}}(x-25)\Big)*\exp\Big(-\mathrm{i}\frac{x}{20}\Big), \tag{4.1}$$

then the exact single-soliton solution is

$$\varphi(x,t)=\frac{1}{\sqrt{2}}\sec\Big(\frac{1}{\sqrt{2}}\Big(x-\frac{t}{10}-25\Big)\Big)*\exp\Big(-\mathrm{i}\Big(\frac{x}{20}+\frac{199}{400}t\Big)\Big).$$

The solitary wave solutions play an important role for understanding these physical problems, which possess special form in space, propagating at a finite constant velocity, and keeping the same shape as time goes on. In deterministic dynamics, this solution is

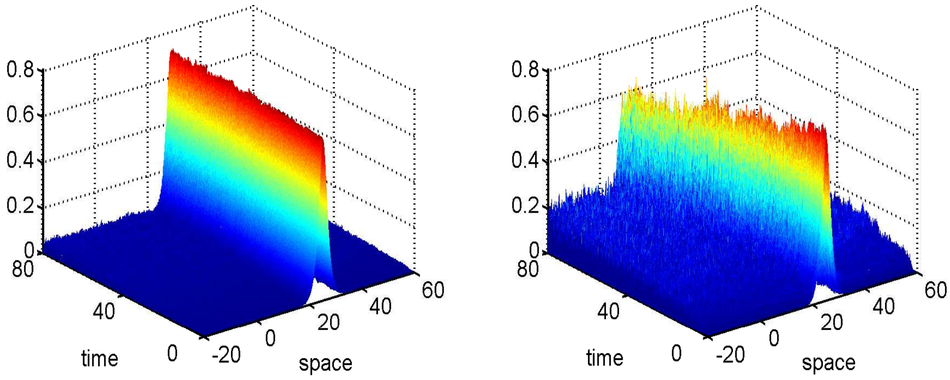

Figure 1: The profile of numerical solution $|\varphi(x,t)|$ for one trajectory as $\varepsilon=0.01$ (left), and $\varepsilon=0.05$ (right).

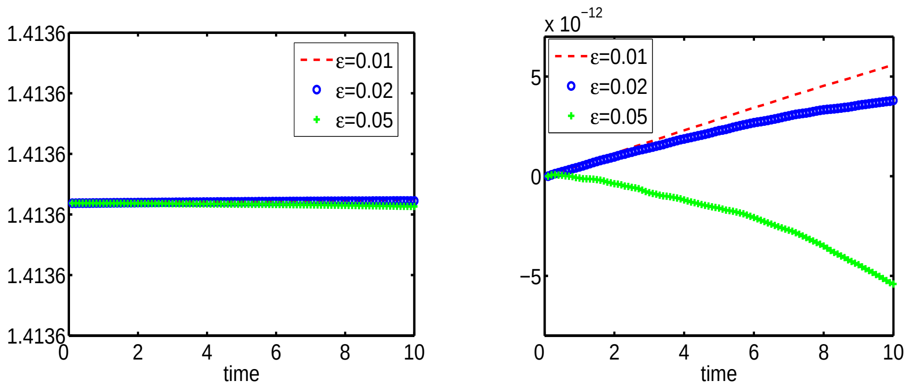

Figure 2: Evolution of the charge conservation law (left), and the global errors of charge conservation law (right), as $\varepsilon=0.01$, $\varepsilon=0.02$, $\varepsilon=0.05$.

stable, so we wish to investigate the situation of its stochastic counterpart. In our numerical calculations, the periodic boundary conditions are considered, i.e., $\varphi|_{x=x_L}=\varphi|_{x=x_R}$. Here, the numerical spatial domain $[x_L,x_R]$ is $[-20,60]$, and the initial value is given by (4.1).

In the following experiments, we take the temporal step-size $\triangle t=0.02$, the spatial meshgrid-size $\triangle x=0.1$, and the longest time interval $[0,80]$. We choose various sizes of noise, such as $\varepsilon=0.01$, $\varepsilon=0.02$, and $\varepsilon=0.05$. From Fig. 1, the profile of the amplitude $|\varphi(x,t)|$ for one trajectory, we see that the solitary wave is weakly perturbed by the noise. But the noise can neither prevent the propagation, nor destroy the solitary. However, as the size of the noise grows, the noise amplitude is higher and indeed influences the velocity of solitary wave.

Fig. 2 shows the evolution of the discrete charge conservation law. As was proved in Theorem 3.1, the multi-symplectic methods preserve the discrete charge conservation law exactly. Although different sizes of noise are chosen, the figures of the charge conservation law remain to be straight horizontal lines approximately, and the global residuals of the discrete charge conservation law, i.e., $(\mathcal{E}_{tq})^n := \mathcal{Q}^n - \mathcal{Q}^0$, all reach the magnitude of 10^{-12} for various ε. Here, $\mathcal{Q}^n$ denotes the discrete charge at time-step t_n.

As was stated in [2, 15], a Stratonovich equation is always equivalent to its Itô form in which the drift is a modified through the addition of a correction term. Based on this, it is not difficult to get the following conclusion about the energy conservation law, though it can not be preserved exactly any more in the presence of noise in the nonlinear Schrödinger equation (2.1).

Remark 4.1. [3] The average energy $\mathrm{E}(H(\varphi(t)))$ of the stochastic Schrödinger equation (2.1) satisfies the following equality:

$$\mathrm{E}(H(\varphi(t))) = \mathrm{E}(H(\varphi_0)) + \frac{\varepsilon^2}{2}\int_0^t \int_{\mathcal{R}} |\varphi(t,x)|^2 \sum_{l\in\mathcal{N}} \Big|\frac{\partial}{\partial x}\Phi e_l(x)\Big|^2 dx, \tag{4.2}$$

where Φ is a linear operator, which is taken to be the identity I_d in this paper.

It can be easily observed from (4.2) that, the average energy conservation law $\mathrm{E}(H(\varphi(t)))$ follows a linear evolution with growth rate $\frac{1}{2}\|\varphi_0\|^2_{L^2}\sum_{l\in\mathcal{N}}|\frac{\partial}{\partial x}\Phi e_l(x)|^2$, if the noise is homogeneous, i.e., $\sum_{l\in\mathcal{N}}|\frac{\partial}{\partial x}\Phi e_l(x)|^2$ does not depend on x. This phenomena is reflected in Fig. 3, where the evolution of the average discrete energy obeys nearly linear growth over 100 trajectories, and so is the discrete energy just over one trajectory with various ε.

In order to investigate the transformation of the discrete energy, we assume it to have the form

$$\mathrm{E}(\mathcal{H}^n) = \mathrm{E}(\mathcal{H}^0) + \varepsilon^2 A(t_n,\varphi)(t_n - t_0), \tag{4.3}$$

where $A(t_n,\varphi)$ represents the growing rate. We exhibit the discrete average energy over 100 trajectories with various ε in Fig. 4. As was mentioned, they all increase linearly. Use $\mathcal{E}^n_{te}(\varepsilon) := \mathrm{E}(\mathcal{H}^n) - \mathrm{E}(\mathcal{H}^0)$ to denote the global errors of the discrete average energy, then if t_n is fixed, this variable would be only related to ε^2. Thus, for different ε, we can define a ratio

$$Ratio(\varepsilon_1,\varepsilon_2) := \frac{\mathcal{E}^n_{te}(\varepsilon_1)}{\mathcal{E}^n_{te}(\varepsilon_2)} \cong \frac{{\varepsilon_1}^2}{{\varepsilon_2}^2}. \tag{4.4}$$

The reason for $\cong$ is that, the numerical solution φ would be a little different for noises of different sizes caused by various ε. Check the ratio in Fig. 4, we find that, the ratio for $\varepsilon = 0.02$ and $\varepsilon = 0.01$, i.e. $Ratio(\varepsilon_1 = 0.02, \varepsilon_2 = 0.01)$ is about 4, and that for $\varepsilon = 0.05$ and $\varepsilon = 0.01$, i.e. $Ratio(\varepsilon_1 = 0.05, \varepsilon_2 = 0.01)$ is about 25, which inversely verifies our assumption (4.3) that the discrete average energy increases linearly over long time.

Fig. 5 shows the evolution of the L^∞ norm of one trajectory, and the average L^∞ norm over 100 trajectories with various ε. It can be seen that, the L^∞ norm decreases evidently in all cases. This phenomena is due to the damping effect on the amplitude of the solitary wave caused by the multiplicative noise.

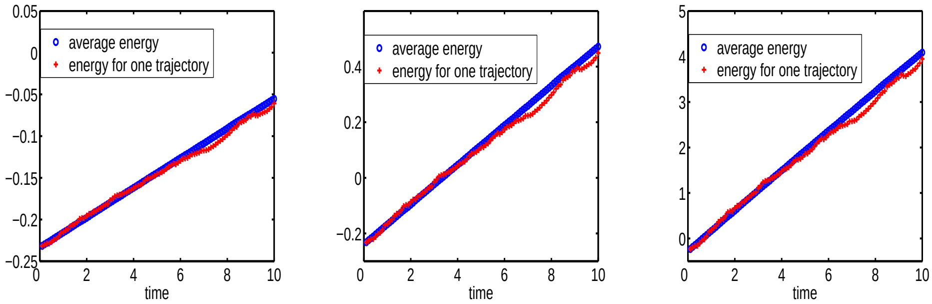

Figure 3: Evolution of average discrete energy over 100 trajectories and discrete energy over one trajectory as $\varepsilon=0.01$ (left), $\varepsilon=0.02$ (middle) and $\varepsilon=0.05$ (right).

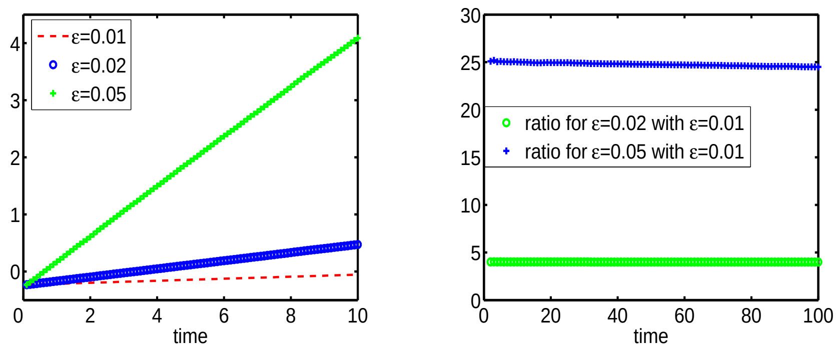

Figure 4: Evolution of global average energy over 100 trajectories as $\varepsilon=0.01$, $\varepsilon=0.02$, $\varepsilon=0.05$ (left), and the ratios of average energy transformation for $\varepsilon=0.02$, $\varepsilon=0.05$ with $\varepsilon=0.01$, respectively (right).

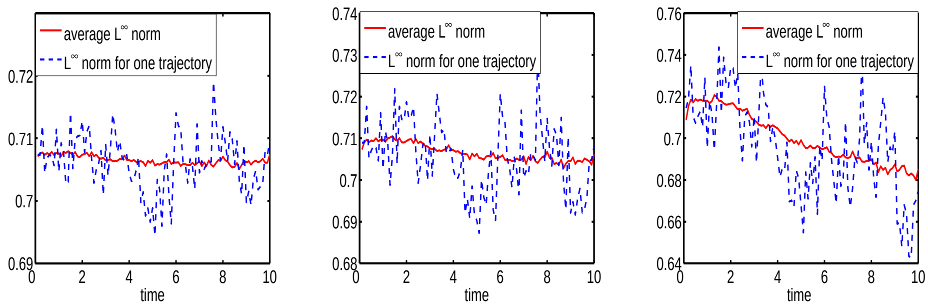

Figure 5: Evolution of L^∞ norm for one trajectory and average L^∞ norm for 100 trajectories as $\varepsilon=0.01$ (left), $\varepsilon=0.02$ (middle) and $\varepsilon=0.05$ (right).

Another interesting case is the collision of two solitons. For the deterministic Schrödinger equation, we consider interacting solitons with initial value

$$\varphi_{t=0}=\sec(x+20)*\exp(-i(2x-20))+\sec(x-20)*\exp(i(2x+20)). \tag{4.5}$$

In this case, the solution includes two solitary waves, which move in the opposite directions. According to the theoretical analysis, the two solitary waves would emerge from their interaction, with shapes and velocities unchanged. For the corresponding stochastic Schrödinger equation (2.1), we take the same initial value (4.5), and also choose the zero boundary condition. The time interval is $[0,10]$, and the space domain is $[-40,40]$, in order to make the boundary condition reasonable.

Fig. 6 shows the profile of the amplitude $|\varphi(x,t)|$ in the case of collision for one trajectory. Similar to the case of solitary wave, two solitary waves are weakly perturbed by the noise. As the size of the noise ε is larger, the noise amplitude becomes higher. Anyway, the numerical solution indicates that, the two solitons propagate in the opposite direction, emerge from interaction, and propagate in their original direction again after collision, which coincides with the theoretical analysis.

Fig. 7 illustrates the evolution of the discrete charge conservation law for soliton collision. It shows that the figures of the charge conservation law remain to be nearly straight horizontal lines for different sizes of noise, and the errors of the charge conservation law all reach the magnitude of 10^{-10} for various ε. All these indicate that, the stochastic multi-symplectic scheme preserves the discrete charge conservation law exactly, also for the soliton-collision.

The linear growth property related to the average energy conservation law for soliton-collision is illustrated in Fig. 8, from which it can be seen that the noise dose not destroy the solitons. And as analysed in soliton case, the growth of discrete average energy is proportion to ε^2.

Up to now, little progress has been made towards error analysis of the numerical methods for stochastic partial differential equations. A semi-discrete version of the scheme of strong order $\frac{1}{2}$ for the stochastic nonlinear Schrödinger equation has been studied in [4]. In [18], the Euler scheme is of weak order $\frac{1}{2}$ in the full discretization of a parabolic stochastic equation. For the multi-symplectic scheme (2.26), which is a space-time discretization of the stochastic nonlinear Schrödinger equation, we presume that it is of order $\frac{1}{2}$, according to its numerical behavior in the simulation of stochastic Hamiltonian ODEs. This certainly needs to be further investigated in the near future.

5 Conclusions

In this paper, we present the multi-symplecticity of the stochastic Hamiltonian PDEs via revealing their preservation of the stochastic multi-symplectic conservation law, which expands the scope of the multi-symplecticity of Hamiltonian PDEs from deterministic to stochastic context. The stochastic nonlinear Schrödinger equation is found to be a stochastic Hamiltonian PDE. For such stochastic Hamiltonian PDEs, the superiority of the newly derived stochastic multi-symplectic numerical methods, preserving the multi-symplecticity, lie not only in the capability of long-time scale computation, but also in the preservation of the charge conservation law and the ratio of the enhanced global energy. Numerical experiments are performed including both the soliton case and the case of

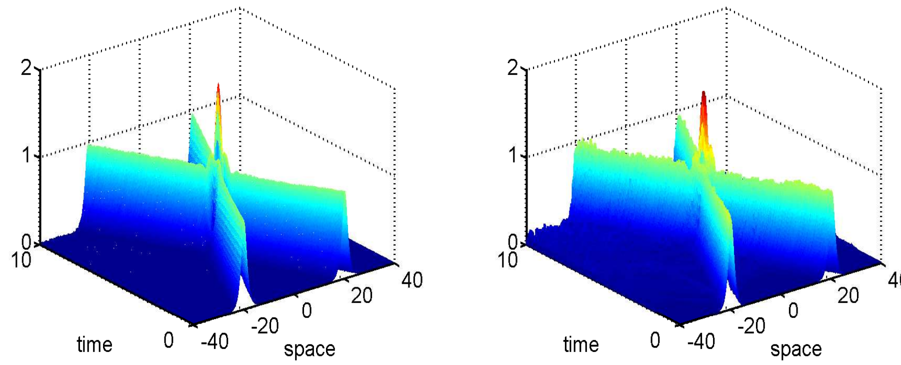

Figure 6: The profile of numerical solution $|\varphi(x,t)|$ for one trajectory as $\varepsilon=0.01$ (left), and $\varepsilon=0.05$ (right) in the case of collision.

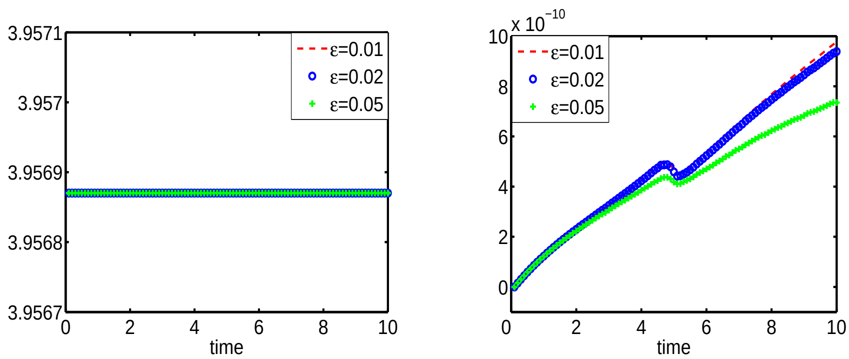

Figure 7: Evolution of the charge conservation law (left), and the global errors of discrete charge conservation law (right), as $\varepsilon=0.01$, $\varepsilon=0.02$, $\varepsilon=0.05$ in the case of collision.

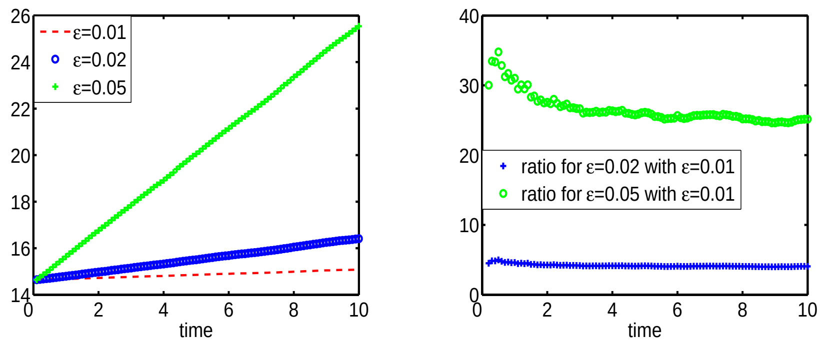

Figure 8: Evolution of global average energy over 100 trajectories with various ε and the ratios of average energy transformation, in the case of collision.

the collision of solitons. It is maybe worthwhile to note that the numerical analysis of stochastic Hamiltonian PDEs is a recently arising subject, and the present paper is just the beginning of the related exploration.

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References

[1] O. Bang, P.L. Christiansen, K.ϕ. Rasmussen, White noise in the two-dimensional nonlinear Schrödinger equation, Appl. Math., 57 (1995), 3-15.
[2] A.De Bouard, A. Debussche, L. Di Menza, Theoretical and numerical aspects of stochastic nonlinear Schrödinger equations, Monte-Carlo Meth. Appl., 7(1-2)(2001), 55-63.
[3] A. Debussche, L. Di Menza, Numerical simulation of focusing stochastic nonlinear Schrödinger equations, Phys. D,162(2002), 131-154.
[4] A. De Bouard, A. Debussche, Weak and strong order of convergence of a semi discrete scheme for the stochastic Nonlinear Schrodinger equation, Appl. Mathe. Optim., 54 (2006), 369-399.
[5] F.Kh. Abdullaev, J.Garuier, Soliton in media with random dispersive perturbations, Physica. D, 134 (1999), 303-315.
[6] T. Bridges, S. Reich, Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity, Phys. Lett. A, 284 (2001), 184-193.
[7] E. Hairer, C. Lubich, G. Wanner, Geometric Numerical Integration, Structure-Preserving Algorithms for Ordinary Differential Equations, Springer-Verlag, 2002.
[8] J. Hong, Y. Liu, H Munthe-Kaas and A Zanna, Globally conservative properties and error estimation of a multi-symplectic scheme for Schrödinger equations with variable coefficients, Appl. Numer. Math., 56 (2006), 814-843.
[9] J. Hong, X. Liu, C. Li, Multi-symplectic Runge-Kutta methods for nonlinear Schrödinger equations with variable coefficients, J. Comput. Phys., 226 (2007), 1968-1984.
[10] J. Hong, R. Scherer, L. Wang, Midpoint Rule for a Linear Stochastic Oscillator with Additive Noise, Neural Parallel and Scientific Computing, 14 (2006), 1-12.
[11] A. Iserles, A First Course in the Numerical Analysis of Differential Equations, Cambridge University Press, Cambridge, 1996.
[12] A. Islas, D. Karpeev, C. Schober, Geometric integrators for the nonlinear Schrödinger equation, J. Comput. Phys., 173 (2001), 116-148.
[13] V.Konotop, L. Vazquez, Nonlinear Random Waves, World Scientific, River Edge. NJ, 1994.
[14] J. Marsden, G. Patrick, S. Shkoller, Multi-symplectic geometry, variational integrators, and nonlinear PDEs, Comm. Math. Phys. 199(1998), 351-395.
[15] G. Milstein, M. Tretyakov, Stochastic Numerics for Mathematical Physics, Kluwer Academic Publishers, 1995.
[16] k.ϕ. Rasmussen, Y.B. Gaididei, O.Bang, P.L. Christiansen, The influence of noise on critical collapse in the nonlinear Schrödinger equation, Phys. Rev. A, 204 (1995), 121-127.
[17] C. Schober, Symplectic integrators for the Ablowitz-Ladik discrete nonlinear Schrödinger equation, Phys. Lett. A, 259 (1999), 140-151.
[18] T. Shardlow, Weak convergence of a numerical method for a stochastic heat equation, BIT, 43 (2003), 179-193.