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Reduction of the Regularization Error of the Method of Regularized Stokeslets for a Rigid Object Immersed in a Three-Dimensional Stokes Flow

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Abstract. We focus on the problem of evaluating the velocity field outside a solid object moving in an incompressible Stokes flow using the boundary integral formulation. For points near the boundary, the integral is nearly singular, and accurate computation of the velocity is not routine. One way to overcome this problem is to regularize the integral kernel. The method of regularized Stokeslet (MRS) is a systematic way to regularize the kernel in this situation. For a specific blob function which is widely used, the error of the MRS is only of first order with respect to the blob parameter. We prove that this is the case for radial blob functions with decay property $\phi(r)=\mathcal{O}(r^{-3-\alpha})$ when $r\to\infty$ for some constant $\alpha>1$. We then find a class of blob functions for which the leading local error term can be removed to get second and third order errors with respect to blob parameter. Since the addition of these terms might give a flow field that is not divergence free, we introduce a modification of these terms to make the divergence of the corrected flow field close to zero while keeping the desired accuracy. Furthermore, these dominant terms are explicitly expressed in terms of blob function and so the computation time is negligible.

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Key words: Stokes flow, regularized Stokeslet, boundary integral equation, nearly singular integral.

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1 Introduction

Incompressible Newtonian Stokesian fluid flows, governed by the Stokes equations

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$$0 = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{f},$$
$$0 = \nabla \cdot \mathbf{u}$$

have been widely used to study biological problems, for example, micro-organism locomotions [11], bioconvection [12], and effective viscosity of suspensions [7,8]. Here, $\mathbf{u}$ is the fluid velocity, p the pressure, $\mathbf{f}$ the external force or the body force, and μ the viscosity of the fluid. The equations are an approximation of the Navier-Stokes equations in the low Reynolds number regime.

When an object moves in and interacts with Stokes flow, the velocity of the exterior flow can be represented in terms of boundary integrals [13]. The boundary integral formulation has the advantage of reducing the full three-dimensional problem of solving for fluid flow to a two-dimensional problem of evaluating surface integrals. In the case of a rigid body immersed in a Stokes flow, we can represent the velocity as a first-kind integral equation with density equal to the traction on the surface and kernel the Stokeslet, the fundamental solution of the Stokes equations in free-space. The velocity can then be computed by numerically evaluating the surface integral. For evaluation points away from the surface, the integrand is smooth and slowly varying and we can use standard quadratures with high accuracy. But for points close to the surface, the integrand becomes nearly singular and accurate computation of the velocity is not routine. There are different approaches to this problem. One commonly used approach is to regularize the kernel.

The method of regularized Stokeslet (MRS), originally introduced by Cortez [5], is a systematic approach to regularize the kernel. The formulation is based on a free-space solution of the Stokes equation with concentrated but smooth forcing of the form

$$\mathbf{f}(\mathbf{x}) = \mathbf{g}\phi^{\epsilon}(\mathbf{x}),$$

where $\mathbf{f}$ is the force, $\mathbf{g}$ is a constant vector, $\phi^{\epsilon}(\mathbf{x})$ is a radially symmetric smooth approximation to the Dirac delta function, and ϵ is the blob parameter that controls the concentration of the force. In our work, we only consider blobs of the form

$$\phi^{\epsilon}(\mathbf{x}) = \frac{1}{\epsilon^3}\phi\left(\frac{\mathbf{x}}{\epsilon}\right), \tag{1.1}$$

where $\phi(\mathbf{x})$ is any radially symmetric function having integral 1 over $\mathbb{R}^3$. The velocity computed by the MRS is automatically divergence free. A boundary integral equation of the first kind based on a regularized Stokeslet together with the trapezoidal quadrature discretization was studied in [6]. In that work, the authors proved and showed by numerical examples that for the blob of the form (1.1) where

$$\phi(\mathbf{x}) = \frac{15}{8\pi(|\mathbf{x}|^2+1)^{7/2}}, \tag{1.2}$$

the error associated with the regularization (referred to as regularization error in this paper) was bounded by $\mathcal{O}(\epsilon^2)$ for evaluation points away from the surface but bounded by only $\mathcal{O}(\epsilon)$ for points close by. This means that in order to improve the accuracy of the numerically evaluated velocity, we need small values of ϵ. As a result, we need a finer grid on the surface and that makes it expensive to compute the velocity. This raises a question of how to improve the regularization error, either by using appropriate blobs or by other methods while ensuring that the computed velocity field is divergence free. The purpose of this paper is to answer this question to some extent.

The structure of our paper will be as follows. In Section 2, we discuss the regularization error of the MRS for a general blob. As detailed in Theorem 2.1, we can construct blobs so that the far field error, the regularization error for points away from the surface, is equal to $\mathcal{O}(\epsilon^\alpha)$ with α as large as we want. We can even make the far field error exactly zero by using appropriate compact support blobs. In contrast, we should only expect $\mathcal{O}(\epsilon)$ near field error because Theorem 2.2 says that, for almost all radially symmetric blobs, the sup-norm of the regularization error cannot be better than $\mathcal{O}(\epsilon)$. However, the leading order term of the regularization error can be isolated and removed to improve the convergence rate of the near field. In Section 3, inspired by the work of Beale [2, 3], we use local analysis to find the leading order terms (called corrections in this paper) of the regularization error and add them back to the computed velocity in order to get $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon^3)$ error. The second order correction is given in Theorem 3.1 and the third order correction is given in Theorem 3.2. The addition of the corrections might result in non-divergence-free flow fields. In Subsection 3.2, we propose a simple analytical modification to the corrections in order to make the divergence of the corrected flow close to zero while maintaining the desire order of accuracy. Our approach to reduce the divergence seems to work well only for the second order correction as shown by numerical results. In Section 4, we present numerical results to support our theoretical prediction in the previous Sections. We end this paper with some conclusions and future work in Section 5.

2 The regularization error

Let us consider the problem of evaluating the velocity field of a Stokesian fluid flow surrounding a moving rigid object. Let $\phi(\mathbf{x})$ be a radially symmetric smooth function having integral 1 over $\mathbb{R}^3$ and define $\phi^\epsilon(\mathbf{x}) = (1/\epsilon^3)\phi(\mathbf{x}/\epsilon)$. In 2005, Cortez et al. [6] showed that one can approximate the velocity at any field point $\mathbf{x}$ by a boundary integral

$$u_j^\epsilon(\mathbf{x}) = -\frac{1}{8\pi\mu}\int_{\partial D} S_{ij}^\epsilon(\mathbf{x},\mathbf{y}) f_i ds(\mathbf{y}), \tag{2.1}$$

where ∂D is the surface of the solid object, $\mathbf{f}$ is the traction on the surface, and $S_{ij}^\epsilon(\mathbf{x},\mathbf{y})$ is the regularized Stokeslet corresponding to $\phi^\epsilon(\mathbf{x}-\mathbf{y})$

$$S_{ij}^\epsilon(\mathbf{x},\mathbf{y}) = \frac{\delta_{ij}}{r}H_1^\epsilon(r) + \frac{(x_i-y_i)(x_j-y_j)}{r^3}H_2^\epsilon(r), \tag{2.2}$$

with $r=|\mathbf{x}-\mathbf{y}|$ and

$$H_1^\epsilon(r)=-8\pi\left[r(B^\epsilon(r))''+(B^\epsilon(r))'\right], \tag{2.3}$$

$$H_2^\epsilon(r)=8\pi\left[r(B^\epsilon(r))''-(B^\epsilon(r))'\right]. \tag{2.4}$$

Here, $B^\epsilon(\mathbf{x})$ is a radially symmetric function satisfying

$$\Delta G^\epsilon=\phi^\epsilon, \tag{2.5}$$

$$\Delta B^\epsilon=G^\epsilon. \tag{2.6}$$

In comparison, the exact velocity can be written in the same form as (2.1) but with the Stokeslet in place of the regularized Stokeslet [13]

$$u_j(\mathbf{x})=-\frac{1}{8\pi\mu}\int_{\partial D}S_{ij}(\mathbf{x},\mathbf{y})f_i ds(\mathbf{y}),$$

where $S_{ij}(\mathbf{x},\mathbf{y})$ is the Stokeslet

$$S_{ij}(\mathbf{x},\mathbf{y})=\frac{\delta_{ij}}{r}+\frac{(x_i-y_i)(x_j-y_j)}{r^3}.$$

The regularization error, defined as

$$E^\epsilon(\mathbf{x})=\mathbf{u}(\mathbf{x})-\mathbf{u}^\epsilon(\mathbf{x}) \tag{2.7}$$

can then be written as

$$E^\epsilon(\mathbf{x})=e_1+e_2, \tag{2.8}$$

with

$$e_1=-\frac{1}{8\pi\mu}\int_{\partial D}\frac{(1-H_1^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|}\mathbf{f}(\mathbf{y})ds(\mathbf{y}), \tag{2.9}$$

$$e_2=-\frac{1}{8\pi\mu}\int_{\partial D}\frac{(1-H_2^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|^3}(\mathbf{f}(\mathbf{y})\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}ds(\mathbf{y}), \tag{2.10}$$

where $\hat{\mathbf{x}}=\mathbf{x}-\mathbf{y}$.

2.1 Far field error

From (2.9) and (2.10), it is clear that, in the far field, the regularization error will depend on how fast $H_1^\epsilon(r)$ and $H_2^\epsilon(r)$ converge to 1 as r/ϵ goes to ∞. As detailed in the following Lemma, $H_i^\epsilon(r)$, $i=1,2$, depend only on the blob and the decay of $(1-H_i^\epsilon(r))$ in the far field can be controlled.

Lemma 2.1. *Suppose that ∂D is C^2, $\mathbf{f}$ is C^1, $\phi^\epsilon(\mathbf{x})=(1/\epsilon^3)\phi(|\mathbf{x}|/\epsilon)$, where $\phi(\mathbf{x})=\phi(|\mathbf{x}|)$ is a piecewise continuous function over $\mathbb{R}^3$ satisfying the following conditions:*

1. *the integral of* $\phi(\mathbf{x})$ *over* $\mathbb{R}^3$ *is 1, and*

2. $|\phi(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-3-\alpha})$ *when* $|\mathbf{x}| \to \infty$ *for some constant* $\alpha > 1$.

For $r > K\epsilon$ *where* K *is some constant depending on the decay rate of* $\phi(\mathbf{x})$*, we have*

$$|1-H_1^\epsilon(r)| \le c_1 \left(\frac{\epsilon}{r}\right)^{1+\beta}, \tag{2.11}$$

$$|1-H_2^\epsilon(r)| \le c_2 \left(\frac{\epsilon}{r}\right)^{1+\beta}, \tag{2.12}$$

where β *is any constant satisfying* $0<\beta\le 1$ *and* $\beta<\alpha-1$*, and* c_1 *and* c_2 *are constants depending on* $\phi(\mathbf{x})$ *and* β*. If, in addition, the function* $\phi(\mathbf{x})$ *has zero second moment, i.e.,*

$$\int_0^\infty s^4\phi(s)ds = 0,$$

we can choose $\beta=\alpha-1$*. Furthermore, if* $\phi(\mathbf{x})$ *also has compact support, we can choose constants* $c_1=c_2=0$ *outside the support of* $\phi(\mathbf{x})$*.*

The proof of Lemma 2.1 is given in Appendix A. The lemma shows that if the second moment of $\phi(\mathbf{x})$ is nonzero, the exponent in (2.11) and (2.12) is at most 2, regardless of how fast $\phi(\mathbf{x})$ decays. As a direct consequence of Lemma 2.1, the far field error can be estimated as in the following theorem.

Theorem 2.1. *Suppose that* ∂D *is* C^2, $\mathbf{f}$ *is* C^1, $\phi^\epsilon(\mathbf{x}) = (1/\epsilon^3)\phi(|\mathbf{x}|/\epsilon)$*, where* $\phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ *is a piecewise continuous function over* $\mathbb{R}^3$ *satisfying the following conditions:*

1. *the integral of* $\phi(\mathbf{x})$ *over* $\mathbb{R}^3$ *is 1, and*

2. $|\phi(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-3-\alpha})$ *when* $|\mathbf{x}| \to \infty$ *for some constant* $\alpha > 1$.

Let β *be such that* $0<\beta\le 1$ *and* $\beta<\alpha-1$*. Then, the regularization error at a point* $\mathbf{x}$ *away from the surface is equal to* $\mathcal{O}(\epsilon^{1+\beta})$*. If, in addition, the function* $\phi(\mathbf{x})$ *has second moment 0, then the regularization error at that point is* $\mathcal{O}(\epsilon^\alpha)$*. Furthermore, if* $\phi(\mathbf{x})$ *also has compact support, the far field error is exactly zero.*

Remark 2.1. Theorem 2.1 holds for any radially symmetric blob. We can construct blobs with $\alpha > 2$ such that the second moment is zero. According to Theorem 2.1, the far field error is $\mathcal{O}(\epsilon^\alpha)$ where α as large as we want. We can even construct compact support blobs so that the far field error is exactly zero.

2.2 Near field error

It is known that an error bound of the regularization error corresponding to a specific radially symmetric blob is $C\epsilon$ for some constant $C>0$ [6]. In this section, we will prove that this is the case for almost all radial blob functions and the bound is sharp, i.e., there exist two constants $C_2 > C_1 > 0$ such that the sup-norm of regularization error lies in the interval $[C_1\epsilon, C_2\epsilon]$. The idea is to use local analysis as in [3] to find the dominant term of the regularization error $E^\epsilon(\mathbf{x})$ and then show that this term is of and no better than $\mathcal{O}(\epsilon)$ for almost all radial blob functions.

Theorem 2.2. *Suppose that ∂D is C^2, $\mathbf{f}$ is C^1, $\phi^\epsilon(\mathbf{x}) = (1/\epsilon^3)\phi(|\mathbf{x}|/\epsilon)$, where $\phi(\mathbf{x}) = \phi(|\mathbf{x}|)$ is a piecewise continuous function over $\mathbb{R}^3$ satisfying the following conditions:*

1. *the integral of $\phi(\mathbf{x})$ over $\mathbb{R}^3$ is 1,*
2. *$|\phi(\mathbf{x})| = \mathcal{O}(|\mathbf{x}|^{-3-\alpha})$ when $|\mathbf{x}| \to \infty$ for some constant $\alpha > 1$.*

Then, the regularization error $E^\epsilon(\mathbf{x})$ defined in (2.7) is as good as $\mathcal{O}(\epsilon)$ but cannot be better, i.e., there exist constants $C_2 > C_1 > 0$ such that

$$C_1\epsilon < \sup_{\mathbf{x}\in\mathbb{R}^3\setminus D} |E^\epsilon(\mathbf{x})| < C_2\epsilon.$$

In order to prove Theorem 2.2, we will need Lemma 2.2 which gives us the exact form of the dominant term of the regularization error $E^\epsilon(\mathbf{x})$ when the evaluation point is close to the surface.

Lemma 2.2. *With $\phi^\epsilon(r)$ as in the Theorem 2.2, choose $0 < \beta \le 1$ such that $\beta < \alpha - 1$. Fix a point $\mathbf{x}$ in the flow field. Let $\mathbf{x}_0$ be a point on the surface that is closest to $\mathbf{x}$ and define $b = |\mathbf{x} - \mathbf{x}_0|$. We assume that $\mathbf{x}$ is close enough to the surface so that $\mathbf{x}_0$ is uniquely determined. Let $\mathbf{N}$ be the unit outer normal vector to the surface at $\mathbf{x}_0$. Let $\mathbf{f}_0$ be the force applied on the surface at $\mathbf{x}_0$. Then the regularization error defined in (2.7) can be written as*

$$E^\epsilon(\mathbf{x}) = -\frac{\pi}{\mu}(\mathbf{f}_0 - (\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_b^\infty s(s-b)^2\phi^\epsilon(s)\,ds + \mathcal{O}(\epsilon^{1+\beta/2}). \tag{2.13}$$

The proof of Lemma 2.2 will be given in Appendix B.

Proof of Theorem 2.2. Fix a point $\mathbf{x}$ close to the surface, let b be the distance from $\mathbf{x}$ to the surface, then, by Lemma 2.2, we have

$$\begin{aligned}
E^\epsilon(\mathbf{x}) &= -\frac{\pi}{\mu}(\mathbf{f}_0 - (\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_b^\infty r(r-b)^2\phi^\epsilon(r)\,dr + \mathcal{O}(\epsilon^{1+\beta/2}) \\
&= \epsilon\left(-\frac{\pi}{\mu}(\mathbf{f}_0 - (\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_\zeta^\infty r(r-\zeta)^2\phi(r)\,dr\right) + \mathcal{O}(\epsilon^{1+\beta/2}) \\
&= \mathcal{O}(\epsilon),
\end{aligned}$$

where $\zeta = b/\epsilon$, and $\beta < \alpha - 1$, $0 < \beta \le 1$.

Furthermore, we have

$$\int_\zeta^\infty r(r-\zeta)^2\phi(r)\,dr = \int_0^\infty r^3\phi(r)\,dr - \frac{\zeta}{2\pi} + \zeta^2\int_0^\infty r\phi(r)\,dr + \int_0^\zeta r(r-\zeta)^2\phi(r)\,dr.$$

Hence, for $\zeta \ll 1$ (equivalently, $b \ll \epsilon$), we have

$$\int_\zeta^\infty r(r-\zeta)^2\phi(r)\,dr = \int_0^\infty r^3\phi(r)\,dr - \frac{\zeta}{2\pi} + \mathcal{O}(\zeta^2).$$

This means that the integral $\int_\zeta^\infty r(r-\zeta)^2\phi(r)\,dr$ is not identically 0 for $\zeta \geq 0$. Therefore, there exist a constant $l>0$ independent of ϵ and some ζ small such that

$$\left|\int_\zeta^\infty r(r-\zeta)^2\phi(r)\,dr\right| > l$$

and

$$\left\|\frac{\pi}{\mu}(\mathbf{f}_0-(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_b^\infty s(s-b)^2\phi^\epsilon(s)\,ds\right\|_2 > \|(\mathbf{f}_0-(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\|_2\frac{\pi\epsilon l}{\mu}.$$

Thus, for ϵ small enough, we have

$$\sup_{\mathbb{R}^3\setminus D}|E^\epsilon(\mathbf{x})| > \|(\mathbf{f}_0-(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\|_2\frac{\pi\epsilon l}{\mu} = C_1\epsilon.$$

In general, the surface force will contain a tangential component so that

$$(\mathbf{f}_0-(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}) \neq \mathbf{0}.$$

Therefore, $C_1>0$. This completes the proof of Theorem 2.2. $\square$

3 Corrections and their divergence

In this section, we will give the explicit forms of what we called the second order correction $E_2^\epsilon(\mathbf{x})$ and the third order correction $E_3^\epsilon(\mathbf{x})$ for an arbitrary blob. As the name suggests, we can write the regularization error (2.7) as

$$E^\epsilon(\mathbf{x}) = E_2^\epsilon(\mathbf{x}) + \mathcal{O}(\epsilon^2), \tag{3.1}$$

$$E^\epsilon(\mathbf{x}) = E_3^\epsilon(\mathbf{x}) + \mathcal{O}(\epsilon^3). \tag{3.2}$$

We then give the explicit form of the divergence of the second order correction. The divergence of the third order correction could be found using the same approach. Since the divergence of the corrections are not zero, we add some high-order terms to make them close to zero. These terms can be found by solving a simple first order ordinary differential equation shown in Subsection 3.2. The second order correction $E_2^\epsilon(\mathbf{x})$ has a much simpler formula than the third order correction, and so does its divergence.

3.1 The corrections

In this Subsection, the explicit forms of the second order and the third order corrections will be given. But only the derivation of the third order correction is presented since the second order correction can be obtained by removing $\mathcal{O}(\epsilon^2)$ terms from the third order correction.

Theorem 3.1 (Second Order Correction). *Suppose that the surface of the solid object is smooth, the force is smooth along the surface. Suppose further that* $\phi^\epsilon(\mathbf{x})=(1/\epsilon^3)\phi(|\mathbf{x}|/\epsilon)$, *where* $\phi(\mathbf{x})=\phi(|\mathbf{x}|)$ *is a smooth function over* $\mathbb{R}^3$ *satisfying the following conditions:*

1. *the integral of* ϕ *over* $\mathbb{R}^3$ *is 1,*
2. $|\phi(r)|\leq Cr^{-7}$ *for* $r\geq 1$ *and some constant* $C>0$,
3. *the second moment of* $\phi(\mathbf{x})$ *is 0.*

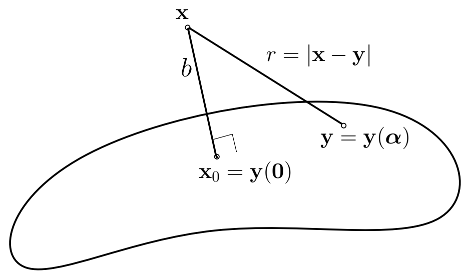


Figure 1: The surface has a parametrization $\mathbf{y}=\mathbf{y}(\boldsymbol{\alpha})$ near $\mathbf{x}_0$.

Fix a point $\mathbf{x}$ *in the flow field. Let* $\mathbf{x}_0$ *be a point on the surface that is closest to* $\mathbf{x}$ *and define* $b=|\mathbf{x}-\mathbf{x}_0|$. *Let* $\mathbf{y}=\mathbf{y}(\boldsymbol{\alpha})$ *with* $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)$ *be a parametrization of the surface near* $\mathbf{x}_0=\mathbf{y}(\mathbf{0})$ *such that* $\{\mathbf{y}_{\alpha_1}(\mathbf{0}),\mathbf{y}_{\alpha_2}(\mathbf{0})\}$ *is an orthonormal set, and define*

$$\mathbf{N}=\frac{\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}}{|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|}$$

at $\boldsymbol{\alpha}=\mathbf{0}$. *Then,*

$$E^\epsilon(\mathbf{x})=E_2^\epsilon(\mathbf{x})+\mathcal{O}(\epsilon^2),$$

where

$$E_2^\epsilon(\mathbf{x})=-\frac{\pi}{\mu}(\mathbf{f}-(\mathbf{f}\cdot\mathbf{N})\mathbf{N})\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds. \tag{3.3}$$

Theorem 3.2 (Third Order Correction). *With the hypotheses and notations as in Theorem* 3.1, *we can write the regularization error* $E^\epsilon(\mathbf{x})=e_1+e_2$ *with*

$$e_1=-\frac{\pi}{2\mu}(2+b\mathbf{N}\cdot\mathbf{y}_{\alpha_i\alpha_i})\mathbf{f}\mathcal{I}_1+\mathcal{O}(\epsilon^3),$$

$$e_2=\frac{\pi}{2\mu}(2+b\mathbf{N}\cdot\mathbf{y}_{\alpha_i\alpha_i})(\mathbf{f}\cdot\mathbf{N})\mathbf{N}\mathcal{I}_1+\frac{\pi}{\mu}\left(-\frac{1}{2}(\mathbf{f}\cdot\mathbf{N})\mathbf{y}_{\alpha_i\alpha_i}+c_N\mathbf{N}+c_{\alpha_i}\mathbf{y}_{\alpha_i}\right)\mathcal{I}_2+\mathcal{O}(\epsilon^3),$$

where

$$\mathcal{I}_1 = \int_b^\infty \left(-bs^2 + \frac{2sb^2}{3} + \frac{s^4}{3b} \right) \phi^\epsilon(s)\, ds,$$
$$\mathcal{I}_2 = \int_b^\infty \left(-\frac{s^4}{3} + s^3 b - s^2 b^2 + \frac{b^3 s}{3} \right) \phi^\epsilon(s)\, ds,$$

and

$$\begin{aligned}
c_N &= -(\mathbf{f}_{\alpha_i} \cdot \mathbf{y}_{\alpha_i}) - \frac{1}{2}(\mathbf{f} \cdot \mathbf{y}_{\alpha_i \alpha_i}) + \frac{1}{2}(\mathbf{f} \cdot \mathbf{y}_{\alpha_i})(\mathbf{y}_{\alpha_i} \cdot \mathbf{y}_{\alpha_j \alpha_j}), \\
c_{\alpha_1} &= \frac{1}{b}(\mathbf{f} \cdot \mathbf{y}_{\alpha_1}) - (\mathbf{f}_{\alpha_1} \cdot \mathbf{N}) + (\mathbf{f} \cdot \mathbf{y}_{\alpha_2})(\mathbf{N} \cdot \mathbf{y}_{\alpha_1 \alpha_2}) \\
&\quad + \frac{1}{2}(\mathbf{f} \cdot \mathbf{N}) \mathbf{y}_{\alpha_1} \cdot \mathbf{y}_{\alpha_i \alpha_i} + \frac{1}{2}(\mathbf{f} \cdot \mathbf{y}_{\alpha_1})(3\mathbf{N} \cdot \mathbf{y}_{\alpha_1 \alpha_1} + \mathbf{N} \cdot \mathbf{y}_{\alpha_2 \alpha_2}), \\
c_{\alpha_2} &= \frac{1}{b}(\mathbf{f} \cdot \mathbf{y}_{\alpha_2}) - (\mathbf{f}_{\alpha_2} \cdot \mathbf{N}) + (\mathbf{f} \cdot \mathbf{y}_{\alpha_1})(\mathbf{N} \cdot \mathbf{y}_{\alpha_1 \alpha_2}) \\
&\quad + \frac{1}{2}(\mathbf{f} \cdot \mathbf{N}) \mathbf{y}_{\alpha_2} \cdot \mathbf{y}_{\alpha_i \alpha_i} + \frac{1}{2}(\mathbf{f} \cdot \mathbf{y}_{\alpha_2})(3\mathbf{N} \cdot \mathbf{y}_{\alpha_2 \alpha_2} + \mathbf{N} \cdot \mathbf{y}_{\alpha_1 \alpha_1}).
\end{aligned}$$

All the derivatives of $\mathbf{y}$ *and* $\mathbf{f}$ *are evaluated at* $\boldsymbol{\alpha} = \mathbf{0}$.

Remark 3.1. When the field point is right on the surface $\mathbf{x} \equiv \mathbf{x}_0$ and $b = 0$, we rewrite the regularization error (2.7) as

$$E^\epsilon(\mathbf{x}) = -\frac{\pi}{\mu} \int_0^\infty s^3 \phi^\epsilon(s)\, ds + \mathcal{O}(\epsilon^3) = -\frac{\pi}{\mu} \epsilon \int_0^\infty s^3 \phi(s)\, ds + \mathcal{O}(\epsilon^3).$$

Thus, if in addition to satisfying conditions in Theorem 3.2, $\phi(r)$ satisfies

$$\int_0^\infty s^3 \phi(s)\, ds = 0,$$

we have

$$E^\epsilon(\mathbf{x}) = \mathcal{O}(\epsilon^3) \quad \text{for } \mathbf{x} \text{ on the surface.}$$

This is useful when we know the velocity distribution on the surface and want to solve for the traction on the surface.

Proof. Fix a small constant $\lambda > \epsilon$. By Lemma 2.1, since $\phi(\mathbf{x})$ has zero second moment, for $r \geq \lambda$, we have

$$\frac{1}{r} |1 - H_1^\epsilon(r)| = \mathcal{O}(\epsilon^4), \tag{3.4}$$
$$\frac{1}{r^3} |1 - H_2^\epsilon(r)| = \mathcal{O}(\epsilon^4). \tag{3.5}$$

Thus, we can rewrite e_1 and e_2 in (2.8) as

$$e_1 = -\frac{1}{8\pi\mu}\int_{\partial D,\ |\hat{\mathbf{x}}|<\lambda} \frac{(1-H_1^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|}\mathbf{f}(\mathbf{y})ds(\mathbf{y}) + \mathcal{O}(\epsilon^4), \tag{3.6}$$

$$e_2 = -\frac{1}{8\pi\mu}\int_{\partial D,\ |\hat{\mathbf{x}}|<\lambda} \frac{(1-H_2^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|^3}(\mathbf{f}(\mathbf{y})\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}ds(\mathbf{y}) + \mathcal{O}(\epsilon^4). \tag{3.7}$$

With the hypothesis as in Theorem 3.2, let $\boldsymbol{\alpha}$-derivatives of $\mathbf{y}$ at $\boldsymbol{\alpha}=\mathbf{0}$ be denoted by $\mathbf{T}_i = \mathbf{y}_{\alpha_i}(\mathbf{0})$ and $\mathbf{T}_{ij} = \mathbf{y}_{\alpha_i\alpha_j}(\mathbf{0})$, then $\{\mathbf{T}_1,\mathbf{T}_2\}$ is an orthonormal set. Expand $\mathbf{y}=\mathbf{y}(\boldsymbol{\alpha})$ near the origin $\mathbf{0}$, we have

$$\mathbf{y}(\boldsymbol{\alpha}) = \mathbf{y}(\mathbf{0}) + \alpha_i\mathbf{T}_i + \frac{1}{2}\alpha_i\alpha_j\mathbf{T}_{ij} + \mathcal{O}(|\boldsymbol{\alpha}|^3). \tag{3.8}$$

Let $\mathbf{x}$ be a point off the surface, and $\mathbf{x}_0 = \mathbf{y}(\mathbf{0})$ a point on the surface that is closest to $\mathbf{x}$, we have

$$\hat{\mathbf{x}} = \mathbf{x} - \mathbf{y} = b\mathbf{N} - \alpha_i\mathbf{T}_i - \frac{1}{2}\alpha_i\alpha_j\mathbf{T}_{ij} + \mathcal{O}(|\boldsymbol{\alpha}|^3) \tag{3.9}$$

and

$$r^2 = b^2 + (1-\kappa b)|\boldsymbol{\alpha}|^2 + \alpha_i\alpha_j\alpha_k\mathbf{T}_i\cdot\mathbf{T}_{jk} + \mathcal{O}(b|\boldsymbol{\alpha}|^3 + |\boldsymbol{\alpha}|^4),$$

where κ is the normal curvature of the surface at $\mathbf{x}_0 = \mathbf{y}(\mathbf{0})$ in the $\boldsymbol{\alpha}$-direction.

$$\kappa = \frac{\alpha_i\alpha_j\mathbf{N}\cdot\mathbf{T}_{ij}}{\alpha_i\alpha_j\mathbf{T}_i\cdot\mathbf{T}_j} = \frac{\alpha_i\alpha_j}{|\boldsymbol{\alpha}|^2}\mathbf{N}\cdot\mathbf{T}_{ij}.$$

To simplify the dependence of r in the integral, we make a further coordinate change $\boldsymbol{\alpha}\mapsto\boldsymbol{\xi}$. We define $\boldsymbol{\xi} = (\xi_1,\xi_2)$ by requiring $\xi_i/|\boldsymbol{\xi}| = \alpha_i/|\boldsymbol{\alpha}|$ and $r^2 = b^2 + |\boldsymbol{\xi}|^2$, or

$$|\boldsymbol{\xi}|^2 = (1-\kappa b)|\boldsymbol{\alpha}|^2 + c|\boldsymbol{\alpha}|^3 + \mathcal{O}(b|\boldsymbol{\alpha}|^3 + |\boldsymbol{\alpha}|^4),$$

so that

$$|\boldsymbol{\xi}| = (1 - \frac{1}{2}\kappa b + \frac{1}{2}c|\boldsymbol{\alpha}|)|\boldsymbol{\alpha}| + \mathcal{O}(b|\boldsymbol{\alpha}|^2 + |\boldsymbol{\alpha}|^3),$$

where

$$c = \frac{\alpha_i}{|\boldsymbol{\alpha}|}\frac{\alpha_j}{|\boldsymbol{\alpha}|}\frac{\alpha_k}{|\boldsymbol{\alpha}|}\mathbf{T}_i\cdot\mathbf{T}_{jk} = \frac{\xi_i}{|\boldsymbol{\xi}|}\frac{\xi_j}{|\boldsymbol{\xi}|}\frac{\xi_k}{|\boldsymbol{\xi}|}\mathbf{T}_i\cdot\mathbf{T}_{jk}.$$

For $\boldsymbol{\alpha}$ near $\mathbf{0}$, we can solve for $|\boldsymbol{\alpha}|$ to get

$$|\boldsymbol{\alpha}| = (1 + \frac{1}{2}\kappa b - \frac{1}{2}c|\boldsymbol{\alpha}|)|\boldsymbol{\xi}| + \mathcal{O}(b|\boldsymbol{\xi}|^2) + \mathcal{O}(|\boldsymbol{\xi}|^3)$$
$$\Rightarrow\ |\boldsymbol{\alpha}| = (1 + \frac{1}{2}\kappa b - \frac{1}{2}c|\boldsymbol{\xi}|)|\boldsymbol{\xi}| + \mathcal{O}(b|\boldsymbol{\xi}|^2) + \mathcal{O}(|\boldsymbol{\xi}|^3),$$

and then

$$\alpha_i = (\xi_i/|\boldsymbol{\xi}|)|\boldsymbol{\alpha}| = (1 + \frac{1}{2}\kappa b - \frac{1}{2}c|\boldsymbol{\xi}|)\xi_i + \mathcal{O}(b|\boldsymbol{\xi}|^2) + \mathcal{O}(|\boldsymbol{\xi}|^3).$$

Thus

$$\left|\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\xi}}\right| = 1+\kappa b-\frac{3}{2}c|\boldsymbol{\xi}|+\mathcal{O}(b|\boldsymbol{\xi}|+|\boldsymbol{\xi}|^2). \tag{3.10}$$

Now, let us look at others term of the integrand in (3.6) and (3.7). With the $\boldsymbol{\alpha}$-derivatives of $\mathbf{f}$ at $\boldsymbol{\alpha}=\mathbf{0}$ denoted by $\mathbf{f}_i$ and $\mathbf{f}_{ij}$, we have

$$\mathbf{f}(\mathbf{y}) = \mathbf{f}_0+\alpha_i\mathbf{f}_i+\mathcal{O}(|\boldsymbol{\alpha}|^2), \tag{3.11}$$

$$|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}| = 1+\alpha_i|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|_i+\mathcal{O}(|\boldsymbol{\alpha}|^2). \tag{3.12}$$

Representing the surface by the new parameter $\boldsymbol{\xi}$, using (3.9), (3.10), (3.11), and (3.12), we have

$$\begin{aligned}&\frac{(1-H_1^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|}\mathbf{f}(\mathbf{y})|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|\left|\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\xi}}\right|\\ =&\frac{(1-H_1^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{\sqrt{|\boldsymbol{\xi}|^2+b^2}}\Big((1+\kappa b)\mathbf{f}_0+\mathbf{w}(\boldsymbol{\xi},b)+\mathbf{R}_2(\boldsymbol{\xi},b)\Big)\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}&\frac{(1-H_2^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|^3}(\mathbf{f}(\mathbf{y})\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|\left|\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\xi}}\right|\\ =&\frac{(1-H_2^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{(\sqrt{|\boldsymbol{\xi}|^2+b^2})^3}\Big(b^2(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}-\frac{b}{2}\xi_i^2(\mathbf{f}_0\cdot\mathbf{y}_{ii})\mathbf{N}-b\xi_i^2(\mathbf{f}_i\cdot\mathbf{T}_i)\mathbf{N}\\ &+(1+\kappa b)\xi_i^2(\mathbf{f}_0\cdot\mathbf{T}_i)\mathbf{T}_i-b\xi_i^2(\mathbf{f}_i\cdot\mathbf{N})\mathbf{T}_i-\frac{1}{2}b\xi_i^2(\mathbf{f}_0\cdot\mathbf{N})\mathbf{y}_{ii}+\kappa b^3(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}\\ &-\kappa b^2\xi_i^2(\mathbf{f}_0\cdot\mathbf{T}_i)\mathbf{T}_i+\frac{3}{2}bc|\boldsymbol{\xi}|\xi_l(\mathbf{f}_0\cdot\mathbf{N})\mathbf{y}_l+\frac{3}{2}bc|\boldsymbol{\xi}|\xi_l(\mathbf{f}_0\cdot\mathbf{y}_l)\mathbf{N}\\ &-b|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|_i\xi_i^2(\mathbf{f}_0\cdot\mathbf{T}_i)\mathbf{N}-b|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|_i\xi_i^2(\mathbf{f}_0\cdot\mathbf{N})\mathbf{T}_i+\mathbf{w}(\boldsymbol{\xi},b)+\mathbf{R}_4(\boldsymbol{\xi},b)\Big),\end{aligned} \tag{3.14}$$

where $\mathbf{w}(\boldsymbol{\xi},b)$ is some vector of odd functions in ξ_1 and ξ_2, $\mathbf{R}_s(\boldsymbol{\xi},b)$, $s\in\mathbb{N}$, is a sum of terms of the form $|\boldsymbol{\xi}|^m b^n\mathbf{q}(\boldsymbol{\xi},b)$ with $m+n=s, m\geq 0, n\geq 0$ and $\mathbf{q}$ is some vector of bounded functions. The term $|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|_i$ can be computed as follow

$$\begin{aligned}|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|_i&=\frac{\partial\sqrt{(\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2})\cdot(\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2})}}{\partial\alpha_i}\\ &=(\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2})\cdot\frac{\partial(\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2})}{\partial\alpha_i}\\ &=(\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2})\cdot\big(\mathbf{y}_{1i}\times\mathbf{y}_2+\mathbf{y}_1\times\mathbf{y}_{2i}\big)\\ &=\mathbf{y}_1\cdot\mathbf{y}_{1i}+\mathbf{y}_2\cdot\mathbf{y}_{2i}.\end{aligned}$$

Now, let (ξ,θ) be the polar coordinate in the $\boldsymbol{\xi}$-plane, i.e., $\xi=|\boldsymbol{\xi}|$. We have

$$\kappa=\cos^2\theta\mathbf{N}\cdot\mathbf{y}_{11}+2\sin\theta\cos\theta\mathbf{N}\cdot\mathbf{y}_{12}+\sin^2\theta\mathbf{N}\cdot\mathbf{y}_{22} \tag{3.15}$$

and

$$c=(\cos\theta\mathbf{y}_1+\sin\theta\mathbf{y}_2)\cdot(\cos^2\theta\mathbf{y}_{11}+2\sin\theta\cos\theta\mathbf{y}_{12}+\sin^2\theta\mathbf{y}_{22}).$$

Using (3.13) and (3.4), we can rewrite (3.6) as

$$\begin{aligned}e_1=&-\frac{1}{8\pi\mu}\int_{\mathbb{R}^2}\frac{(1-H_1^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{\sqrt{|\boldsymbol{\xi}|^2+b^2}}(1+\kappa b)\mathbf{f}_0 d\boldsymbol{\xi}\\&-\frac{1}{8\pi\mu}\int_{|\xi|^2+b^2<\lambda^2}\frac{(1-H_1^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{\sqrt{|\boldsymbol{\xi}|^2+b^2}}\mathbf{w}(\boldsymbol{\xi},b)d\boldsymbol{\xi}\\&-\frac{1}{8\pi\mu}\int_{|\xi|^2+b^2<\lambda^2}\frac{(1-H_1^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{\sqrt{|\boldsymbol{\xi}|^2+b^2}}\mathbf{R}_2(\boldsymbol{\xi},b)d\boldsymbol{\xi}+\mathcal{O}(\epsilon^4).\end{aligned}$$

Since, $\mathbf{w}(\boldsymbol{\xi},b)$ is odd with respect to ξ_1 and/or ξ_2, the second integral of e_1 equals to zero. By changing to polar coordinates, the first integral of e_1 can be rewritten as

$$-\frac{1}{8\pi\mu}\mathbf{f}_0\int_0^{2\pi}(1+\kappa b)d\theta\int_0^\infty\frac{(1-H_1^\epsilon(\sqrt{\xi^2+b^2}))}{\sqrt{\xi^2+b^2}}\xi d\xi. \tag{3.16}$$

Recall that κ is a function of θ only as shown in (3.15). Using the integral form of $H_1^\epsilon(r)$ developed in Appendix A (Eq. (A.5)), making a change of variables, and changing the order of integration, we can replace (3.16) by

$$-\frac{\pi}{2\mu}\Big(2+b(\mathbf{N}\cdot\mathbf{y}_{11}+\mathbf{N}\cdot\mathbf{y}_{22})\Big)\int_b^\infty\left(-bs^2+\frac{2sb^2}{3}+\frac{s^4}{3b}\right)\phi^\epsilon(s)ds.$$

Similarly, we can represent e_2 as in Theorem 3.2. What is left is to prove that the remaining terms in e_1 and e_2 are really $\mathcal{O}(\epsilon^3)$. For e_1, the remaining terms are $\mathcal{O}(\epsilon^3)$ if, for $m,n\in\mathbf{z}$, $m\geq1$, $n\geq0$, $m+n=3$, we have

$$\int_0^\infty\frac{|1-H_1^\epsilon(\sqrt{\xi^2+b^2})|}{(\xi^2+b^2)^{1/2}}b^n\xi^m d\xi=\mathcal{O}(\epsilon^3). \tag{3.17}$$

For e_2, the remaining terms are $\mathcal{O}(\epsilon^3)$ if, for $m,n\in\mathbf{z}$, $m\geq1$, $n\geq0$, $m+n=5$, we have

$$\int_0^\infty\frac{|1-H_2^\epsilon(\sqrt{\xi^2+b^2})|}{(\xi^2+b^2)^{3/2}}b^n\xi^m d\xi=\mathcal{O}(\epsilon^3). \tag{3.18}$$

The proofs of (3.17) and (3.18) are based on formulas (2.11) and (2.12) with $\beta=\alpha-1=(7-3)-1=3$, because $\phi(\mathbf{x})$ has zero second moment and decays as $|\mathbf{x}|^{-7}$. We only give the proof of (3.17), since the proof of (3.18) is basically the same. To prove (3.17), we first substitute $r^2=\xi^2+b^2$ and rewrite the integral on the left hand side of (3.17) as

$$\int_b^\infty\frac{|1-H_1^\epsilon(r)|}{r}b^n(r^2-b^2)^{(m-1)/2}r dr=\int_b^\infty|1-H_1^\epsilon(r)|b^n(r^2-b^2)^{(m-1)/2}dr. \tag{3.19}$$

Since $r^2-b^2<r^2$ and $m\geq 1$, we have $(r^2-b^2)^{(m-1)/2}\leq (r^2)^{(m-1)/2}=r^{m-1}$. Also, because the integral is from b to ∞, we have $b\leq r$, and, thus, $b^n\leq r^n$. Therefore, the last integral in (3.19) is bounded by

$$\int_b^\infty \left|1-H_1^\epsilon(r)\right|r^n r^{m-1}dr \overset{m+n=3}{=\!=\!=} \int_b^\infty \left|1-H_1^\epsilon(r)\right|r^2 dr \leq \int_0^\infty \left|1-H_1^\epsilon(r)\right|r^2 dr. \tag{3.20}$$

Now let substitute $r=\epsilon t$, the last integral in (3.20) becomes

$$\int_0^\infty \left|1-H_1^\epsilon(\epsilon t)\right|(\epsilon t)^2\epsilon dt=\epsilon^3\int_0^\infty \left|1-H_1^1(t)\right|t^2 dt. \tag{3.21}$$

Since $\phi(\mathbf{x})$ has zero second moment, according to Lemma 2.1, $\left|1-H_1^\epsilon(t)\right|=\mathcal{O}(t^{-4})$ for t large enough, and, hence, the last integral in (3.20) converges. We have proved (3.17).

This completes the proof of Theorem 3.2. □

3.2 Divergence of the computed flow field

The advantage of the MRS is that the computed flow field is automatically divergence free. This is a desired property in applications. On the other hand, the correction to the flow field computed by using the MRS is not necessarily divergence free, and could be $\mathcal{O}(1)$. Although we have not been able to derive a divergence free correction formula, in many cases, the corrections can be modified to get a flow field with small divergence while keeping the magnitude of the error induced by adding corrections to the computed flow field. The idea is to represent the divergence of the correction in terms of the distance from the field point to the surface, and the surface and force information at the point on the surface that is closest to that field point. We then add a modification in the normal direction, which could be found by solving a simple ordinary differential equation in b, the distance from field points to the surface, to make the total divergence close to zero.

Modified Second Order Correction. *In addition to the conditions in Theorem* 3.1*, we assume that at every point on the surface, the principal curvatures with respect to the outer normal vector are non-positive. Then we can write the flow velocity* $\mathbf{u}(\mathbf{x})$ *at* $\mathbf{x}$ *as*

$$\mathbf{u}(\mathbf{x})=\tilde{\mathbf{u}}(\mathbf{x})+\mathcal{O}(\epsilon^2),$$

with

$$\tilde{\mathbf{u}}(\mathbf{x})=\mathbf{u}^\epsilon(\mathbf{x})+E_2^\epsilon(\mathbf{x})+\mathcal{V}_2(\mathbf{x}),$$

where

$$\begin{aligned}\mathcal{V}_2(\mathbf{x})=&-\frac{\pi\mathbf{N}}{\mu(1-b\kappa_1)(1-b\kappa_2)}\int db\left(\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds\right)\\&\times\Big((1-b\kappa_2)\mathbf{f}_1\cdot\mathbf{T}_1+(1-b\kappa_1)\mathbf{f}_2\cdot\mathbf{T}_2+(\kappa_1+\kappa_2-2b\kappa_1\kappa_2)\mathbf{f}_0\cdot\mathbf{N}\Big)\\=&\,\mathcal{O}(\epsilon^2)\end{aligned} \tag{3.22}$$

is a term with divergence being a 'good' approximation (numerically verified) to the negative of the divergence of $E^\epsilon(\mathbf{x})$ *and* $\mathbf{T}_i,\kappa_i$, $i=1,2$ *are the principal directions and principal curvatures at* $\mathbf{x}_0$*,* $\mathbf{f}_i$*'s are the derivatives of the force* $\mathbf{f}$ *at* $\mathbf{x}_0$ *in the directions of* $\mathbf{T}_i$*'s.*

To validate the above statement, we first find the divergence of the correction (3.3) at $\mathbf{x}$. We have to assume that the coordinates have the directions of principal curvatures, more special than in the previous Sections. Assuming that the origin of the coordinate system is $\mathbf{x}_0$, in that special coordinates system, locally near $\mathbf{x}_0=\mathbf{0}$ the surface can be described as

$$\mathbf{y}=\mathbf{y}(\boldsymbol{\alpha})=\alpha_1\mathbf{T}_1+\alpha_2\mathbf{T}_2+\left(\frac{1}{2}\kappa_1\alpha_1^2+\frac{1}{2}\kappa_2\alpha_2^2\right)\mathbf{N}+\mathcal{O}(|\boldsymbol{\alpha}|^3).$$

Let $\mathbf{z}=z_1\mathbf{T}_1+z_2\mathbf{T}_2+z_3\mathbf{N}$ be a field point that is close to $\mathbf{x}$, i.e., $z_1\ll 1$, $z_2\ll 1$, and $z_3\approx b$. Let $\mathbf{z}_0=\mathbf{y}(\boldsymbol{\alpha})$ be a point on the surface that is closest to $\mathbf{z}$. The unit outer normal vector at $\mathbf{z}_0$ is given by

$$\mathbf{n}=-\kappa_1\alpha_1\mathbf{T}_1-\kappa_2\alpha_2\mathbf{T}_2+\mathbf{N}+\mathcal{O}(|\boldsymbol{\alpha}|^2).$$

Let t be the distance from $\mathbf{z}$ to the surface. We then have $\mathbf{z}=\mathbf{z}_0+t\mathbf{n}$, i.e.,

$$\begin{cases} z_1=(1-t\kappa_1)\alpha_1+\mathcal{O}(|\boldsymbol{\alpha}|^2),\\ z_2=(1-t\kappa_2)\alpha_2+\mathcal{O}(|\boldsymbol{\alpha}|^2),\\ z_3=t+\mathcal{O}(|\boldsymbol{\alpha}|^2).\end{cases}$$

Thus

$$\begin{cases} \alpha_1=\dfrac{z_1}{1-z_3\kappa_1}+\mathcal{O}(z_1^2+z_2^2),\\ \alpha_2=\dfrac{z_2}{1-z_3\kappa_1}+\mathcal{O}(z_1^2+z_2^2),\\ t=z_3+\mathcal{O}(z_1^2+z_2^2).\end{cases}$$

Hence, the unit outer normal vector at $\mathbf{z}_0$ can be written as

$$\mathbf{n}=-\frac{\kappa_1 z_1}{1-z_3\kappa_1}\mathbf{T}_1-\frac{\kappa_2 z_2}{1-z_3\kappa_2}\mathbf{T}_2+\mathbf{N}+\mathcal{O}(z_1^2+z_2^2).$$

Expanding $\mathbf{f}(\mathbf{z}_0)$ near $\mathbf{x}_0$, we have

$$\begin{aligned}\mathbf{f}(\mathbf{z}_0)&=\mathbf{f}(\mathbf{x}_0)+\alpha_1\frac{\partial\mathbf{f}}{\partial\alpha_1}(\mathbf{x}_0)+\alpha_2\frac{\partial\mathbf{f}}{\partial\alpha_2}(\mathbf{x}_0)+\mathcal{O}(|\boldsymbol{\alpha}|^2)\\ &=\mathbf{f}_0+\frac{z_1}{1-z_3\kappa_1}\mathbf{f}_1+\frac{z_2}{1-z_3\kappa_2}\mathbf{f}_2+\mathcal{O}(z_1^2+z_2^2).\end{aligned}$$

When $\mathbf{z}=z_1\mathbf{T}_1+b\mathbf{N}$, we have

$$t=b+\mathcal{O}(z_1^2), \tag{3.23}$$

$$\mathbf{n}=-\frac{\kappa_1 z_1}{1-b\kappa_1}\mathbf{T}_1+\mathcal{O}(z_1^2), \tag{3.24}$$

$$\mathbf{f}(\mathbf{z}_0)=\mathbf{f}_0+\frac{z_1}{1-b\kappa_1}\mathbf{f}_1+\mathcal{O}(z_1^2). \tag{3.25}$$

Hence,

$$\mathbf{f}(\mathbf{z}_0)-(\mathbf{f}(\mathbf{z}_0)\cdot\mathbf{n})\mathbf{n}=\mathbf{f}_0-(\mathbf{f}_0\cdot\mathbf{n})\mathbf{n}+\frac{z_1}{1-b\kappa_1}(\mathbf{f}_1-(\mathbf{f}_1\cdot\mathbf{n})\mathbf{n})+\mathcal{O}(z_1^2), \tag{3.26}$$

and the correction at $\mathbf{z}$ is

$$\begin{aligned}E_2^\epsilon(\mathbf{z})&=-\frac{\pi}{\mu}\big(\mathbf{f}(\mathbf{z}_0)-(\mathbf{f}(\mathbf{z}_0)\cdot\mathbf{n})\mathbf{n}\big)\int_t^\infty s(s-t)^2\phi^\epsilon(s)ds\\&=-\frac{\pi}{\mu}\big(\mathbf{f}(\mathbf{z}_0)-(\mathbf{f}(\mathbf{z}_0)\cdot\mathbf{n})\mathbf{n}\big)\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds+\mathcal{O}(z_1^2).\end{aligned} \tag{3.27}$$

From (3.24), (3.26), and (3.27), the component of $E_2^\epsilon(\mathbf{z})$ along $\mathbf{T}_1$-direction can be written as

$$\begin{aligned}(E_2^\epsilon(\mathbf{z}))_1=-\frac{\pi}{\mu}\left((\mathbf{f}_0\cdot\mathbf{T}_1)+\frac{\kappa_1 z_1}{1-b\kappa_1}(\mathbf{f}_0\cdot\mathbf{N})+\frac{z_1}{1-b\kappa_1}(\mathbf{f}_1\cdot\mathbf{T}_1)\right)\\\times\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds+\mathcal{O}(z_1^2).\end{aligned} \tag{3.28}$$

The component of $E_2^\epsilon(\mathbf{x})$ along $\mathbf{T}_1$-direction is

$$(E_2^\epsilon(\mathbf{x}))_1=-\frac{\pi}{\mu}(\mathbf{f}_0\cdot\mathbf{T}_1)\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds. \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\frac{(E_2^\epsilon(\mathbf{z}))_1-(E_2^\epsilon(\mathbf{x}))_1}{z_1}=-\frac{\pi}{\mu}\cdot\frac{(\mathbf{f}_1\cdot\mathbf{T}_1)+\kappa_1(\mathbf{f}_0\cdot\mathbf{N})}{1-b\kappa_1}\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds+\mathcal{O}(z_1).$$

Letting z_1 go to zero, we have

$$\frac{\partial(E_2^\epsilon(\mathbf{x}))_1}{\partial z_1}=-\frac{\pi}{\mu}\cdot\frac{(\mathbf{f}_1\cdot\mathbf{T}_1)+\kappa_1(\mathbf{f}_0\cdot\mathbf{N})}{1-b\kappa_1}\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds.$$

Similarly, we have

$$\frac{\partial(E_2^\epsilon(\mathbf{x}))_2}{\partial z_2}=-\frac{\pi}{\mu}\cdot\frac{(\mathbf{f}_2\cdot\mathbf{T}_2)+\kappa_2(\mathbf{f}_0\cdot\mathbf{N})}{1-b\kappa_2}\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds.$$

When $\mathbf{z}=z_3\mathbf{N}$, the component of $E_2^\epsilon(\mathbf{z})$ along the normal direction $\mathbf{N}$ is identically zero. Therefore, the divergence of the correction at $\mathbf{x}$ is

$$\nabla\cdot E_2^\epsilon(\mathbf{x})=-\frac{\pi}{\mu}\left(\frac{(\mathbf{f}_1\cdot\mathbf{T}_1)+\kappa_1(\mathbf{f}_0\cdot\mathbf{N})}{1-b\kappa_1}+\frac{(\mathbf{f}_2\cdot\mathbf{T}_2)+\kappa_2(\mathbf{f}_0\cdot\mathbf{N})}{1-b\kappa_2}\right)\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds. \tag{3.30}$$

Similarly, let $D(b)$ be a function of b only. The divergence of the term $D(b)\mathbf{n}$ is

$$D'(b)-\left(\frac{\kappa_1}{1-b\kappa_1}+\frac{\kappa_2}{1-b\kappa_2}\right)D(b). \tag{3.31}$$

Solving the following differential equation for $D(b)$

$$D'(b) - \left(\frac{\kappa_1}{1-b\kappa_1} + \frac{\kappa_2}{1-b\kappa_2}\right) D(b) = \nabla \cdot E_2^\epsilon(\mathbf{x}) \tag{3.32}$$

and writing

$$\mathcal{V}_2(\mathbf{x}) = D(b)\mathbf{N},$$

we have our main result (3.22).

Note that, in order to arrive at Eq. (3.32), we assume that D is a function of b only. But the solution of (3.32) actually depends on $\mathbf{x}$ too. As we will see in the numerical results in Section 4, this approach works quite well for modified second order correction but does a terrible job for modified third order correction.

Now let us prove that $\mathcal{V}_2(\mathbf{x}) = \mathcal{O}(\epsilon^2)$. Consider the integral $\mathcal{I}$ in the definition of $\mathcal{V}_2(\mathbf{x})$

$$\begin{aligned}
\mathcal{I} &= \int db \left(\int_b^\infty s(s-b)^2 \phi^\epsilon(s)\,ds\right) \\
&\qquad \times \Big((1-b\kappa_2)\mathbf{f}_1 \cdot \mathbf{T}_1 + (1-b\kappa_1)\mathbf{f}_2 \cdot \mathbf{T}_2 + (\kappa_1+\kappa_2-2b\kappa_1\kappa_2)\mathbf{f}_0 \cdot \mathbf{N}\Big) \\
&= \int (a_1 + a_2 b) \left(\int_b^\infty s(s-b)^2 \phi^\epsilon(s)\,ds\right) db,
\end{aligned} \tag{3.33}$$

where $a_1 = \mathbf{f}_1 \cdot \mathbf{T}_1 + \mathbf{f}_2 \cdot \mathbf{T}_2 + (\kappa_1+\kappa_2)\mathbf{f}_0 \cdot \mathbf{N}$ and $a_2 = -\kappa_2 \mathbf{f}_1 \cdot \mathbf{T}_1 - \kappa_1 \mathbf{f}_2 \cdot \mathbf{T}_2 - 2\kappa_1\kappa_2 \mathbf{f}_0 \cdot \mathbf{N}$ are two constants.

Making the change of variables $b = \epsilon b'$ and $s = \epsilon s'$, we have

$$\begin{aligned}
\mathcal{I} &= \int (a_1 + a_2 \epsilon b') \left(\int_{b'}^\infty \epsilon^3 s'(s'-b')^2 \frac{1}{\epsilon^3} \phi(s') \epsilon\, ds'\right) \epsilon\, db' \\
&= \epsilon^2 \int (a_1 + \epsilon a_2 b') \left(\int_b^{\prime\infty} s'(s'-b')^2 \phi(s')\,ds'\right) db'.
\end{aligned}$$

Therefore,

$$|\mathcal{I}| \le \epsilon^2 \int_0^\infty (|a_1| + \epsilon |a_2| b') \left(\int_{b'}^\infty s'(s'-b')^2 \phi(s')\,ds'\right) db'. \tag{3.34}$$

Fix a constant K such that $|\phi(s')| \le C(s')^{-7}$ for $s' \ge K$ and rewrite the double integral on the right hand side of (3.34) as a sum of two double integrals, we have

$$\begin{aligned}
\epsilon^{-2}|\mathcal{I}| \le &\int_0^K (|a_1| + \epsilon |a_2| b') \left(\int_{b'}^\infty s'(s'-b')^2 \phi(s')\,ds'\right) db' \\
&+ \int_K^\infty (|a_1| + \epsilon |a_2| b') \left(\int_{b'}^\infty s'(s'-b')^2 \phi(s')\,ds'\right) db'.
\end{aligned} \tag{3.35}$$

Clearly, the first integral in (3.35) is bounded. Since $|\phi(s')| \le C(s')^{-7}$ for $s' \ge K$, the second integral in (3.35) is bounded by

$$\begin{aligned}\int_K^\infty (|a_1|+\epsilon|a_2|b')\left(\int_{b'}^\infty (s')^3\frac{C}{(s')^7}ds'\right)db' &= \int_K^\infty (|a_1|+\epsilon|a_2|b')\frac{C}{3(b')^3}db' \\ &= \frac{C}{3}\left(\frac{|a_1|}{2K^2}+\frac{\epsilon|a_2|}{K}\right). \end{aligned} \tag{3.36}$$

Therefore, $\epsilon^{-2}|\mathcal{I}| = \mathcal{O}(1)$ and $\mathcal{I} = \mathcal{O}(\epsilon^2)$.

4 Numerical results

4.1 Test problem

We will use the above numerical correction formulas to determine the flow field around a prolate spheroid translating parallel to its axis of symmetry. We choose this problem as a test problem for our numerical method because the exact solution is well known and non-trivial [4]. We parametrized the surface of a prolate spheroid whose axis of symmetry is aligned with the z-direction as

$$\mathbf{y}(\phi,\theta) = a\sin\phi\cos\theta\vec{x} + a\sin\phi\sin\theta\vec{y}, + c\cos\phi\vec{z}$$

where $\phi \in [0,\pi]$ and $\theta \in [0,2\pi]$. For translation velocity $\mathbf{U} = U\vec{z}$ parallel to the axis of symmetry, the surface traction is of the form $\mathbf{f} = f(\phi)\vec{z}$ with

$$f(\phi) = -\frac{4U\mu e^3}{a((1+e^2)L-2e)} \cdot \frac{1}{\sqrt{1-e^2\cos^2\phi}},$$

where $e = \sqrt{c^2-a^2}/c$ and $L = \log[(1+e)/(1-e)]$. In all of our numerical tests, we use $\mu = 1$, $U = 1$, $a = 1$, $c = 2$.

4.2 Blobs in use

We compare the error corresponding to 3 different blob functions. The first blob which is widely used in literature [1,5,6,14] is

$$\phi^\epsilon(\mathbf{x}) = \frac{15\epsilon^4}{8\pi(r^2+\epsilon^2)^{7/2}}. \tag{4.1}$$

When $r \to \infty$ (ϵ fixed) or, equivalently, $\epsilon \to 0$ (r fixed), the regularized Stokeslet corresponding to this blob converges to the singular one as ϵ^2/r^2 at best, regardless of how fast the blob decays.

The second blob, which has been used in other computations [9,10], has second moment 0. The regularized Stokeslet corresponding to this blob, when $r \to \infty$ (ϵ fixed) or, equivalently, $\epsilon \to 0$ (r fixed), converges to the singular one as $(\epsilon/r)^{m-3}$ if the blob decays as ϵ^{m-3}/r^m for $m>5$

$$\phi^\epsilon(r) = \frac{5\epsilon^2 - 2r^2}{2\pi^{3/2}\epsilon^5} e^{-r^2/\epsilon^2}. \tag{4.2}$$

The third blob will be useful in the situation where we know the velocity of the boundary surface and want to solve for the traction distribution on the surface.

$$\phi^\epsilon(r) = \frac{1}{\epsilon^3} \begin{cases} \left(\dfrac{525}{2\pi} - \dfrac{1050}{\pi}\dfrac{r}{\epsilon} + \dfrac{945}{\pi}\dfrac{r^2}{\epsilon^2} \right) \left(1 - \dfrac{r}{\epsilon}\right)^2, & \text{if } 0 \le \dfrac{r}{\epsilon} \le 1, \\ 0, & \dfrac{r}{\epsilon} \ge 1. \end{cases} \tag{4.3}$$

Beside having second moment 0 as the second blob function, the third blob function also satisfies

$$\int_0^\infty r^3 \phi^\epsilon(r) dr = 0$$

and

$$S_{ij}^\epsilon(\mathbf{x},\mathbf{y}) = S_{ij}(\mathbf{x},\mathbf{y}) \quad \text{when} \quad |\mathbf{x}-\mathbf{y}| > \epsilon.$$

4.3 Numerical quadrature

For the purpose of numerically evaluating the exterior flow field, the surface of the prolate spheroid is discretized with $M_\phi + 1$ points in the ϕ-direction and $M_\theta + 1$ points in the θ-direction. Accounting for the redundancy of the poles, the total number of points is $M = (M_\phi - 1)M_\theta + 2$. We will use trapezoidal quadrature in the ϕ- and θ- directions to approximate the integrals. Thus, depending on the correction used, the total error can be written as [6]

$$\mathcal{O}(\epsilon^p) + \mathcal{O}\left(\frac{h^2}{\epsilon^3}\right), \qquad p = 2,3. \tag{4.4}$$

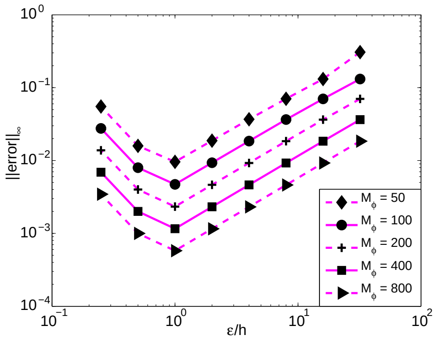
The first term in (4.4) is the regularization error and the second term is the quadrature error. From (4.4), one can see that there is an optimal value of ϵ for given h. For a fixed discretization of the surface, when ϵ is small compared to h (smaller than the optimized value of ϵ), the total error (4.4) is dominated by the quadrature error and we will not see the effect of adding corrections to the computed exterior velocity. When ϵ is large compared to h (larger than the optimized value of ϵ), the total error is dominated by the regularization error and we can see the effect of adding corrections to the computed velocity by looking at the total error.

In all of our numerical tests, we observe that for a fixed M_ϕ, the error when $M_\theta \ll M_\phi$ is worse than when $M_\theta \approx M_\phi$, and when $M_\theta \gg M_\phi$, we haven't seen much improvement. So the best choice in this case is $M_\theta = M_\phi$.

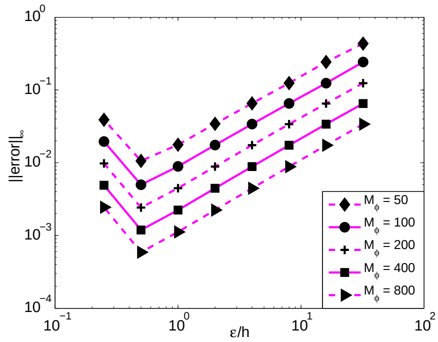
Note that, since the computational time for corrections is $\mathcal{O}(1)$, it is not affected by the refinement of the grid on the surface.

4.4 Numerical results

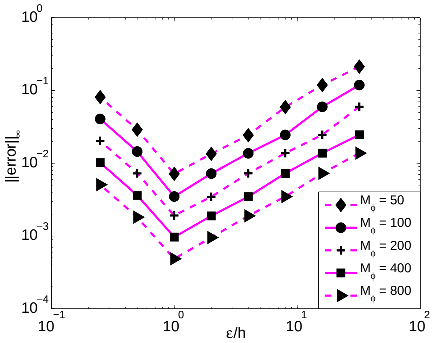
Because of symmetry, we only evaluate the flow field on the half-plane $\{y=0,x>0\}$ and compare to the exact solution. Since the regularization error and the quadrature error are very small when the distance from the field points to the surface is bigger than 1, we only evaluate the velocity inside a region of distance less than 1 from the surface. The sup-norm in this section is computed on this region.



(a) the first blob (4.1)



(b) the second blob (4.2)



(c) the third blob (4.3)

Figure 2: log-log plot of the sup-norm of the error without using any of the corrections developed in Section 3 versus the ratio ϵ/h, $h=2\pi/M_\phi$, M_ϕ ranges from 50 to 800.

In Figs. 2(a), 2(b), and 2(c), we draw log-log plots of the sup-norm of the error without using any correction versus the ratio ϵ/h for various values of ϵ and $h=\pi/M_\phi$, $M_\phi=M_\theta$ is in the range 50–800. Each line corresponding to a fixed value of h and varying values of ϵ. We can see that we only get error of the first order with respect to ϵ in all cases.

Next, we examine the effect of the second and third order corrections. Fig. 3 show the log-log plot of the sup-norm of the error corresponding to the second blob when using corrections versus the ratio ϵ/h for various values of ϵ and $h=\pi/M_\phi$, $M_\phi=M_\theta$ is in the range 50–800. Each line corresponding to a fixed value of h and varying values of ϵ. Dash lines and solid lines correspond to applying $E_2^\epsilon(\mathbf{x})$ and $E_3^\epsilon(\mathbf{x})$ to the second blob, respectively. The slopes of these lines equal the order of convergence of the regularization error. By comparing with reference lines, we can see that we get the right order as

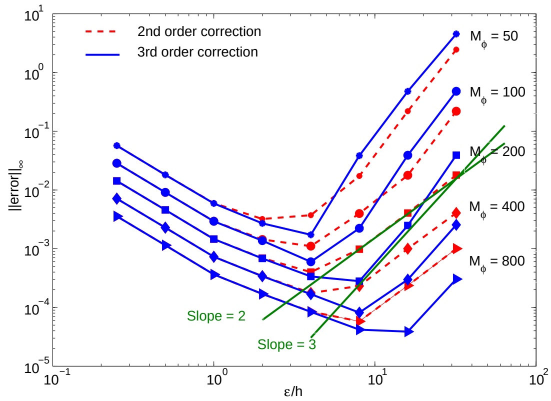


Figure 3: log-log plot of the sup-norm of the error corresponding to the second blob when using the second and third order corrections versus versus the ratio ϵ/h, $h=2\pi/M_\phi$, M_ϕ ranges from 50 to 800.

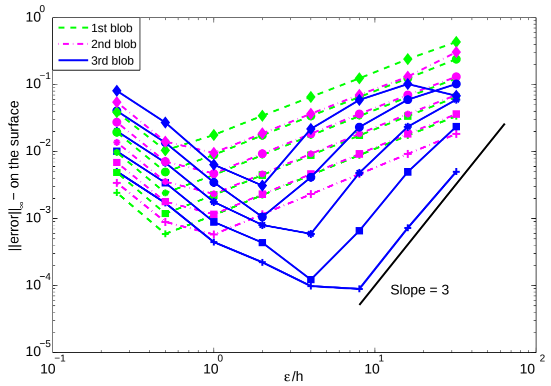


Figure 4: log-log plot of the maximum of the absolute value of the errors on the surface versus the ratio ϵ/h. Each curve corresponds to a fixed value of $h=\pi/M_\phi$, $M_\phi=M_\theta$ ranges from 50 to 800. Lines with $+$-markers corresponds to $M_\phi=800$, $\Box$-markers $M_\phi=400$, $*$-markers $M_\phi=200$, $\circ$-markers $M_\phi=100$, and $\diamond$-markers $M_\phi=50$. The black line is a line of slope 3 drawn for reference.

predicted by the theorems in Section 3. When ϵ is large compared to h, the regularization error is dominant and is of expected order. When ϵ goes to zero, after some point, the quadrature error will be dominant and we do not see the effect of the corrections to the regularization error. The graph also suggests that there might be some optimal value of the ratio ϵ/h that give the best approximation. The same results hold when we use modified corrections.

When we only have information of the velocity on the surface, in order to evaluate the exterior velocity field, we need to solve the boundary integral equation for the traction on the surface and use the MRS. As mentioned above the third blob (4.3) will be of advantage in this situation because of high order of accuracy with respect to the blob parameters. In Fig. 4, we look at errors right on the surface when using 3 different blobs. The solid line corresponds to the third blob (4.3), the dash lines the second blob (4.2), the dash-dot lines the first blob (4.1). The first and the second blob give only first order error while the third blob gives third order error as expected.

Now let verify the argument in Subsection 3.2 numerically. In Fig. 5, we graph the sup-norm of the divergence of the computed flow field with and without corrections. We use finite difference to approximate the derivatives of the velocity. The divergence of the computed flow field before adding correction is close to zero but after adding corrections, the divergence is large compared to zero. For the case of adding modified second order correction, the divergence is as good as without correction, and is even better if the blob parameter ϵ is small. But the modified third order correction does not work as expected. This phenomenon can be explained as following. As mentioned earlier in Subsection 3.2, we need to assume that the function $D(b)$ depends on the distance b only. But when we solve (3.32), the function $D(b)$ now depends on the information about the force and the surface at $\mathbf{x}_0$ which is not completely independent of b. This dependence in the modified third order correction is stronger than in the modified second order correction. The modified third order correction requires values of the third derivatives of the surface at $\mathbf{x}_0$ and the second derivatives of the force at $\mathbf{x}_0$.

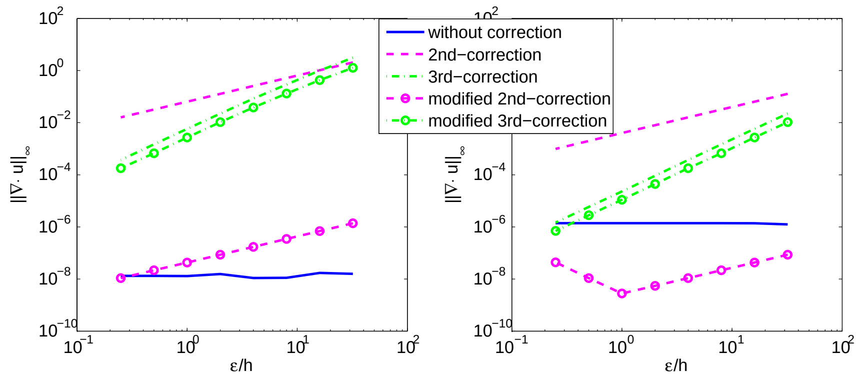


Figure 5: log-log plot of the sup-norm of divergence versus the ratio ϵ/h. Each curve corresponds to a fixed value of $h=\pi/M_\phi$. The figure on the left corresponds to $M_\phi=50$, and the figure on the right $M_\phi=800$.

5 Conclusions and future work

In this paper we focused on the problem of accurately evaluating the velocity field outside a solid object moving in an incompressible Stokes flow using the method of regularized Stokeslet. The original goal was to find a class of blobs whose corresponding regularization error (2.7) was of at least second order with respect to blob parameter ϵ. We proved that this was nearly impossible to achieve with radially symmetric blobs. Though we cannot make the 'constant' in (2.13) equal to zero, in practice, we could find a class of blobs that minimizes that constant. This could be the topic of another analysis.

Then, we found a class of blobs that the local analysis was feasible and found the corresponding error corrections. Since these error corrections may not be divergence free, we proposed a simple approach to modify the error corrections in order to make the divergence of the corrected flow close to zero. However, our approach to reduce divergence seemed to work well only for the second order correction as shown by numerical results in Section 4.

Acknowledgments

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A Proof of Lemma 2.1

We will now give a proof of Lemma 2.1 by representing the regularized Stokeslet explicitly as a sum of integrals involving the $\phi^\epsilon(r)$ blob.

Solve (2.5) for G^ϵ by changing to spherical coordinates, we have

$$G^\epsilon(r)=G^\epsilon(0)-\frac{1}{r}\int_0^r s^2\phi^\epsilon(s)ds+\int_0^r s\phi^\epsilon(s)ds. \tag{A.1}$$

We want $G^\epsilon(r)$ to behave like the Green's function as $r\to\infty$, i.e.,

$$\lim_{r\to\infty}\big(G^\epsilon(r)+\frac{1}{4\pi r}\big)=0.$$

Add $1/(4\pi r)$ to both sides of Eq. (A.1) and use condition 1 of $\phi(r)$ in the statement of Theorem 2.1, we have

$$G^\epsilon(r)+\frac{1}{4\pi r}=G^\epsilon(0)+\frac{1}{r}\int_r^\infty s^2\phi^\epsilon(s)ds+\int_0^r s\phi^\epsilon(s)ds. \tag{A.2}$$

Let r goes to infinity, the left hand side of (A.2) goes to zero as needed, and the right hand side goes to

$$G^\epsilon(0)+\int_0^\infty s\phi^\epsilon(s)ds.$$

Therefore, (A.1) becomes

$$G^\epsilon(r)=-\frac{1}{r}\int_0^r s^2\phi^\epsilon(s)ds-\int_r^\infty s\phi^\epsilon(s)ds. \tag{A.3}$$

Solve (2.6) for $(B^\epsilon(r))'$, we have

$$(B^\epsilon(r))'=\frac{1}{r^2}\int_0^r r^2 G^\epsilon(r)dr. \tag{A.4}$$

Using (2.6), (A.4), and (A.3), we can rewrite (2.3) as

$$H_1^\epsilon(r)=4\pi\int_0^r s^2\phi^\epsilon(s)ds+\frac{16\pi r}{3}\int_r^\infty s\phi^\epsilon(s)ds+\frac{4\pi}{3r^2}\int_0^r s^4\phi^\epsilon ds. \tag{A.5}$$

In a similar manner, (2.4) becomes

$$H_2^\epsilon(r)=4\pi\int_0^r s^2\phi^\epsilon(s)ds-\frac{4\pi}{r^2}\int_0^r s^4\phi^\epsilon(s)ds. \tag{A.6}$$

It is easy to see that $H_1^\epsilon(r)$ and $H_2^\epsilon(r)$ are continuous. We will now prove the boundedness properties (2.11) and (2.12). We will prove (2.11) only since the proof of (2.12) is similar. Using (A.5), we write

$$1-H_1^\epsilon(r)=4\pi\int_r^\infty s^2\phi^\epsilon ds-\frac{16\pi r}{3}\int_r^\infty s\phi^\epsilon(s)ds-\frac{4\pi}{3r^2}\int_0^r s^4\phi^\epsilon ds. \tag{A.7}$$

Fix a constant $K>1$ such that

$$|\phi(r)|\le Cr^{-3-\alpha} \quad \text{when} \quad r>K.$$

There exists such a constant K because of the second condition of $\phi(r)$. For $r>K\epsilon$, since $\beta<\alpha-1$, the first term is bounded by

$$4\pi\int_r^\infty s^2\frac{C\epsilon^\alpha}{s^{\alpha+3}}ds=4\pi C\frac{\epsilon^\alpha}{(\alpha)r^\alpha}<4\pi C\left(\frac{\epsilon}{r}\right)^{1+\beta},$$

and the second term is bounded by

$$\frac{16\pi r}{3}\int_r^\infty s\frac{C\epsilon^\alpha}{s^{\alpha+3}}ds=\frac{16\pi Cr}{3}\frac{\epsilon^\alpha}{(\alpha+1)r^{\alpha+1}}<\frac{16\pi C}{3}\left(\frac{\epsilon}{r}\right)^{1+\beta}.$$

Now let us look at the last term of (A.7). For $r>K\epsilon$, the last term in $1-H_1^\epsilon(r)$ can be rewritten as

$$\frac{4\pi}{3r^2}\left(\int_0^{K\epsilon}s^4\phi^\epsilon(s)ds+\int_{K\epsilon}^r s^4\phi^\epsilon(s)ds\right)=\frac{4\pi}{3}\left(\frac{\epsilon}{r}\right)^2\int_0^K s^4\phi(s)ds+\frac{4\pi}{3r^2}\int_{K\epsilon}^r s^4\phi^\epsilon(s)ds.$$

Since $\beta\le 1$, the first term on the right hand side is bounded by

$$\frac{4\pi C_K}{3}\left(\frac{\epsilon}{r}\right)^2\le\frac{4\pi C_K}{3}\left(\frac{\epsilon}{r}\right)^{1+\beta},$$

where C_K is some constant. The second term is bounded by

$$\frac{4\pi}{3r^2}\int_{K\epsilon}^r s^4\frac{C\epsilon^\alpha}{s^{\alpha+3}}ds=\begin{cases}\dfrac{4\pi C}{2-\alpha}\left(\dfrac{\epsilon}{r}\right)^\alpha-\dfrac{4\pi CK^{2-\alpha}}{2-\alpha}\left(\dfrac{\epsilon}{r}\right)^2, & \text{for } \alpha\neq 2,\\[2ex] 4\pi C\left(\dfrac{\ln(r/\epsilon)}{(r/\epsilon)^2}\right)-4\pi C\ln K\left(\dfrac{\epsilon}{r}\right)^2, & \text{for } \alpha=2.\end{cases}$$

Thus, in order to prove that the left hand side is bounded by $C_1\left(\frac{\epsilon}{r}\right)^{1+\beta}$ for some constant $C_1>0$, we only need to prove that

$$\frac{\ln r}{r^2}<C_2\left(\frac{\epsilon}{r}\right)^{1+\beta} \quad \text{for } \ r>K\epsilon,$$

where $C_2>0$ is some constant. Since

$$\lim_{r\to+\infty} r^\gamma \ln r=0 \quad \text{if } \ \gamma<0,$$

there exists some constant C_γ such that

$$r^\gamma \ln r<C_\gamma \quad \text{if } r>1.$$

Because $\beta<\alpha-1$, $1+\beta<\alpha$. Put $\gamma=(1+\beta)-\alpha<0$ and simplify to get

$$\frac{\ln r}{r^\alpha}<C_\gamma\frac{1}{r^{1+\beta}} \quad \text{if } \ r>1.$$

Therefore, when $\alpha=2$, since $r/\epsilon>K>1$, we have

$$\frac{\ln(r/\epsilon)}{(r/\epsilon)^2}<C_\gamma\left(\frac{\epsilon}{r}\right)^{1+\beta}.$$

If, in addition, the function $\phi(\mathbf{x})$ has second moment 0, then we can change the integral in the last term of (A.7) to the integral from r to ∞ and proving the boundedness of the last term becomes straight forward as for the first two terms.

B Proof of Lemma 2.2

Fix a small constant $\lambda>\epsilon$, using (2.11) and (2.12), for $r\geq\lambda$, we have

$$\begin{aligned}
&\frac{1}{r}|1-H_1^\epsilon(r)|=\mathcal{O}(\epsilon^{1+\beta}),\\
&\frac{1}{r^3}|1-H_2^\epsilon(r)|=\mathcal{O}(\epsilon^{1+\beta}).
\end{aligned}$$

Therefore, we can rewrite e_1 and e_2 in (2.8) as

$$e_1=-\frac{1}{8\pi\mu}\int_{\partial D,\ |\hat{\mathbf{x}}|<\lambda}\frac{(1-H_1^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|}\mathbf{f}(\mathbf{y})ds(\mathbf{y})+\mathcal{O}(\epsilon^{1+\beta}), \tag{B.1}$$

$$e_2=-\frac{1}{8\pi\mu}\int_{\partial D,\ |\hat{\mathbf{x}}|<\lambda}\frac{(1-H_2^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|^3}(\mathbf{f}(\mathbf{y})\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}ds(\mathbf{y})+\mathcal{O}(\epsilon^{1+\beta}). \tag{B.2}$$

With the same notation as in the proof of Theorem 3.2, we can write

$$\frac{(1-H_1^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|}\mathbf{f}(\mathbf{y})|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|\left|\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\xi}}\right|=\frac{(1-H_1^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{\sqrt{|\boldsymbol{\xi}|^2+b^2}}(\mathbf{f}_0+|\boldsymbol{\xi}|\mathbf{q}(\boldsymbol{\xi},b)) \tag{B.3}$$

with some vector of bounded functions $\mathbf{q}$, and

$$\begin{aligned}&\frac{(1-H_2^\epsilon(|\hat{\mathbf{x}}|))}{|\hat{\mathbf{x}}|^3}(\mathbf{f}(\mathbf{y})\cdot\hat{\mathbf{x}})\hat{\mathbf{x}}|\mathbf{y}_{\alpha_1}\times\mathbf{y}_{\alpha_2}|\left|\frac{\partial\boldsymbol{\alpha}}{\partial\boldsymbol{\xi}}\right|\\ =&\frac{(1-H_2^\epsilon(\sqrt{|\boldsymbol{\xi}|^2+b^2}))}{(\sqrt{|\boldsymbol{\xi}|^2+b^2})^3}\Big(b^2(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}+\xi_i^2(\mathbf{f}_0\cdot\mathbf{T}_i)\mathbf{T}_i+\mathbf{w}(\boldsymbol{\xi},b)+\mathbf{R}(\boldsymbol{\xi},b)\Big),\end{aligned} \tag{B.4}$$

where $\mathbf{w}(\boldsymbol{\xi},b)$ is an vector of odd functions in ξ_1 and ξ_2, $\mathbf{R}$ is a sum of terms of the form $|\boldsymbol{\xi}|^m b^n\mathbf{q}(\boldsymbol{\xi},b)$ with $m+n=3, m\geq 0, n\geq 0$ and $\mathbf{q}$ is some vector of bounded functions.

Now, let (ξ,θ) be the polar coordinate in the $\boldsymbol{\xi}$-plane, i.e., $\xi=|\boldsymbol{\xi}|$. Using (B.3) and changing to polar coordinate, we can rewrite (B.1) as

$$\begin{aligned}e_1=&-\frac{\mathbf{f}_0}{4\mu}\int_0^{\sqrt{\lambda^2-b^2}}\frac{(1-H_1^\epsilon(\sqrt{\xi^2+b^2}))}{\sqrt{\xi^2+b^2}}\xi d\xi\\ &-\frac{1}{4\mu}\int_0^{\sqrt{\lambda^2-b^2}}\frac{(1-H_1^\epsilon(\sqrt{\xi^2+b^2}))}{\sqrt{\xi^2+b^2}}\xi^2\mathbf{q}(\boldsymbol{\xi},b)d\xi+\mathcal{O}(\epsilon^{1+\beta}).\end{aligned} \tag{B.5}$$

The first integral in (B.5) can, by using (2.11), be replaced by

$$-\frac{1}{4\mu}\int_0^\infty\mathbf{f}_0\frac{(1-H_1^\epsilon(\sqrt{\xi^2+b^2}))}{\sqrt{\xi^2+b^2}}\xi d\xi+\mathcal{O}(\epsilon^{1+\beta}). \tag{B.6}$$

After a change of variable $r=\sqrt{\xi^2+b^2}$ and changing the order of integration, we can replace (B.6) by

$$-\frac{\pi}{\mu}\mathbf{f}_0\int_b^\infty\left(-bs^2+\frac{2sb^2}{3}\right)\phi^\epsilon(s)ds+\frac{\pi}{3b\mu}\mathbf{f}_0\int_0^b s^4\phi^\epsilon(s)ds+\mathcal{O}(\epsilon^{1+\beta}). \tag{B.7}$$

Now, let us look at the second integral in (B.5). Since $0<\xi\leq\lambda<1$ and $1+\beta/2<2$, we have $\xi^2\leq\xi^{1+\beta/2}$. Using (2.11) and making change of variable $\epsilon r=\sqrt{\xi^2+b^2}$, the absolute value of the second integral in (B.5) is, with C denoting an arbitrary positive constant independent of ϵ, bounded by

$$\begin{aligned}C\int_0^\infty\frac{|1-H_1^\epsilon(\sqrt{\xi^2+b^2})|}{\sqrt{\xi^2+b^2}}\xi^{1+\beta/2}d\xi\leq& C\epsilon^{1+\beta/2}\int_0^\infty|1-H_1^1(r)|r^{\beta/2}dr\\ \leq& C\epsilon^{1+\beta/2}.\end{aligned} \tag{B.8}$$

Combining (B.7) and (B.8), we can write e_1 as

$$e_1 = -\frac{\pi}{\mu}\mathbf{f}_0 \int_b^\infty \left(-bs^2 + \frac{2sb^2}{3}\right)\phi^\epsilon(s)ds + \frac{\pi}{3b\mu}\mathbf{f}_0 \int_0^b s^4\phi^\epsilon(s)ds + \mathcal{O}(\epsilon^{1+\beta/2}). \tag{B.9}$$

We now turn to e_2. Using (B.4) and changing to polar coordinate, we can rewrite (B.2) as

$$\begin{aligned} e_2 = &-\frac{1}{4\mu}\int_0^{\sqrt{\lambda^2-b^2}} \frac{(1-H_2^\epsilon(\sqrt{\xi^2+b^2}))}{(\xi^2+b^2)^{3/2}} b^2(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}\xi d\xi \\ &-\frac{1}{8\mu}\Big((\mathbf{f}_0\cdot\mathbf{T}_1)\mathbf{T}_1 + (\mathbf{f}_0\cdot\mathbf{T}_2)\mathbf{T}_2\Big)\int_0^{\sqrt{\lambda^2-b^2}} \frac{(1-H_2^\epsilon(\sqrt{\xi^2+b^2}))}{(\xi^2+b^2)^{3/2}}\xi^3 d\xi \\ &-\frac{1}{4\mu}\int_0^{\sqrt{\lambda^2-b^2}} \frac{(1-H_2^\epsilon(\sqrt{\xi^2+b^2}))}{(\xi^2+b^2)^{3/2}}\mathbf{w}(\boldsymbol{\xi},b)\xi d\xi \\ &-\frac{1}{4\mu}\int_0^{\sqrt{\lambda^2-b^2}} \frac{(1-H_2^\epsilon(\sqrt{\xi^2+b^2}))}{(\xi^2+b^2)^{3/2}}\mathbf{R}(\boldsymbol{\xi},b)\xi d\xi + \mathcal{O}(\epsilon^{1+\beta}). \end{aligned} \tag{B.10}$$

Using (2.12), similarly to the case of e_1, we can write the first integral of (B.10) as

$$-\frac{\pi}{\mu}(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}\int_b^\infty \left(bs^2 - \frac{2}{3}b^2 s\right) ds - \frac{\pi}{3b\mu}(\mathbf{f}_0\cdot\mathbf{N})\mathbf{N}\int_0^b s^4\phi^\epsilon(s)ds + \mathcal{O}(\epsilon^{1+\beta}) \tag{B.11}$$

and the second integral of e_2 as

$$\begin{aligned} &-\frac{\pi}{\mu}(\mathbf{f}_0 - (\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_b^\infty \left(s^3 - bs^2 + \frac{1}{3}b^2 s\right) ds \\ &\qquad -\frac{\pi}{3b\mu}(\mathbf{f}_0 - (\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_0^b s^4\phi^\epsilon(s)ds + \mathcal{O}(\epsilon^{1+\beta}). \end{aligned} \tag{B.12}$$

Since $\mathbf{w}$ is a vector of odd functions in ξ_1 and ξ_2, the third integral of e_2 equals to zero. We will now prove that the last integral of e_2 is $\mathcal{O}(\epsilon^{1+\beta/2})$, i.e., we will prove that for $m\geq 0, n\geq 0, m+n=3$, we have

$$-\frac{1}{4\mu}\int_0^{\sqrt{\lambda^2-b^2}} \frac{|1-H_2^\epsilon(\sqrt{\xi^2+b^2})|}{(\xi^2+b^2)^{3/2}}\xi^m b^n \xi d\xi = \mathcal{O}(\epsilon^{1+\beta/2}). \tag{B.13}$$

Similarly to proving (B.8), using (2.12) and the fact that $1-\beta/2>0$, the above integral can be, with C denoting an arbitrary constant independent of ϵ, bounded by

$$\begin{aligned} C\int_0^\infty \frac{|1-H_2^\epsilon(\sqrt{\xi^2+b^2})|}{(\xi^2+b^2)^{3/2}}\xi^{m+\beta/2}b^n d\xi &\leq C\epsilon^{1+\beta/2}\int_0^\infty |1-H_2^1(r)|r^{\beta/2}dr \\ &\leq C\epsilon^{1+\beta/2}. \end{aligned} \tag{B.14}$$

Combining (B.11), (B.12), (B.14), and (B.9), we have

$$E^\epsilon(\mathbf{x}) = -\frac{\pi}{\mu}(\mathbf{f}_0 - (\mathbf{f}_0\cdot\mathbf{N})\mathbf{N})\int_b^\infty s(s-b)^2\phi^\epsilon(s)ds + \mathcal{O}(\epsilon^{1+\beta/2}).$$

This completes the proof the Lemma 2.2.

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