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Approximation of Derivative for a Singularly Perturbed Second-Order ODE of Robin Type with Discontinuous Convection Coefficient and Source Term

R. Mythili Priyadharshini and N. Ramanujam*

Department of Mathematics, Bharathidasan University, Tiruchirappalli – 620 024, Tamilnadu, India.

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Abstract. In this paper, a singularly perturbed Robin type boundary value problem for second-order ordinary differential equation with discontinuous convection coefficient and source term is considered. A robust-layer-resolving numerical method is proposed. An ε-uniform global error estimate for the numerical solution and also to the numerical derivative are established. Numerical results are presented, which are in agreement with the theoretical predictions.

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Key words: Singular perturbation problem, piecewise uniform mesh, discrete derivative, discontinuous convection coefficient, Robin boundary conditions, discontinuous source term.

1. Introduction

The theory of singular perturbation is not a settled direction in mathematics and the path of its development is a dramatic one. In the intensive development of science and technology, many practical problems, such as the mathematical boundary layer theory or approximation of solution of various problems described by differential equations involving; large or small parameters, become more complex. In some problems, the perturbations are operative over a very narrow region across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in Fluid Mechanics, edge layers in Solid Mechanics, skin layers in Electrical Applications and shock layers in Fluid and Solid Mechanics.

Various methods for the numerical solution of problem involving singularly perturbed second-order ordinary differential equations with non-smooth data (discontinuous source

*Corresponding author. *Email addresses:* `mythiliroy777@yahoo.co.in` (R. M. Priyadharshini), `matram2k3@yahoo.com` (N. Ramanujam)

term/convection coefficient) using special piecewise uniform meshes (Shishkin mesh and Bakhvalov mesh) have been considered widely in the literature (see [11–15] and references therein). While many finite difference methods have been proposed to approximate such solutions, there has been much less research into the finite-difference approximation of their derivatives, even though such approximations are desirable in certain applications. It should be noted that for convection-diffusion problems, the attainment of high accuracy in a computed solution does not automatically lead to good approximation of derivatives of the true solution.

Li, Shishkin and Shishkina [5] obtained an approximation of the solution and its derivative for the singularly perturbed Black-Scholes equation with non-smooth initial data. In [2–4, 6], approximations to the normalized derivative $\varepsilon(\partial/\partial x)u(x,t)$, that is, the first order spatial derivative multiplied by the parameter ε, were considered. In [1], for singularly perturbed convection-diffusion problems with continuous convection coefficient and source term estimates for numerical derivatives have been derived. Here the scaled derivative is taken on whole domain where as Kopteva and Stynes [8] have obtained approximation of derivatives with scaling in the boundary layer region and without scaling in the outer region. It may be noted that the source term and convection coefficient are smooth for the problem considered in [1,8]. Priyadharshini and Ramanujam [9] estimated the scaled derivative for a singularly perturbed reaction-convection-diffusion problem with two parameters. To the best of our knowledge, it seems no work has been reported in the literature for finding approximation to scaled derivatives of the solution for problems having discontinuous convection coefficient for both upwind and hybrid finite difference schemes on Shishkin mesh.

Motivated by the works given in [10, 12], the present paper considers singularly perturbed second order ordinary differential equation with discontinuous coefficients. Since derivatives are related to flux or drag in physical and chemical applications, we obtain parameter-uniform approximations not only to the solution but also to its derivatives. Thus in this paper, motivated by the work of [8], bounds on the errors in approximating the first derivative of the solution with weight in the fine mesh where as without weight in the coarse mesh are obtained.

Note: Through out this paper, C denotes a generic constant (sometimes subscripted) is independent of the singular perturbation parameter ε and the dimension of the discrete problem N. Let $y : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}$. The appropriate norm for studying the convergence of numerical solution to the exact solution of a singular perturbation problem is the supremum norm $\| y \| = \sup_{x\in D} |y(x)|$.

2. Continuous problem

A singularly perturbed convection-diffusion equation in one dimension with discontinuous convection coefficient and source term is considered on $\Omega = (0,1)$. A single discontinuity is assumed to occur at a point $d \in \Omega$. It is convenient to introduce the notation

$\Omega^- = (0,d)$ and $\Omega^+ = (d,1)$ and the jump at d in any function with

$$[w](d) = w(d+) - w(d-).$$

The corresponding problem is

$$(P_\varepsilon): \begin{cases} \text{Find} \quad u \in Y \equiv C^1(\overline{\Omega}) \cap C^2(\Omega^- \cup \Omega^+) \quad \text{such that} \\ Lu(x) \equiv \varepsilon u''(x) + a(x)u'(x) = f(x), \quad \forall\, x \in \Omega^- \cup \Omega^+, \\ B_0 u(0) \equiv \beta_1 u(0) - \varepsilon \beta_2 u'(0) = A, \\ B_1 u(1) \equiv \gamma_1 u(1) + \gamma_2 u'(1) = B, \\ a(x) \le -\alpha_1 < 0, \quad \text{for } x < d; \quad a(x) \ge \alpha_2 > 0, \quad \text{for } x > d, \\ \beta_1 - \varepsilon\beta_2 \ge 1, \quad \alpha_1 \ge 1, \quad \gamma_1 - \gamma_2 \ge 1, \\ |[a](d)| \le C, \quad |[f](d)| \le C, \end{cases} \tag{2.1}$$

where $0 < \varepsilon \ll 1$ is a small positive parameter, $\overline{\Omega} = [0,1]$, $d \in \Omega$, $\Omega^- = (0,d)$, $\Omega^+ = (d,1)$. For the functions $a(x)$ and $f(x)$ we assume they are sufficiently smooth on $\Omega^- \cup \Omega^+$ and has a jump discontinuity at $x = d$. Further it is assumed that $f(x)$ and $a(x)$ has right and left limits at $x = d$. We denote the jump at d in any function with $[w](d) = w(d+) - w(d-)$.

In the following, the maximum principle for (2.1) is established. Then using this principle, a stability result is derived.

Theorem 2.1. *Suppose that a function $u \in Y$ satisfies*

$$B_0 u(0) \ge 0, \;\; B_1 u(1) \ge 0, \;\; Lu(x) \le 0, \quad \forall\, x \in \Omega^- \cup \Omega^+ \quad \textit{and} \;\; [u'](d) \le 0,$$

then $u(x) \ge 0$, $\forall\, x \in \overline{\Omega}$.

Proof. Define $s(x)$ as

$$s(x) = \begin{cases} 1/2 + x/8 - d/8, & x \in \Omega^- \cup \{0, d\}, \\ 1/2 - x/4 + d/4, & x \in \Omega^+ \cup \{1\}, \end{cases}$$

where $s \in Y$. Then $s(x) > 0,\ x \in \overline{\Omega}$, and

$$\begin{aligned} &B_0 s(0) \equiv \beta_1 s(0) - \varepsilon\beta_2 s'(0) = \beta_1(1/2 - d/8) - \varepsilon\beta_2/8 > 0, \\ &Ls(x) \equiv \varepsilon s''(x) + a(x)s'(x) = \begin{cases} \frac{a(x)}{8} \\ \frac{-a(x)}{4} \end{cases} < \begin{cases} 0, & x \in \Omega^-, \\ 0, & x \in \Omega^+, \end{cases} \\ &B_1 s(1) \equiv \gamma_1 s(1) + \gamma_2 s'(1) = \gamma_1(1/2 - 1/4 + d/4) - \gamma_2/4 > 0. \end{aligned}$$

We define

$$\mu = \max\left\{ \max_{x \in \overline{\Omega}} \left(\frac{-u}{s} \right)(x) \right\}.$$

Assume that the theorem is not true. Then $\mu > 0$ and there exists a point $x_0 \in \overline{\Omega}$, such that $(u+\mu s)(x_0) = 0$. Also,

$$(u+\mu s)(x) \geq 0, \quad \text{for } x \in \overline{\Omega}.$$

We consider the following cases:

Case (i): $(u+\mu s)(x_0) = 0$, for $x_0 = 0$. It implies that $(u+\mu s)$ attains its minimum at x_0. Therefore,

$$0 < B_0(u+\mu s)(x_0) = \beta_1(u+\mu s)(x_0) - \varepsilon\beta_2(u+\mu s)'(x_0) \leq 0,$$

which is a contradiction.

Case (ii):$(u+\mu s)(x_0) = 0$, for $x_0 \in (\Omega^- \cap \Omega^+)$. It implies that $(u+\mu s)$ attains its minimum at x_0. Therefore,

$$0 > L(u+\mu s)(x_0) = \varepsilon(u+\mu s)''(x_0) + a(x_0)(u+\mu s)'(x_0) \geq 0,$$

which is a contradiction.

Case (iii): $(u+\mu s)(x_0) = 0$, for $x_0 = 1$. It implies that $(u+\mu s)$ attains its minimum at x_0. Therefore,

$$0 < B_1(u+\mu s)(x_0) = (u+\mu s)(x_0) + (u+\mu s)'(x_0) \leq 0,$$

which is a contradiction.

For $x_0 = d$, we have

$$0 \leq [u+\mu s]'(d) = [u'](d) + \mu[s'](d) \leq \mu\left[-\frac{1}{4} - \frac{1}{8}\right] < 0,$$

which is a contradiction. Hence the proof of the theorem is complete. □

Lemma 2.1. *If $u \in Y$, then*

$$\| u \| \leq C \max\left\{|B_0 u(0)|, |B_1 u(1)|, \frac{1}{\gamma} \| Lu \|_{\Omega^- \cup \Omega^+}\right\},$$

where $\gamma = \min\{\alpha_1/d, \alpha_2/(1-d)\}$.

Proof. Set $C_1 = C\max\{|B_0 u(0)|, |B_1 u(1)|, \| Lu \|_{\Omega^- \cup \Omega^+}\}$. Define two functions,

$$w^{\pm}(x) = C_1 s(x) \pm u(x),$$

where

$$s(x) = \begin{cases} 1/2 + x/8 - d/8, & x \in \Omega^- \cup \{0, d\}, \\ 1/2 - x/4 + d/4, & x \in \Omega^+ \cup \{1\}. \end{cases}$$

We observe that

$$\begin{aligned}
&B_0 w^{\pm}(0) > 0, \quad Lw^{\pm}(x) \le 0, \quad \text{for } x \in \Omega^- \cup \Omega^+, \\
&B_1 w^{\pm}(1) \ge 0, \quad [w'^{\pm}](d) < 0.
\end{aligned}$$

Applying Theorem 2.1 to the function $w^{\pm}(x)$, we get $w^{\pm}(x) \ge 0$, for all $x \in \overline{\Omega}$, which completes the proof. □

Consider the following decomposition of the solution $u = v + w$ into a non-layer component v and an interior layer component w. Define the functions v_0 and v_1 respectively by

$$\begin{aligned}
&a(x)v_0'(x) = f(x), && x \in \Omega^- \cup \Omega^+, \\
&\beta_1 v_0(0) = A, && \gamma_1 v_0(1) + \gamma_2 v'(1) = B
\end{aligned}$$

and

$$\begin{aligned}
&a(x)v_1'(x) = -v_0'', && x \in \Omega^- \cup \Omega^+, \\
&\beta_1 v_1(0) - \varepsilon\beta_2 v_1'(0) = \beta_2 v_0'(0), && \gamma_1 v_1(1) + \gamma_2 v_1'(1) = 0.
\end{aligned}$$

We now define the function v as the solution of problem

$$Lv(x) = f(x), \qquad x \in \Omega^- \cup \Omega^+, \tag{2.2}$$

$$\beta_1 v(0) - \varepsilon\beta_2 v'(0) = A, \qquad v(d-) = v_0(d-) + \varepsilon v_1(d-), \tag{2.3}$$

$$v(d+) = v_0(d+) + \varepsilon v_1(d+), \qquad \gamma_1 v(1) + \gamma_2 v'(1) = B. \tag{2.4}$$

Define the function w, which is the layer component of the decomposition, as follows :

$$Lw(x) = 0, \qquad x \in \Omega^- \cup \Omega^+, \tag{2.5}$$

$$\beta_1 w(0) - \varepsilon\beta_2 w'(0) = 0, \qquad [w](d) = -[v](d), \tag{2.6}$$

$$[w'](d) = -[v'](d), \qquad \gamma_1 w(1) + \gamma_2 w'(1) = 0. \tag{2.7}$$

Hence $w(d-) = u(d-) - v(d-)$ and $w(d+) = u(d+) - v(d+)$. Note also that since there is a unique solution to (2.1), then $u = v + w$.

Lemma 2.2. *For each integer k, satisfying $0 \le k \le 3$, the solutions v and w of (2.2)-(2.4) and (2.5)-(2.7) respectively satisfy the following bounds*

$$\begin{aligned}
&\| v \| \le C, \quad \| v^{(k)} \|_{\Omega^- \cup \Omega^+} \le C(1 + \varepsilon^{2-k}), \\
&|[v](d)|, |[v'](d)|, |[v''](d)| \le C
\end{aligned}$$

and

$$|w^{(k)}(x)| \le \begin{cases} C\varepsilon^{-k} e^{-(d-x)\alpha_1/\varepsilon}, & x \in \Omega^-, \\ C\varepsilon^{-k} e^{-(x-d)\alpha_2/\varepsilon}, & x \in \Omega^+. \end{cases}$$

Proof. Using the technique adopted in [10, 12] and applying the argument separately on each of the subintervals Ω^- and Ω^+, the present theorem can be proved. □

3. Discrete problem

A fitted mesh method for the Problem (2.1) is now introduced. On Ω a piecewise uniform mesh of N mesh interval is constructed as follows. The domain $\overline{\Omega}$ is subdivided into the four subintervals $[0,d-\sigma_1]\cup[d-\sigma_1,d]\cup[d,d+\sigma_2]\cup[d+\sigma_2,1]$ for some σ_1,σ_2 that satisfy $0<\sigma_1\le d/2,\quad 0<\sigma_2\le(1-d)/2$. On each subinterval a uniform mesh with $N/4$ mesh-intervals is placed. The interior points of the mesh are denoted by

$$\Omega^N=\Big\{x_i\,:\,0\le i\le\frac{N}{2}-1\Big\}\cup\Big\{x_i\,:\,\frac{N}{2}+1\le i\le N\Big\}.$$

Clearly $x_{N/2}=d$ and $\overline{\Omega}^N=\{x_i\}_0^N$. We now introduce the following notations for the four mesh widths

$$h_1=\frac{4(d-\sigma_1)}{N},\quad h_2=\frac{4\sigma_1}{N},\quad h_3=\frac{4\sigma_2}{N}\quad\text{and}\quad h_4=\frac{4(1-d-\sigma_2)}{N}.$$

It is fitted to the singular perturbation problem (2.1) by choosing σ_1 and σ_2 to be the following functions of N and ε

$$\sigma_1=\min\Big\{\frac{d}{2},\frac{2\varepsilon}{\alpha}\ln N\Big\}\quad\text{and}\quad\sigma_2=\min\Big\{\frac{1-d}{2},\frac{2\varepsilon}{\alpha}\ln N\Big\},$$

where $\alpha=\min\{\alpha_1,\alpha_2\}$. Then the fitted mesh method for the problem (2.1) is

$$(P_\varepsilon^N):\begin{cases}L^NU(x_i)\equiv\varepsilon\delta^2U(x_i)+a(x_i)DU(x_i)=f(x_i),\quad\text{for }\ x_i\in\Omega^N,\\ B_0U(x_0)\equiv\beta_1U(x_0)-\varepsilon\beta_2D^+U(x_0)=A,\\ B_NU(x_N)\equiv\gamma_1U(x_N)+\gamma_2D^-U(x_N)=B,\\ D^-U(x_{N/2})=D^+U(x_{N/2}),\end{cases}\tag{3.1}$$

where

$$\delta^2Z_i=\frac{D^+Z_i-D^-Z_i}{(x_{i+1}-x_{i-1})/2},\qquad DZ_i=\begin{cases}D^-Z_i,\ i<N/2,\\ D^+Z_i,\ i>N/2.\end{cases}$$

Here D^+ and D^- are the standard forward and backward finite difference operators, respectively. Analogous to the continuous results stated in Theorem 2.1 and Lemma 2.1 one can prove the following results.

Theorem 3.1. *Suppose that a mesh function $Z(x_i)$ satisfies*

$$B_0Z(x_0)\ge0,\quad B_NZ(x_N)\ge0,\quad L^NZ(x_i)\le0,\quad\textit{for}\ \ x_i\in\Omega^N$$

and

$$D^+Z(d)-D^-Z(d)\le0.$$

Then $Z(x_i)\ge0$, for all $x_i\in\overline{\Omega}^N$.

Proof. Define $S(x_i)$ as

$$S(x_i) = \begin{cases} \dfrac{1}{2} + \dfrac{x_i}{8} + \dfrac{d}{8}, & x_i \in \overline{\Omega} \cap [0,d], \\ \dfrac{1}{2} - \dfrac{x_i}{4} + \dfrac{d}{4}, & x_i \in \overline{\Omega} \cap (d,1]. \end{cases}$$

Then

$$S(x_i) > 0, \quad x_i \in \overline{\Omega}^N, \quad B_0 S(x_0) \equiv \beta_1 S(0) - \varepsilon\beta_2 D^+ S(0) > 0,$$
$$B_1 S(x_N) \equiv \gamma_1 S(x_N) + \gamma_2 D^- S(x_N) > 0$$

and

$$L^N S(x_i) \equiv \varepsilon\delta^2 S(x_i) + a(x_i) D^+ S(x_i) = \begin{cases} \frac{a(x_i)}{8} \\ \frac{-a(x_i)}{4} \end{cases} < \begin{cases} 0, & x_i \in \Omega^N \cap (0,d), \\ 0, & x_i \in \Omega^N \cap (d,1). \end{cases}$$

We define

$$\mu = \max\left\{ \max_{0\le i\le N} \left(\frac{-Z}{S}\right)(x_i) \right\}.$$

Assume that the theorem is not true. Then $\mu > 0$ and $(Z+\mu S)(x_i) = 0$. Further there exists a $i^* \in \{0,1,2,\cdots,N\}$ such that $(Z+\mu S)(x_{i^*}) = 0$ and we consider the following cases:

Case (i): $(Z+\mu S)(x_{i^*}) = 0$, for $i^* = 0$. In this case,

$$\begin{aligned} 0 &\le B_0(Z+\mu S)(x_{i^*}) \\ &= \beta_1(Z+\mu S)(x_{i^*}) - \varepsilon\beta_2 D^+(Z+\mu S)(x_{i^*}) \\ &= -\varepsilon\beta_2 \frac{(Z+\mu S)(x_{i^*+1}) - (Z+\mu S)(x_{i^*})}{x_{i^*+1} - x_{i^*}} < 0, \end{aligned}$$

which is a contradiction.

Case (ii): $(Z+\mu S)(x_{i^*}) = 0$, for $0 < i^* < N$. In this case,

$$\begin{aligned} 0 &\ge L^N(Z+\mu S)(x_{i^*}) \\ &= \varepsilon\delta^2(Z+\mu S)(x_{i^*}) + a(x_{i^*}) D^+(Z+\mu S)(x_{i^*}) > 0, \end{aligned}$$

which is a contradiction.

Case (iii): $(Z+\mu S)(x_{i^*}) = 0$, for $i^* = N$. Therefore,

$$\begin{aligned} 0 &\le B_N(Z+\mu S)(x_{i^*}) \\ &= \gamma_1(Z+\mu S)(x_{i^*}) + \gamma_2 D^-(Z+\mu S)(x_{i^*}) \\ &= \gamma_2 \frac{(Z+\mu S)(x_{i^*}) - (Z+\mu S)(x_{i^*-1})}{x_{i^*} - x_{i^*-1}} < 0, \end{aligned}$$

which is a contradiction.

For $i^* = N/2$, we have

$$0 \geq D^+(Z+\mu S)(x_{i^*}) - D^-(Z+\mu S)(x_{i^*}) = \frac{(Z+\mu S)(x_{i^*+1})}{x_{i^*+1}-x_{i^*}} + \frac{(Z+\mu S)(x_{i^*-1})}{x_{i^*}-x_{i^*-1}} > 0,$$

which is a contradiction. Hence the theorem is proved. □

To bound the nodal error $|(U-u)(x_i)|$, we define mesh functions V_L and V_R, which approximate v respectively to the left and right of the point of discontinuity $x = d$. Then, we construct mesh functions W_L and W_R, so that the amplitude of the jump $W_R(d) - W_L(d)$ is determined by the size of the jump $|[v](d)|$. Also W_L and W_R, are sufficiently small away from the interior layer region. Using these mesh functions the nodal error $|(U-u)(x_i)|$ is then bounded separately outside and inside the layer. Define the mesh functions V_L and V_R to be the solutions of the following discrete problems

$$L^N V_L(x_i) = f(x_i), \qquad \text{for } x_i \in \Omega^N \cap \Omega^-, \tag{3.2}$$

$$B_0 V_L(x_0) \equiv \beta_1 V_L(x_0) - \varepsilon\beta_2 D^+ V_L(x_0) = A, \qquad V_L(x_{N/2}) = v(d-) \tag{3.3}$$

and

$$L^N V_R(x_i) = f(x_i), \qquad \text{for } x_i \in \Omega^N \cap \Omega^+, \tag{3.4}$$

$$V_R(x_{N/2}) = v(d+), \qquad B_N V_R(x_N) \equiv \gamma_1 V_R(x_N) + \gamma_2 D^- V_R(x_N) = B. \tag{3.5}$$

Now, we define the mesh functions W_L and W_R to be the solutions of the following system of finite difference equations

$$L^N W_L(x_i) = 0, \quad \text{for } x_i \in \Omega^N \cap \Omega^-, \tag{3.6}$$

$$L^N W_R(x_i) = 0, \quad \text{for } x_i \in \Omega^N \cap \Omega^+, \tag{3.7}$$

$$\beta_1 W_L(x_0) - \varepsilon\beta_2 D^+ W_L(x_0) = 0, \quad \gamma_1 W_R(x_N) + \gamma_2 D^- W_R(x_N) = 0, \tag{3.8}$$

$$W_R(x_{N/2}) + V_R(x_{N/2}) = W_L(x_{N/2}) + V_L(x_{N/2}), \tag{3.9}$$

$$D^+ W_R(x_{N/2}) + D^+ V_R(x_{N/2}) = D^- W_L(x_{N/2}) + D^- V_L(x_{N/2}). \tag{3.10}$$

Now, we can define $U(x_i)$ to be

$$U(x_i) = V(x_i) + W(x_i) = \begin{cases} V_L(x_i) + W_L(x_i), & \text{for } x_i \in \{0\} \cup (\Omega^N \cap \Omega^-), \\ V_L(x_i) + W_L(x_i) = V_R(x_i) + W_R(x_i), & \text{for } x_i = d, \\ V_R(x_i) + W_R(x_i), & \text{for } x_i \in (\Omega^N \cap \Omega^+) \cup \{1\}. \end{cases} \tag{3.11}$$

Lemma 3.1. *At each mesh points $x_i \in \overline{\Omega}^N \setminus \{d\}$, the smooth component of the error satisfies the estimate*

$$|(V-v)(x_i)| \leq \begin{cases} C(d+x_i)N^{-1}, & \textit{for } x_i \in \{0\} \cup (\Omega^N \cap \Omega^-), \\ C(2-x_i)N^{-1}, & \textit{for } x_i \in (\Omega^N \cap \Omega^+) \cup \{1\}. \end{cases} \tag{3.12}$$

Proof. We have the inequalities

$$\begin{aligned} |\, B_0(V-v)(x_0)\,| &= |\beta_1(V-v)(x_0)-\beta_2\varepsilon D^+(V-v)(x_0)| \\ &\le C\beta_2\varepsilon(x_{i+1}-x_i)\parallel v^{(2)}\parallel\le CN^{-1} \end{aligned}$$

and

$$\begin{aligned} |\, B_N(V-v)(x_N)\,| &= |\gamma_1(V-v)(x_N)+\gamma_2 D^-(V-v)(x_N)| \\ &\le C\gamma_2(x_i-x_{i-1})\parallel v^{(2)}\parallel\le CN^{-1}. \end{aligned}$$

By standard local truncation error estimate and Lemma 2.2, we have

$$|L^N(V-v)(x_i)|\le CN^{-1}.$$

Using the two mesh functions

$$\Psi^{\pm}(x_i)=\phi(x_i)\pm(V-v)(x_i),$$

where

$$\phi(x_i)=\begin{cases} C(d+x_i)N^{-1}, & \text{for } x_i=0, \\ \dfrac{Cx_iN^{-1}}{\alpha_1 d}, & \text{for } x_i\in\Omega^N\cap\Omega^-, \\ \dfrac{C(1-x_i)N^{-1}}{\alpha_2(1-d)}, & \text{for } x_i\in\Omega^N\cap\Omega^+, \\ C(2-x_i)N^{-1}, & \text{for } x_i=1. \end{cases}$$

We have

$$B_0\Psi^{\pm}(x_0)\ge\beta_1CN^{-1}-\varepsilon\beta_2CN^{-1}\pm CN^{-1}\ge 0,$$

and

$$L^N\Psi^{\pm}(x_i)\le-\alpha_1CN^{-1}\pm CN^{-1}\le 0,\quad \text{for } x_i\in\Omega^N\cap\Omega^-.$$

Similarly,

$$L^N\Psi^{\pm}(x_i)\le 0,\quad \text{for } x_i\in\Omega^N\cap\Omega^+,$$

and

$$\begin{aligned} &B_N\Psi^{\pm}(x_N)\ge\gamma_1CN^{-1}-\gamma_2CN^{-1}\pm CN^{-1}\ge 0, \\ &D^+\Psi^{\pm}(x_{N/2})-D^-\Psi^{\pm}(x_{N/2})=D^+\phi(x_{N/2})-D^-\phi(x_{N/2})<0. \end{aligned}$$

Applying Theorem 3.1, we get $\Psi^{\pm}(x_i)\ge 0$, for all $x_i\in\overline{\Omega}^N$, which completes the proof. □

Theorem 3.2. *Let w be the solution of* (2.5)-(2.7) *and W the corresponding numerical solution of* (3.6)-(3.10). *Then at each mesh point $x_i\in\overline{\Omega}^N$, we have*

$$|(W-w)(x_i)|\le CN^{-1}(\ln N)^2.$$

Proof. First we consider the case $\sigma = \sigma_1 = \sigma_2 = \varepsilon\alpha^{-1}\ln N$. Since $|U(x_{N/2})| \le C$ and with (3.12), we can easily deduce that

$$|W_L(x_{N/2})| \le C, \quad |W_R(x_{N/2})| \le C.$$

Using the arguments in [10], for $x_i \le d-\sigma$ and $x_i \ge d+\sigma$ respectively we have

$$\begin{aligned}|W_L(x_i)| &\le |W_L(x_{N/2})|N^{-2} \le CN^{-2},\\ |W_R(x_i)| &\le |W_R(x_{N/2})|N^{-2} \le CN^{-2}.\end{aligned}$$

Thus for $x_i \le d-\sigma$,

$$|(W_L - w)(x_i)| \le |W_L(x_i)| + |w(x_i)| \le CN^{-2} + Ce^{-\alpha_1\sigma/\varepsilon} \le CN^{-2}. \tag{3.13}$$

Similarly for $x_i \ge d+\sigma$, we have

$$|(W_R - w)(x_i)| \le |W_R(x_i)| + |w(x_i)| \le CN^{-2} + Ce^{-\alpha_2\sigma/\varepsilon} \le CN^{-2}. \tag{3.14}$$

For $i = N/4+1,\cdots,N/2-1$, by standard local truncation error estimate and Lemma 2.2, we have

$$|L^N(W_L - w)(x_i)| \le \varepsilon h_2|w^{(3)}(x_i)| + h_2|w^{(2)}(x_i)| \le \frac{Ch_2}{\varepsilon^2}.$$

Similarly for $i = N/2+1,\cdots,3N/4-1$, we obtain

$$|L^N(W_R - w)(x_i)| \le \varepsilon h_3|w^{(3)}(x_i)| + h_3|w^{(2)}(x_i)| \le \frac{Ch_3}{\varepsilon^2}.$$

At the mesh point $x_{N/2} = d$, let $h = \max\{h_2, h_3\}$. Thus

$$\begin{aligned}&\left|(D^+ - D^-)(W - w)(x_{N/2})\right| = \left|(D^+ - D^-)w(x_{N/2})\right|\\ \le& \left|\Big(D^+ - \frac{d}{dx}\Big)w(x_{N/2})\right| + \left|\Big(D^- - \frac{d}{dx}\Big)w(x_{N/2})\right|\\ \le& \frac{1}{2}h_3\left|w^{(2)}(x_i)\right| + \frac{1}{2}h_2\left|w^{(2)}(x_i)\right| \le \frac{Ch}{\varepsilon^2}.\end{aligned}$$

Consider the discrete barrier functions $\Phi^{\pm}(x_i) = \Psi(x_i) \pm (W-w)(x_i)$, where

$$\Psi(x_i) = CN^{-1} + \frac{CN^{-1}\sigma}{\varepsilon^2}\begin{cases}1+\sigma-d+x_i, & x_i \in \Omega^N \cap (d-\sigma, d)\\ 1+d+\sigma-x_i, & x_i \in \Omega^N \cap (d, d+\sigma).\end{cases}$$

We have

$$\begin{aligned}&B_0\Psi(x_{N/4}) = \beta_1\Big(CN^{-1} + \frac{CN^{-1}\sigma}{\varepsilon^2}\Big) - \beta_2\frac{CN^{-1}\sigma}{\varepsilon} > 0,\\ &L^N\Psi(x_i) \le -\alpha_1\frac{CN^{-1}\sigma}{\varepsilon^2} < 0, \quad \text{for} \quad x_i \in \Omega^N \cap (d-\sigma, d).\end{aligned}$$

Similarly, $L^N\Psi(x_i) < 0$, for $x_i \in \Omega^N \cap (d, d+\sigma)$,

$$B_N\Psi(x_{3N/4}) = \gamma_1\Big(CN^{-1} + \frac{CN^{-1}\sigma}{\varepsilon^2}\Big) - \gamma_2\frac{CN^{-1}\sigma}{\varepsilon^2} > 0$$

and

$$D^+\Psi^\pm(x_{N/2}) - D^-\Psi^\pm(x_{N/2}) < 0.$$

Applying Theorem 3.1 to $\Phi^\pm(x_i)$ over the interval $[d-\sigma_1, d+\sigma_2]$, we get the desired result. Thus, for $x_i \in \overline{\Omega}^N$,

$$|(W-w)(x_i)| \le CN^{-1}(\ln N)^2.$$

Now we consider the case $\sigma_1 = d/2$ and $\sigma_2 = (1-d)/2$. In this case $\varepsilon^{-1} \le C\ln N$. We have the inequalities

$$\begin{aligned}&|B_0(W-w)(x_0)| = |\beta_1(W-w)(x_0) - \beta_2\varepsilon D^+(W-w)(x_0)|\\ \le\ &\beta_2\varepsilon(x_{i+1}-x_i)|w''(x_i)| \le CN^{-1}\ln N\end{aligned}$$

and

$$\begin{aligned}&|B_N(W-w)(x_N)| = |\gamma_1(W-w)(x_N) + \gamma_2 D^-(W-w)(x_N)|\\ \le\ &\gamma_2(x_i - x_{i-1})|w''(x_i)| \le CN^{-1}(\ln N)^2.\end{aligned}$$

By standard local truncation error estimate and Lemma 2.2, we have

$$|L^N(W-w)(x_i)| \le CN^{-1}(\ln N)^2.$$

Consider the mesh functions

$$\Psi^\pm(x_i) = \phi(x_i) \pm (W-w)(x_i),$$

where

$$\phi(x_i) = CN^{-1}(\ln N)^2\begin{cases}1+x_i, & x_i \in \Omega^N \cap [0,d),\\ 2-x_i, & x_i \in \Omega^N \cap (d,1].\end{cases}$$

We have

$$\begin{aligned}&B_0\Psi^\pm(x_0) \ge \beta_1 CN^{-1}(\ln N)^2 - \varepsilon\beta_2 CN^{-1}(\ln N)^2 \pm CN^{-1}\ln N > 0,\\ &L^N\Psi^\pm(x_i) \le -\alpha_1 CN^{-1}(\ln N)^2 \pm CN^{-1}(\ln N)^2 \le 0, \quad \text{for } x_i \in \Omega^N \cap (d-\sigma, d).\end{aligned}$$

Similarly, $L^N\Psi^\pm(x_i) \le 0$, for $x_i \in \Omega^N \cap \Omega^-$,

$$B_N\Psi^\pm(x_N) \ge \gamma_1 CN^{-1}(\ln N)^2 - \gamma_2 CN^{-1}(\ln N)^2 \pm CN^{-1}(\ln N)^2 > 0$$

and

$$D^+\Psi^\pm(x_{N/2}) - D^-\Psi^\pm(x_{N/2}) < 0.$$

Applying Theorem 3.1 to $\Psi^\pm(x_i)$ over the entire domain, we get

$$|(W-w)(x_i)| \le CN^{-1}(\ln N)^2, \quad \text{for } x_i \in \overline{\Omega}^N$$

which is the desired result. $\square$

Theorem 3.3. *Let u be the solution of Problem* (2.1) *and U be the solution of the corresponding discrete Problem* (3.1)*. Then we have*

$$\| U - u \| \leq CN^{-1}(\ln N)^2.$$

Proof. Proof follows immediately, if one applies the above Lemmas 3.1 and 3.2 to $U - u = (V - v) + (W - w)$. □

Remark 3.1. Following the procedure adopted in [1, §3.5] and applying it separately on the intervals $[0, d]$ and $[d, 1]$, one can extend the above result to obtain the global error bound

$$\sup_{0<\varepsilon\leq 1} \| \overline{U} - u \| \leq CN^{-1}(\ln N)^2,$$

where $\overline{U}$ is the piecewise linear interpolant of U on $\overline{\Omega}^N$.

4. Analysis on derivative estimate

In this section, we give an ε-uniform error estimate between the scaled derivative of the continuous solution and the corresponding numerical solution in the fine mesh region. Further, in the coarse mesh, an estimate is obtained without scaling the derivative. We note that the errors $e(x_i) \equiv U(x_i) - u(x_i)$, satisfy the equations

$$[\varepsilon\delta^2 + a(x_i)D^+]e(x_i) = \text{truncation error},$$

where, by Theorem 3.3, $e(x_i) = \mathcal{O}(N^{-1}(\ln N)^2)$. In the proofs of the following lemmas and theorems, we use the above equations. Hence the analysis carried out in [1, §3.5] can be applied immediately with a slight modifications where ever necessary. Therefore, proofs for some lemmas are omitted; for some of the them short proves are given.

Lemma 4.1. *At each mesh point $x_i \in \Omega^N$ and all $x \in \bar{\Omega}_i = [x_{i-1}, x_i]$, we have*

$$\begin{aligned}
&|D^- u(x_i) - u'(x)| \leq CN^{-1}, && \text{for } x_i \leq d - \sigma_1,\\
&|\varepsilon(D^- u(x_i) - u'(x))| \leq CN^{-1}\ln N, && \text{for } x_i \in (d - \sigma_1, d),\\
&|\varepsilon(D^+ u(x_i) - u'(x))| \leq CN^{-1}\ln N, && \text{for } x_i \in (d, d + \sigma_2),\\
&|D^+ u(x_i) - u'(x)| \leq CN^{-1}, && \text{for } x_i \geq d + \sigma_2,
\end{aligned}$$

where $u(x)$ is the solution of (2.1).

Lemma 4.2. *At each mesh point $x_i \in \Omega^N$, we have*

$$\begin{aligned}
&\max_{0<i\leq N/4} |D^-(V_L - v)(x_i)| \leq CN^{-1},\\
&\max_{N/4<i\leq N/2} |\varepsilon(D^-(V_L - v)(x_i))| \leq CN^{-1},\\
&\max_{N/2<i\leq 3N/4} |\varepsilon(D^+(V_R - v)(x_i))| \leq CN^{-1},\\
&\max_{3N/4<i\leq N} |D^+(V_R - v)(x_i)| \leq CN^{-1},
\end{aligned}$$

where v and V_L^N, V_R^N are the solutions of (2.2)-(2.4) *and* (3.2)-(3.5) *respectively.*

Proof. We denote the error and the local truncation error respectively at each mesh point by

$$e(x_i) = V(x_i) - v(x_i) \quad \text{and} \quad \tau(x_i) = L^N e(x_i).$$

First, we prove that for all i, $N/2 \leq i \leq 3N/4 - 1$, $|\varepsilon D^+ e_i| \leq CN^{-1}$. We have

$$|\varepsilon D^+ e(x_{3N/4-1})| \leq C\varepsilon N^{-1}. \tag{4.1}$$

Now we write $\tau(x_i) = L^N e(x_i)$ in the form

$$\begin{aligned} &\varepsilon D^+ e(x_i) - \varepsilon D^+ e(x_{i-1}) + \frac{1}{2}(x_{i+1} - x_{i-1}) a(x_i) D^+ e(x_i) \\ &= \frac{1}{2}(x_{i+1} - x_{i-1}) \tau(x_i). \end{aligned} \tag{4.2}$$

Summing and rearranging for each i, $N/2 \leq i \leq 3N/4 - 2$, we get

$$\begin{aligned} |\varepsilon D^+ e(x_i)| \leq &\left| \varepsilon D^+ e(x_{3N/4-1})| + \frac{1}{2} \sum_{j=i}^{3N/4-1} (x_{j+1} - x_{j-1}) |\tau(x_j) \right| \\ &+ \left| \frac{1}{2} \sum_{j=i}^{3N/4-1} (x_{j+1} - x_{j-1}) a(x_j) D^+ e(x_j) \right|. \end{aligned}$$

Using the telescopic effect of the last term, $|e(x_i)| \leq CN^{-1}$ and $\| a' \| \leq C$, we get

$$|\varepsilon D^+ (V_R - v)(x_i)| \leq CN^{-1}.$$

Similarly, we can obtain

$$|\varepsilon D^- (V_L - v)(x_i)| \leq CN^{-1}, \quad \text{for } N/4 < i \leq N/2.$$

We can rewrite (4.2) in the form

$$(1 + \rho_j) D^+ e(x_j) = D^+ e(x_{j-1}) + \frac{\rho_j}{a(x_j)} \tau(x_j), \tag{4.3}$$

where

$$\rho_j = a(x_j)(x_{j+1} - x_{j-1})/\varepsilon.$$

Summing the equations (4.3) from $j = 3N/4$ to $j = i < N - 1$ gives

$$|D^+ e(x_i)| \leq |D^+ e(d + \sigma_2)| \frac{(1 + \bar{\rho})^{-(i - \frac{3N}{4} - 1)}}{1 + \rho_i} + CN^{-1} \leq CN^{-1},$$

where $\bar{\rho} = \alpha_2 h_4 / \varepsilon$. For $j = N - 1$,

$$|D^+ e(x_{N-1})| \leq |D^+ e(d + \sigma_2)| \frac{(1 + \bar{\rho})^{-(\frac{N}{4} - 2)}}{1 + \rho_{N-1}} + CN^{-1} \leq CN^{-1}.$$

Similarly for $i \leq N/4$, we get the desired result. $\square$

Lemma 4.3. *Let w and W be the solutions of* (2.5)-(2.7) *and* (3.6)-(3.10) *respectively. Then, we have*

$$\max_{0<i\le N/4} |D^-(W_L - w)(x_i)| \le CN^{-1},$$
$$\max_{N/4<i<N/2} |\varepsilon(D^-(W_L - w)(x_i))| \le CN^{-1}(\ln N)^2$$

and

$$\max_{N/2<i<3N/2} |\varepsilon(D^+(W_R - w)(x_i))| \le CN^{-1}(\ln N)^2,$$
$$\max_{3N/2\le i<N} |D^+(W_R - w)(x_i)| \le CN^{-1}.$$

Proof. Suppose

$$\sigma_1 = 2\varepsilon\alpha^{-1}\ln N, \quad \sigma_2 = 2\varepsilon\alpha^{-1}\ln N.$$

We have

$$|W_L(x_i)| \le CN^{-2}, \quad \text{for } x_i \le d - \sigma_1,$$
$$|W_R(x_i)| \le CN^{-2}, \quad \text{for } x_i \ge d + \sigma_2$$

and $|w(x_i)| \le CN^{-2}$. This implies

$$\max_{0<i\le N/4} |D^-(W_L - w)(x_i)| \le CN^{-1},$$
$$\max_{3N/2\le i<N} |D^+(W_R - w)(x_i)| \le CN^{-1}.$$

For $x_i = d + \sigma_2$, we write

$$L^N W_R(d+\sigma_2) = 0$$

in the form

$$\varepsilon D^+ W_R(x_{3N/4-1}) = (\varepsilon + a(d+\sigma_2)(h_3 + h_4))D^+ W_R(d+\sigma_2) \le CN^{-1}.$$

Similarly one can obtain

$$\varepsilon D^- W_L(x_{N/4+1}) = (\varepsilon - a(\sigma_1)(h_1 + h_2))D^- W_L(\sigma_1) \le CN^{-1}.$$

Let

$$\hat{e}(x_i) = (W_R - w)(x_i), \quad \hat{\tau}(x_i) = L^N \hat{e}(x_i).$$

Then on the interval $[d, 1-\sigma_2)$, we write the equation $\hat{\tau}(x_i) = L^N \hat{e}(x_i)$ in the form

$$\varepsilon D^+ \hat{e}(x_j) - \varepsilon D^+ \hat{e}(x_{j-1}) + a(x_j)(\hat{e}(x_{j+1}) - \hat{e}(x_j)) = h_3 \hat{\tau}(x_j).$$

Summing from $x_j = x_i > d$ to $x_j = d + \sigma_2 - h_3$ and rearranging the resulting equations yield

$$\begin{aligned}\varepsilon D^+\hat{e}(x_i) &= \varepsilon D^+\hat{e}(x_{3N/4-1}) + a(x_{3N/4-1})\hat{e}(x_{3N/4}) - a(x_{i-1})\hat{e}(x_i)\\ &\quad - \sum_{j=i}^{3N/4-1}(a(x_j) - a(x_{j-1}))\hat{e}(x_j) - \varepsilon h_3 \sum_{j=i}^{3N/4-1}\hat{\tau}(x_j)\\ &\leq CN^{-1}(\ln N)^2 + Ch_3\sigma_2\varepsilon^{-2}N^{-1}\sum_{j=i}^{3N/4-1} e^{-(j-1)\alpha_2 h_1/\varepsilon}.\end{aligned}$$

Thus, we have

$$\varepsilon D^+\hat{e}(x_i) \leq CN^{-1}\Big(\ln^2 N + \frac{\sigma_2}{\varepsilon}\frac{\alpha h_3/\varepsilon}{1 - e^{-\alpha h_3/\varepsilon}}\Big).$$

But $y = \alpha h/\varepsilon = 4N^{-1}\ln N$ and $B(y) = y/(1 - e^{-y})$ is bounded and it follows that

$$|\varepsilon D^+\hat{e}(x_i)| \leq CN^{-1}\ln^2 N$$

as desired. Finally over the range $(d - \sigma_1, d]$, we repeat the above procedure to complete the proof. □

Theorem 4.1. *Let u be the solution of* (2.1) *and U the corresponding numerical solution of* (3.1)*. Then for $x \in \overline{\Omega}_i = [x_i, x_{i+1}]$, we have*

$$\begin{aligned}&\| D^-U(x_i) - u' \|_{\overline{\Omega}_i} \leq CN^{-1}, && 0 < i \leq N/4,\\ &\| \varepsilon(D^-U(x_i) - u') \|_{\overline{\Omega}_i} \leq CN^{-1}(\ln N)^2, && N/4 + 1 < i \leq N/2\end{aligned}$$

and

$$\begin{aligned}&\| \varepsilon(D^+U(x_i) - u') \|_{\overline{\Omega}_i} \leq CN^{-1}(\ln N)^2, && N/2 \leq i \leq 3N/4 - 1,\\ &\| D^+U(x_i) - u' \|_{\overline{\Omega}_i} \leq CN^{-1}, && 3N/4 \leq i \leq N - 1.\end{aligned}$$

Proof. The desired results can be obtained using triangular inequality and Lemmas 4.1-4.3. □

Remark 4.1. Since $\bar{U}$ is a linear function in the open interval $\Omega_i = (x_i, x_{i+1})$ for each $i, 0 \leq i \leq N - 1$, we have

$$\varepsilon\bar{U}'(x) = \varepsilon D^+U(x_i) \quad \forall\ x \in \Omega_i.$$

It then follows, from Theorem 4.1, that $\varepsilon\bar{U}'$ is an ε-uniform approximation to $\varepsilon u'(x)$ for each $x \in (x_i, x_{i+1})$. We now show that this approximation can be extended in a natural way to the entire domain $\overline{\Omega}$. We define the piecewise constant function $\bar{D}^+U$ on $[0,1)$ by

$$\varepsilon\bar{D}^+U(x) = \varepsilon D^+U(x_i), \quad \text{for } x \in [x_i, x_{i+1}),\ \ i = 0, \cdots, N - 1$$

and at the point $x = 1$ by

$$\varepsilon \bar{D}^+ U(1) = \varepsilon D^+ U(x_{N-1}).$$

Then, from the above theorem, $\bar{D}^+ U$ is an ε-uniform global approximation to u' in the sense that

$$\sup_{0<\varepsilon\le 1} \| \varepsilon(\bar{D}^+ U - u') \|_{\overline{\Omega}} \le CN^{-1}(\ln N)^2.$$

5. Numerical results

In this section, the following example is given to illustrate the numerical methods discussed in this paper:

$$\begin{aligned} &\varepsilon u''(x) + a(x)u'(x) = f(x), \quad x \in (0,1), \\ &3u(0) - \varepsilon u'(0) = 3, \quad 3u(1) + u'(1) = 2, \end{aligned} \tag{5.1}$$

where

$$a(x) = \begin{cases} -1, & x \le 0.5, \\ 1, & x \ge 0.5; \end{cases} \qquad f(x) = \begin{cases} 1, & x \le 0.5, \\ -1, & x \ge 0.5. \end{cases}$$

For all integers N, satisfying N, $2N \in R_N = [64, 128, 256, 512, 1024]$ and for a finite set of values $\varepsilon \in R_\varepsilon = [2^{-12}, 2^{-1}]$, we compute the maximum pointwise two-mesh differences for the solution and first derivative respectively as

$$E_\varepsilon^N = \| U^N - \overline{U}^{2N} \|_{\Omega^N} \quad \text{and} \quad SD_\varepsilon^N = \| \varepsilon(DU^N - D\bar{U}^{2N}) \|,$$

where U^N and $\overline{U}^{2N}$ denote respectively, the numerical solutions obtained using N and $2N$ mesh intervals and

$$DU_i = \begin{cases} D^- U_i, & i < N/2, \\ D^+ U_i, & i > N/2. \end{cases}$$

From these values the ε-uniform maximum pointwise two-mesh difference

$$E^N = \max_{\varepsilon \in R_\varepsilon} E_\varepsilon^N \quad \text{and} \quad SD^N = \max_{\varepsilon \in R_\varepsilon} SD_\varepsilon^N$$

are formed for each available value of N satisfying N, $2N \in R_N$. Approximations of ε-uniform order of local convergence are defined, for all N, $4N \in R_N$, by

$$p^N = \log_2\left(\frac{E^N}{E^{2N}}\right) \quad \text{and} \quad s^N = \log_2\left(\frac{SD^N}{SD^{2N}}\right).$$

We compute the maximum pointwise two-mesh difference for the derivative of the solution as

$$D_\varepsilon^N = \begin{cases} \max |(D^- U^N - \bar{D}^- U^{2N})(x_i)|, & \text{for } 1 \le i \le N/4, \\ \max |\varepsilon(D^- U^N - \bar{D}^- U^{2N})(x_i)|, & \text{for } N/4+1 \le i \le N/2, \\ \max |\varepsilon(D^+ U^N - \bar{D}^+ U^{2N})(x_i)|, & \text{for } N/2+1 \le i \le 3N/4-1, \\ \max |(D^+ U^N - \bar{D}^+ U^{2N})(x_i)|, & \text{for } 3N/4 \le i \le N-1. \end{cases}$$

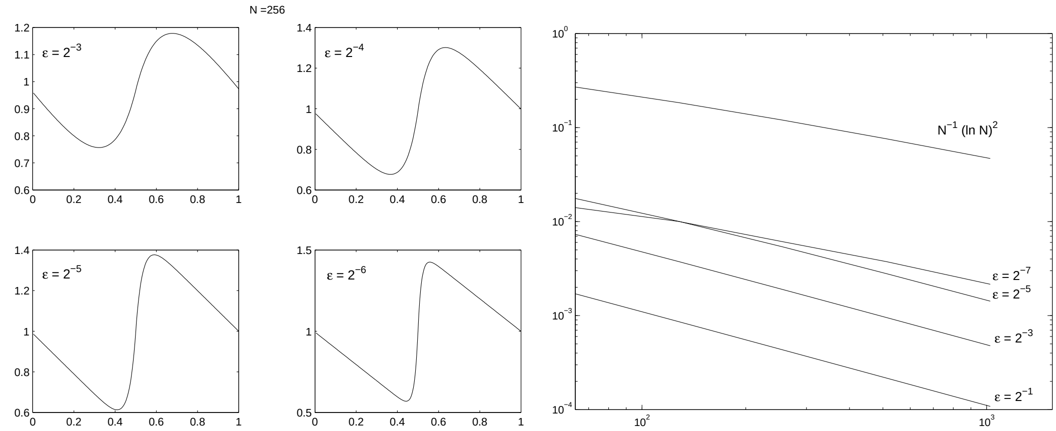

Figure 1: Graphs of the numerical solution and loglog plot of maximum point-wise errors respectively for the solution of problem 5.1 for various values of ε and N.

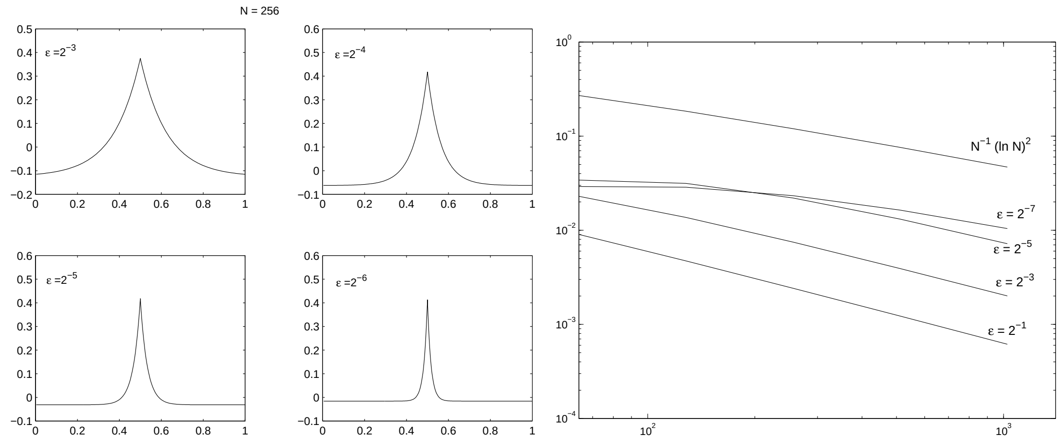

Figure 2: Graphs of the numerical scaled first derivative and loglog plot of maximum point-wise errors respectively for the scaled first derivative of problem 5.1 for various values of ε and N.

From these values the ε-uniform maximum pointwise two-mesh difference $D^N = \max_{\varepsilon\in R_\varepsilon} D^N_\varepsilon$ are formed for each available value of N satisfying $N, 2N \in R_N$. Approximations of ε-uniform order of local convergence are defined, for all N, $4N \in R_N$, by

$$dp^N = \log_2\Big(\frac{D^N}{D^{2N}}\Big).$$

Table 1 presents values of E^N, p^N and SD^N, sp^N for the solution u and the scaled derivative $\varepsilon u'$ throughout the domain. The computed maximum pointwise two-mesh differences D^N and order of local convergence dp^N for the scaled derivative in the fine mesh region and without scaling the derivative in the coarse mesh, are given in Table 2.

Table 1: Values of E^N, SD^N, p^N and s^N for the solution u and the scaled first derivative $\varepsilon u'$.

N	$U-u$		$\varepsilon(DU-u')$	
	Error	Rate	Error	Rate
64	1.7610e-2	8.1164e-1	6.7363e-2	5.8498e-1
128	1.0033e-2	7.1249e-1	4.4908e-2	6.0172e-1
256	6.1228e-3	7.0292e-1	2.9593e-2	6.5291e-1
512	3.7614e-3	8.0051e-1	1.8821e-2	7.3858e-1
1024	2.1596e-3	-	1.1280e-2	-

Table 2: Values of D^N and dp^N for the first derivative of the solution u on $(0, x_{N/4}]$, $(x_{N/4}, d]$, $[d, x_{3N/4})$ and $[x_{3N/4}, 1)$ respectively.

N	D^-U-u'		$\varepsilon(D^-U-u')$		$\varepsilon(D^+U-u')$		D^+U-u'	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
64	1.07	2.3	2.92e-2	0.3	2.92e-2	0.3	1.07	2.3
128	2.14 e-1	2.3	2.86e-2	0.3	2.86e-2	0.3	2.14e-1	2.3
256	4.30e-2	2.3	2.32e-2	0.5	2.32e-2	0.5	4.30e-2	2.3
512	8.74e-3	2.3	1.63e-2	0.7	1.63e-2	0.7	8.74 e-3	2.3
1024	1.76e-3	-	1.04e-2	-	1.04e-2	-	1.76e-3	-

6. Conclusion

A singularly perturbed convection-diffusion problem, with a discontinuous convection coefficient was examined. Due to the discontinuity an interior layer appears in the solution. A finite difference scheme was constructed for solving this problem which generates ε-uniform convergent numerical approximation not only to the solution but also to the scaled first derivative of the solution. The method uses a piecewise uniform mesh, which is fitted to the interior layers and the standard finite difference operator on this mesh. The main theoretical result is the ε-uniform convergence in the supremum norm of the approximations generated by this finite difference method. Numerical results were presented, which are in agreement with the theoretical predictions.

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