Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2010.m99027

Vol. **3**, No. 4, pp. 499-522
November 2010

Alternating Direction Finite Volume Element Methods for Three-Dimensional Parabolic Equations

Tongke Wang*

School of Mathematical Sciences, Tianjin Normal University, Tianjin 300387, China.

Received 30 September 2009; Accepted (in revised version) 18 March 2010

Available online 9 September 2010

Abstract. This paper presents alternating direction finite volume element methods for three-dimensional parabolic partial differential equations and gives four computational schemes, one is analogous to Douglas finite difference scheme with second-order splitting error, the other two schemes have third-order splitting error, and the last one is an extended LOD scheme. The L^2 norm and H^1 semi-norm error estimates are obtained for the first scheme and second one, respectively. Finally, two numerical examples are provided to illustrate the efficiency and accuracy of the methods.

AMS subject classifications: 65M08, 65M12, 65M15

Key words: Three-dimensional parabolic equation, alternating direction method, finite volume element method, error estimate.

1. Introduction

Finite volume element methods (FVEMs) [1–3] or generalized difference methods [4] discretize the integral form of conservation laws of differential equation by choosing linear or high order finite element space as the trial space. The method lies in between finite element method and finite difference method in concept and implementation. In recent years, some literature focused on the error estimates of finite volume element methods, especially for two dimensional problems, see the references [5–13]. Recently, the author [14] combines finite volume element methods and alternating direction methods for two dimensional parabolic differential equations and presents some alternating direction finite volume element schemes. Here, we further extend the method to three-dimensional partial differential equations. As an efficient technique, alternating direction method [15, 16] successfully converts multidimensional problems to a collection of one dimensional problems, which can be solved very easily. Because ADI finite difference methods and alternating direction finite element methods are unconditionally stable and highly efficient, they have been applied in many areas of applied sciences [17, 18]. It is worth mentioning that

*Corresponding author. *Email address:* wangtke@sina.com (T. Wang)

Professor Douglas et al. [16] presented an LOD finite difference scheme with third-order perturbation term. In this paper, we write the finite volume element method as tensor product form by perturbing the differential equations, so we can convert the method to a series of one dimensional problems. We give four kinds of alternating direction schemes, the first one is similar to Douglas scheme [15] with second-order splitting error, the second and third are also Douglas schemes with third order splitting error [16]. The last one is an extended locally one dimensional (LOD) scheme [19]. It is worth mentioning that the LOD scheme in this paper completely decomposes multidimensional problems to a collection of one dimensional problems and the method is valid for nonhomogeneous differential equations with nonhomogeneous boundary conditions.

The remainder of the article is outlined as follows. In Section 2, we obtain a class of finite volume element method with tensor product form by perturbing the differential equation. We present four kinds of computation schemes. In Section 3, taking the fist scheme and the second one as two examples, we further analyze these schemes. By defining discrete L^2 norm and H^1 semi-norm, we obtain L^2 norm and H^1 semi-norm error estimates for the first scheme and the second one. Finally, in Section 4, we provides two numerical examples to illustrate the effectiveness of the four schemes.

Throughout the article C will denote a generic (sometimes large) constant and ϵ a generic small one independent of mesh-size h, where C and ϵ can have different values in different places.

2. Alternating direction FVEM for 3D parabolic equations

Consider the following three-dimensional parabolic problem on domain $\Omega = [0,1]^3$

$$\frac{\partial u}{\partial t} - \Delta u = f(x,y,z,t), \quad (x,y,z) \in \Omega, \quad t \in (0,T], \tag{2.1}$$

$$u|_{\partial\Omega} = 0, \quad u(x,y,z,0) = u_0(x,y,z), \tag{2.2}$$

where $f(x,y,z,t)$ is sufficiently smooth.

First, give a cuboidal partition Q_h for Ω and the nodes are denoted by (x_i, y_j, z_k), $i(j,k) = 0,1,\cdots,N_x(N_y,N_z)$. Let

$$h_i^x = x_i - x_{i-1}, \quad h_j^y = y_j - y_{j-1}, \quad h_k^z = z_k - z_{k-1},$$
$$h_x = \max_{1\le i\le N_x} h_i^x, \quad h_y = \max_{1\le j\le N_y} h_j^y, \quad h_z = \max_{1\le k\le N_z} h_k^z, \quad h = \max(h_x, h_y, h_z).$$

Further let

$$x_{i-\frac{1}{2}} = x_i - \frac{1}{2}h_i^x, \quad x_{i+\frac{1}{2}} = x_i + \frac{1}{2}h_{i+1}^x,$$
$$y_{j-\frac{1}{2}} = y_j - \frac{1}{2}h_j^y, \quad y_{j+\frac{1}{2}} = y_j + \frac{1}{2}h_{j+1}^y,$$
$$z_{k-\frac{1}{2}} = z_k - \frac{1}{2}h_k^z, \quad z_{k+\frac{1}{2}} = z_k + \frac{1}{2}h_{k+1}^z.$$

Then

$$V_{ijk} = \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right] \times \left[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}\right] \times \left[z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}}\right]$$

is a control volume or dual element of node (x_i, y_j, z_k). All control volumes constitute the dual partition Q_h^* of domain Ω. Next, make a uniform partition for $[0, T]$ with time step Δt and denote by $t^n = n\Delta t$. Integrate (2.1) over $\left[t^{n-1}, t^n\right]$, then by trapezoidal rule, we have

$$\frac{u^n - u^{n-1}}{\Delta t} - \frac{\Delta u^n + \Delta u^{n-1}}{2} = \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} f(x,y,z,t)dt + R_1^{n-\frac{1}{2}}, \tag{2.3}$$

where

$$R_1^{n-\frac{1}{2}} = \frac{1}{2\Delta t}\int_{t^{n-1}}^{t^n} \Delta u_{tt}\left(\theta_1(t)\right)(t - t^n)\left(t - t^{n-1}\right)dt. \tag{2.4}$$

Denote by

$$dV = dxdydz, \quad \partial V_{ij} = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}],$$

etc.. Integrate (2.3) over V_{ijk} $(i(j,k) = 1, \cdots, N_x - 1(N_y - 1, N_z - 1))$. Then by Gauss formula for diffusive term, the conservative integral form of (2.3) reads, finding $u^n \in H_0^1(\Omega)$, such that

$$\int_{V_{ijk}} \frac{u^n - u^{n-1}}{\Delta t}dV + \frac{1}{2}B_{ijk}(u^n + u^{n-1}) = \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\int_{V_{ijk}} f dVdt + \int_{V_{ijk}} R_1^{n-\frac{1}{2}}dV, \tag{2.5}$$

where

$$\begin{aligned} B_{ijk}(u) = &\int_{\partial V_{jk}}\left[\frac{\partial u}{\partial x}(x_{i-\frac{1}{2}}) - \frac{\partial u}{\partial x}(x_{i+\frac{1}{2}})\right]dydz + \int_{\partial V_{ki}}\left[\frac{\partial u}{\partial y}(y_{j-\frac{1}{2}}) - \frac{\partial u}{\partial y}(y_{j+\frac{1}{2}})\right]dzdx \\ &+ \int_{\partial V_{ij}}\left[\frac{\partial u}{\partial z}(z_{k-\frac{1}{2}}) - \frac{\partial u}{\partial z}(z_{k+\frac{1}{2}})\right]dxdy. \end{aligned}$$

In order to derive alternating direction schemes from (2.3), we need to add some necessary perturbing terms to (2.3), which are

$$\frac{\Delta t}{4}\left(\frac{\partial^4}{\partial y^2 \partial z^2} + \frac{\partial^4}{\partial z^2 \partial x^2} + \frac{\partial^4}{\partial x^2 \partial y^2}\right)(u^n - u^{n-1}), \quad \frac{\Delta t^2}{8}\frac{\partial^6}{\partial x^2 \partial y^2 \partial z^2}(u^n - u^{n-1}).$$

These terms also need to be integrated over the control volume V_{ijk}. For convenience, denote by

$$
\begin{aligned}
D_{ijk}(u)=&\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}}\left[\frac{\partial^2 u}{\partial y\partial z}(x,y_{j+\frac{1}{2}},z_{k+\frac{1}{2}})-\frac{\partial^2 u}{\partial y\partial z}(x,y_{j-\frac{1}{2}},z_{k+\frac{1}{2}})\right.\\
&\left.-\frac{\partial^2 u}{\partial y\partial z}(x,y_{j+\frac{1}{2}},z_{k-\frac{1}{2}})+\frac{\partial^2 u}{\partial y\partial z}(x,y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})\right]dx\\
&+\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}}\left[\frac{\partial^2 u}{\partial z\partial x}(x_{i+\frac{1}{2}},y,z_{k+\frac{1}{2}})-\frac{\partial^2 u}{\partial z\partial x}(x_{i-\frac{1}{2}},y,z_{k+\frac{1}{2}})\right.\\
&\left.-\frac{\partial^2 u}{\partial z\partial x}(x_{i+\frac{1}{2}},y,z_{k-\frac{1}{2}})+\frac{\partial^2 u}{\partial z\partial x}(x_{i-\frac{1}{2}},y,z_{k-\frac{1}{2}})\right]dy\\
&+\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}}\left[\frac{\partial^2 u}{\partial x\partial y}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},z)-\frac{\partial^2 u}{\partial x\partial y}(x_{i-\frac{1}{2}},y_{j+\frac{1}{2}},z)\right.\\
&\left.-\frac{\partial^2 u}{\partial x\partial y}(x_{i+\frac{1}{2}},y_{j-\frac{1}{2}},z)+\frac{\partial^2 u}{\partial x\partial y}(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z)\right]dz,\\
W_{ijk}(u)=&\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},z_{k+\frac{1}{2}})-\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i-\frac{1}{2}},y_{j+\frac{1}{2}},z_{k+\frac{1}{2}})\\
&-\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i+\frac{1}{2}},y_{j-\frac{1}{2}},z_{k+\frac{1}{2}})-\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},z_{k-\frac{1}{2}})\\
&+\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k+\frac{1}{2}})+\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i-\frac{1}{2}},y_{j+\frac{1}{2}},z_{k-\frac{1}{2}})\\
&+\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i+\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}})-\frac{\partial^3 u}{\partial x\partial y\partial z}(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}}).
\end{aligned}
$$

Further denote πu by the piecewise trilinear interpolating function of u over Q_h. Replacing u by πu in (2.5) and perturbing (2.5), we have

$$
\begin{aligned}
&\int_{V_{ijk}}\frac{\pi u^n-\pi u^{n-1}}{\Delta t}dV+\frac{1}{2}B_{ijk}(\pi u^n+\pi u^{n-1})\\
&\quad+\frac{\Delta t}{4}D_{ijk}(\pi u^n-\pi u^{n-1})+\frac{\Delta t^2}{8}W_{ijk}(\pi u^n-\pi u^{n-1})\\
=&\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\int_{V_{ijk}}f\,dV\,dt+\int_{V_{ijk}}R_1^{n-\frac{1}{2}}dV+R_2^{n-\frac{1}{2}}+R_3^{n-\frac{1}{2}},
\end{aligned}\tag{2.6}
$$

where ($R_3^{n-\frac{1}{2}}$ is the perturbing error)

$$
\begin{aligned}
R_2^{n-\frac{1}{2}}=&\int_{V_{ijk}}\frac{\pi u^n-\pi u^{n-1}}{\Delta t}dV-\int_{V_{ijk}}\frac{u^n-u^{n-1}}{\Delta t}dV\\
&+\frac{1}{2}B_{ijk}(\pi u^n+\pi u^{n-1})-\frac{1}{2}B_{ijk}(u^n+u^{n-1}),
\end{aligned}\tag{2.7}
$$

$$R_3^{n-\frac{1}{2}} = \frac{\Delta t}{4} D_{ijk}(\pi u^n - \pi u^{n-1}) + \frac{\Delta t^2}{8} W_{ijk}(\pi u^n - \pi u^{n-1}). \tag{2.8}$$

Suppose that $U_h \subset H_0^1(\Omega)$ is a finite element space over partition Q_h, which is obtained by piecewise trilinear interpolating functions. Assume that $\{\alpha_i(x)\}$ $(i = 0, 1, \cdots, N_x)$, $\{\beta_j(y)\}$ $(j = 0, 1, \cdots, N_y)$ and $\{\gamma_k(z)\}$ $(k = 0, 1, \cdots, N_z)$ are the base functions of interpolation in x, y and z directions, respectively, then $\{\alpha_i(x)\beta_j(y)\gamma_k(z)\}$ are the base functions for U_h. That is, for $u_h \in U_h$, we have ($u_{ijk} = u_h\left(x_i, y_j, z_k\right)$)

$$u_h = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y}\sum_{k=0}^{N_z} \alpha_i(x)\beta_j(y)\gamma_k(z)u_{ijk}.$$

Dropping the error terms and substituting πu by $u_h \in U_h$ in (2.6), we obtain the finite volume element scheme for (2.1) and (2.2)

$$\begin{aligned}
&\int_{V_{ijk}} \frac{u_h^n - u_h^{n-1}}{\Delta t} dV + \frac{1}{2} B_{ijk}(u_h^n + u_h^{n-1}) + \frac{\Delta t}{4} D_{ijk}(u_h^n - u_h^{n-1}) + \frac{\Delta t^2}{8} W_{ijk}(u_h^n - u_h^{n-1}) \\
=& \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\int_{V_{ijk}} f(x,y,z,t)dVdt.
\end{aligned} \tag{2.9}$$

Let U be a vector composed of u_{ijk}, arranged k first from 0 to N_z, j second from 0 to N_y and i last from 0 to N_x and

$$\begin{aligned}
C_x &= \left[\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \alpha_m(x)dx\right]_{(N_x-1)\times(N_x+1)}, \\
A_x &= \left[\alpha_m'\left(x_{i-\frac{1}{2}}\right) - \alpha_m'\left(x_{i+\frac{1}{2}}\right)\right]_{(N_x-1)\times(N_x+1)}, \\
\Gamma^{n-\frac{1}{2}} &= \left[\frac{1}{\Delta t}\int_{t^{n-1}}^{t^n}\int_{V_{ijk}} f(x,y,z,t)dVdt\right]^{\mathrm{T}}_{(N_x-1)\times(N_y-1)\times(N_z-1)},
\end{aligned}$$

where $\Gamma^{n-\frac{1}{2}}$ is a vector arranged k first from 1 to N_z-1, j second from 1 to N_y-1 and i last from 1 to N_x-1. C_y, C_z are defined analogously to C_x and A_y, A_z are defined analogously to A_x, respectively. Use the above notations, (2.9) can be written as

$$\begin{aligned}
C_{xyz}\frac{U^n - U^{n-1}}{\Delta t} + Q\frac{U^n + U^{n-1}}{2} + \frac{\Delta t}{4} S(U^n - U^{n-1}) \\
+ \frac{\Delta t^2}{8} A_{xyz}(U^n - U^{n-1}) = \Gamma^{n-\frac{1}{2}},
\end{aligned} \tag{2.10}$$

where

$$
\begin{aligned}
&C_{xyz} = C_x \otimes C_y \otimes C_z, \quad A_{xyz} = A_x \otimes A_y \otimes A_z,\\
&Q = A_x \otimes C_y \otimes C_z + C_x \otimes A_y \otimes C_z + C_x \otimes C_y \otimes A_z,\\
&S = A_x \otimes A_y \otimes C_z + C_x \otimes A_y \otimes A_z + A_x \otimes C_y \otimes A_z
\end{aligned}
$$

and $\otimes$ represents Kronecker tensor product. For piecewise trilinear interpolation, a straightforward calculation shows

$$
C_x = \frac{1}{8}\begin{pmatrix}
h_1^x & 3(h_1^x + h_2^x) & h_2^x & & & \\
& h_2^x & 3(h_2^x + h_3^x) & h_3^x & & \\
& & \ddots & \ddots & \ddots & \\
& & & h_{N_x-1}^x & 3(h_{N_x-1}^x + h_{N_x}^x) & h_{N_x}^x
\end{pmatrix}, \tag{2.11}
$$

$$
A_x = \begin{pmatrix}
-\dfrac{1}{h_1^x} & \dfrac{1}{h_1^x} + \dfrac{1}{h_2^x} & -\dfrac{1}{h_2^x} & & & \\
& -\dfrac{1}{h_2^x} & \dfrac{1}{h_2^x} + \dfrac{1}{h_3^x} & -\dfrac{1}{h_3^x} & & \\
& & \ddots & \ddots & \ddots & \\
& & & -\dfrac{1}{h_{N_x-1}^x} & \dfrac{1}{h_{N_x-1}^x} + \dfrac{1}{h_{N_x}^x} & -\dfrac{1}{h_{N_x}^x}
\end{pmatrix}. \tag{2.12}
$$

C_y, C_z are analogous to C_x and A_y, A_z are analogous to A_x, respectively.

(2.10) can convert to a collection of one dimensional problems, which can be solved alternately. Give the following four computational schemes.

1. By introducing intermediate vectors $U^{n,1}$ and $U^{n,2}$, (2.10) is decomposed as

$$
\begin{aligned}
\left(C_x + \frac{\Delta t}{2}A_x\right) \otimes I \otimes I U^{n,1} = &\left(C_x - \frac{\Delta t}{2}A_x\right) \otimes C_y \otimes C_z U^{n-1}\\
&- \Delta t \left[C_x \otimes A_y \otimes C_z + C_x \otimes C_y \otimes A_z\right] U^{n-1} + \Delta t G^{n-\frac{1}{2}},
\end{aligned} \tag{2.13}
$$

$$
I \otimes \left(C_y + \frac{\Delta t}{2}A_y\right) \otimes I U^{n,2} = U^{n,1} + \frac{\Delta t}{2} I \otimes A_y \otimes C_z U^{n-1}, \tag{2.14}
$$

$$
I \otimes I \otimes \left(C_z + \frac{\Delta t}{2}A_z\right) U^n = U^{n,2} + \frac{\Delta t}{2} I \otimes I \otimes A_z U^{n-1}, \tag{2.15}
$$

where $G^{n-\frac{1}{2}} = \Gamma^{n-\frac{1}{2}}$. The splitting error of scheme (2.13)-(2.15) is second-order accuracy with respect to Δt.

2. In order to raise the splitting error to $\mathcal{O}(\Delta t^3)$, let

$$
G^{n-\frac{1}{2}} = \Gamma^{n-\frac{1}{2}} + \frac{\Delta t}{4} S(U^{n-1} - U^{n-2}) + \frac{\Delta t^2}{8} A_{xyz}(U^{n-1} - U^{n-2}). \tag{2.16}
$$

By eliminating the intermediate vectors $U^{n,1}$ and $U^{n,2}$, the splitting error of scheme (2.13)-(2.15) with (2.16) has truly third-order accuracy with respect to Δt.

3. Noting that the third term of the right hand side of (2.16) has already third-order accuracy with respect to Δt, we further let

$$G^{n-\frac{1}{2}} = \Gamma^{n-\frac{1}{2}} + \frac{\Delta t}{4} S(U^{n-1} - U^{n-2}). \tag{2.17}$$

Then scheme (2.13)-(2.15) with (2.17) has also third-order splitting accuracy with respect to Δt.

Scheme (2.13)-(2.15) with (2.16) or (2.17) is a three-level scheme. In practical computation, we must first compute U^1 in an efficient and accurate manner, which can be stated as follows.

(i) Compute the approximate vector $\widetilde{U}^1$ to U^1 by two-level scheme (2.13)-(2.15).

(ii) Compute

$$G^{\frac{1}{2}} = \Gamma^{\frac{1}{2}} + \frac{\Delta t}{4} S(\widetilde{U}^1 - U^0) + \frac{\Delta t^2}{8} A_{xyz}(\widetilde{U}^1 - U^0).$$

(iii) Compute new approximate vector of U^1 by (2.13)-(2.15), still denote by $\widetilde{U}^1$.

(iv) Repeat (ii), (iii) until the accuracy is satisfied, then we get U^1.

---

4. Decompose (2.10) as locally one dimensional scheme. Because $\Gamma^{n-\frac{1}{2}} \neq \mathbf{0}$ in (2.10), it must be treated first. Approximate $\Gamma^{n-\frac{1}{2}}$ as $C_{xyz}F^{n-\frac{1}{2}}$, where

$$F^{n-\frac{1}{2}} = \left(F^n + F^{n-1}\right)/2, \quad F^n = \left[f^n(x_i, y_j, z_k)\right]_{i,j,k=0}^{N_x,N_y,N_z}.$$

Adding perturbation term $\frac{1}{4}\Delta t^2 S F^{n-\frac{1}{2}}$ for $\Gamma^{n-\frac{1}{2}}$ and introducing intermediate vectors $U^{n,1}$ and $U^{n,2}$, we obtain the following locally one dimensional (LOD) scheme:

$$\left(C_x + \frac{\Delta t}{2}A_x\right) \otimes I \otimes I U^{n,1} = \left(C_x - \frac{\Delta t}{2}A_x\right) \otimes I \otimes I U^{n-1} + \Delta t C_x \otimes I \otimes I F^{n-\frac{1}{2}}, \tag{2.18}$$

$$I \otimes \left(C_y + \frac{\Delta t}{2}A_y\right) \otimes I U^{n,2} = I \otimes \left(C_y - \frac{\Delta t}{2}A_y\right) \otimes I U^{n,1} - \Delta t I \otimes C_y \otimes I F^{n-\frac{1}{2}}, \tag{2.19}$$

$$I \otimes I \otimes \left(C_z + \frac{\Delta t}{2}A_z\right) U^n = I \otimes I \otimes \left(C_z - \frac{\Delta t}{2}A_z\right) U^{n,2} + \Delta t I \otimes I \otimes C_z F^{n-\frac{1}{2}}. \tag{2.20}$$

For the purpose of practical computing, we further rewrite (2.18)-(2.20) as component forms

$$\begin{aligned}
&\frac{1}{8}h_i^x \frac{u_{i-1,j,k}^{n,1} - u_{i-1,j,k}^{n-1}}{\Delta t} + \frac{3}{8}(h_i^x + h_{i+1}^x)\frac{u_{i,j,k}^{n,1} - u_{i,j,k}^{n-1}}{\Delta t} + \frac{1}{8}h_{i+1}^x \frac{u_{i+1,j,k}^{n,1} - u_{i+1,j,k}^{n-1}}{\Delta t} \\
&\quad - \frac{1}{h_i^x}\frac{u_{i-1,j,k}^{n,1} + u_{i-1,j,k}^{n-1}}{2} + \left(\frac{1}{h_i^x} + \frac{1}{h_{i+1}^x}\right)\frac{u_{i,j,k}^{n,1} + u_{i,j,k}^{n-1}}{2} - \frac{1}{h_{i+1}^x}\frac{u_{i+1,j,k}^{n,1} + u_{i+1,j,k}^{n-1}}{2} \\
&= \frac{1}{8}h_i^x f_{i-1,j,k}^{n-\frac{1}{2}} + \frac{3}{8}(h_i^x + h_{i+1}^x) f_{i,j,k}^{n-\frac{1}{2}} + \frac{1}{8}h_{i+1}^x f_{i+1,j,k}^{n-\frac{1}{2}},
\end{aligned} \tag{2.21}$$

$$
\begin{aligned}
&\frac{1}{8}h_j^y\frac{u_{i,j-1,k}^{n,2}-u_{i,j-1,k}^{n,1}}{\Delta t}+\frac{3}{8}(h_j^y+h_{j+1}^y)\frac{u_{i,j,k}^{n,2}-u_{i,j,k}^{n,1}}{\Delta t}+\frac{1}{8}h_{j+1}^y\frac{u_{i,j+1,k}^{n,2}-u_{i,j+1,k}^{n,1}}{\Delta t}\\
&\quad-\frac{1}{h_j^y}\frac{u_{i,j-1,k}^{n,2}+u_{i,j-1,k}^{n,1}}{2}+\left(\frac{1}{h_j^y}+\frac{1}{h_{j+1}^y}\right)\frac{u_{i,j,k}^{n,2}+u_{i,j,k}^{n,1}}{2}-\frac{1}{h_{j+1}^y}\frac{u_{i,j+1,k}^{n,2}+u_{i,j+1,k}^{n,1}}{2}\\
&=-\frac{1}{8}h_j^y f_{i,j-1,k}^{n-\frac{1}{2}}-\frac{3}{8}(h_j^y+h_{j+1}^y)f_{i,j,k}^{n-\frac{1}{2}}-\frac{1}{8}h_{j+1}^y f_{i,j+1,k}^{n-\frac{1}{2}},
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
&\frac{1}{8}h_k^z\frac{u_{i,j,k-1}^{n}-u_{i,j,k-1}^{n,2}}{\Delta t}+\frac{3}{8}(h_k^z+h_{k+1}^z)\frac{u_{i,j,k}^{n}-u_{i,j,k}^{n,2}}{\Delta t}+\frac{1}{8}h_{k+1}^z\frac{u_{i,j,k+1}^{n}-u_{i,j,k+1}^{n,2}}{\Delta t}\\
&\quad-\frac{1}{h_k^z}\frac{u_{i,j,k-1}^{n}+u_{i,j,k-1}^{n,2}}{2}+\left(\frac{1}{h_k^z}+\frac{1}{h_{k+1}^z}\right)\frac{u_{i,j,k}^{n}+u_{i,j,k}^{n,2}}{2}-\frac{1}{h_{k+1}^z}\frac{u_{i,j,k+1}^{n}+u_{i,j,k+1}^{n,2}}{2}\\
&=\frac{1}{8}h_k^z f_{i,j,k-1}^{n-\frac{1}{2}}+\frac{3}{8}(h_k^z+h_{k+1}^z)f_{i,j,k}^{n-\frac{1}{2}}+\frac{1}{8}h_{k+1}^z f_{i,j,k+1}^{n-\frac{1}{2}}.
\end{aligned}
\tag{2.23}
$$

By eliminating the intermediate vectors $U^{n,1}$ and $U^{n,2}$ in (2.18)-(2.20), we get

$$
\begin{aligned}
&C_{xyz}\frac{U^n-U^{n-1}}{\Delta t}+Q\frac{U^n+U^{n-1}}{2}+\frac{\Delta t}{4}S(U^n-U^{n-1})+\frac{\Delta t^2}{8}A_{xyz}(U^n+U^{n-1})\\
&=C_{xyz}F^{n-\frac{1}{2}}+\frac{1}{4}\Delta t^2 SF^{n-\frac{1}{2}}.
\end{aligned}
\tag{2.24}
$$

We see from (2.24) that the splitting errors of U and F are both second-order accuracy with respect to Δt. In general, if the source term $f(x,y,z,t)$ is smooth enough, then we can modify the scheme to eliminate the perturbing effect of $f(x,y,z,t)$ partly, that is, let

$$
f-\frac{\Delta t^2}{4}\left(\frac{\partial^4 f}{\partial x^2\partial y^2}+\frac{\partial^4 f}{\partial y^2\partial z^2}+\frac{\partial^4 f}{\partial z^2\partial x^2}\right)\Rightarrow f.
$$

The advantage of scheme (2.18)-(2.20) or (2.21)-(2.23) is that (2.18) or (2.21) is solved only in x direction and (2.19) or (2.22) is solved only in y direction, etc.. In practical computation, we must treat the boundary values of $U^{n,1}$ and $U^{n,2}$ first. For example, for scheme (2.21)-(2.23), let $u_{i,j,k}^{n,1}=0$ for $i=0,N_x,\ j=0,N_y,\ k=0,1,\cdots,N_z$, $u_{i,j,k}^{n,2}=0$ for $i=0,N_x,\ j=0,1,\cdots,N_y,\ k=0,N_z$ and $i=0,1,\cdots,N_x,\ j=0,N_y,\ k=0,N_z$. By solving (2.23), we can obtain the boundary conditions for $u_{i,j,k}^{n,2}$ and by solving (2.22), we can obtain the boundary conditions for $u_{i,j,k}^{n,1}$. Then by solving (2.21) we get $U^{n,1}$ and again by solving (2.22) we get $U^{n,2}$. Finally, we obtain the final results U^n by solving (2.23).

3. Error estimates

We have derived four alternating direction finite volume element schemes in Section 2. In this section, we further analyze the convergence of these schemes with respect to L^2 norm or H^1 semi-norm. We first consider scheme (2.13)-(2.15) with $G^{n-\frac{1}{2}}=\Gamma^{n-\frac{1}{2}}$, which

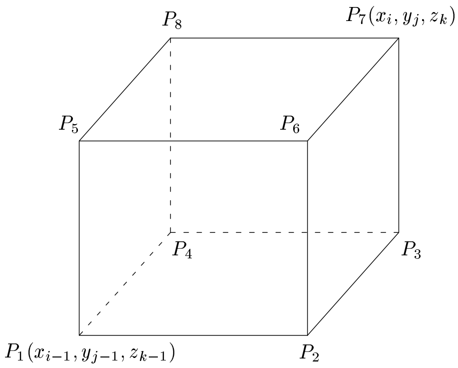

Figure 1: Illustration for an element and its nodes.

is equivalent to (2.9). Let $e = \pi u - u_h$ and subtract (2.9) from (2.6), the error equation reads

$$\begin{aligned}&\int_{V_{ijk}} \frac{e^n - e^{n-1}}{\Delta t} dV + \frac{1}{2} B_{ijk}(e^n + e^{n-1}) + \frac{\Delta t}{4} D_{ijk}(e^n - e^{n-1}) + \frac{\Delta t^2}{8} W_{ijk}(e^n - e^{n-1}) \\ =& \int_{V_{ijk}} R_1^{n-\frac{1}{2}} dV + R_2^{n-\frac{1}{2}} + R_3^{n-\frac{1}{2}}. \end{aligned} \tag{3.1}$$

Multiply $(e^n_{ijk} + e^{n-1}_{ijk})/2$ on the two sides of (3.1) and add from 1 to $N_x - 1$, $N_y - 1$ and $N_z - 1$ for i, j, k respectively. Denote the terms of the left hand side of the result by L_1, L_2, L_3, L_4 and the terms of right hand side by T_1, T_2, T_3, sequentially. Assume that Q_h is a quasi-uniformly regular cuboidal partition. Denote $\|\cdot\|_s$ or $\|\cdot\|_{s,\Omega}$ and $|\cdot|_s$ or $|\cdot|_{s,\Omega}$ by continuous norm and continuous semi-norm of order s in Sobolev space respectively. Further define discrete H^1 semi-norm and discrete L^2 norm respectively by

$$|\varphi_h|_{1,h} = \Big\{ \sum_{E \in Q_h} |\varphi_h|^2_{1,h,E} \Big\}^{\frac{1}{2}}, \quad \|\varphi_h\|_{0,h} = \Big\{ \sum_{E \in Q_h} \|\varphi_h\|^2_{0,h,E} \Big\}^{\frac{1}{2}}, \quad \forall \varphi_h \in U_h, \tag{3.2}$$

where

$$E = \overline{P_1 P_2 \cdots P_7 P_8} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k],$$

depicted as in Fig. 1 and

$$\begin{aligned}|\varphi_h|^2_{1,h,E} =& h_i^x h_j^y h_k^z \frac{1}{4} \sum_{l=1,3,5,7} \Big(\frac{\varphi_h(P_{l+1}) - \varphi_h(P_l)}{h_i^x} \Big)^2 + h_i^x h_j^y h_k^z \frac{1}{4} \Big\{ \sum_{l=2,6} \Big(\frac{\varphi_h(P_{l+1}) - \varphi_h(P_l)}{h_j^y} \Big)^2 \\ &+ \sum_{l=4,8} \Big(\frac{\varphi_h(P_l) - \varphi_h(P_{l-3})}{h_j^y} \Big)^2 \Big\} + h_i^x h_j^y h_k^z \frac{1}{4} \sum_{l=1,2,3,4} \Big(\frac{\varphi_h(P_{l+4}) - \varphi_h(P_l)}{h_k^z} \Big)^2, \end{aligned} \tag{3.3}$$

$$\|\varphi_h\|^2_{0,h,E} = \frac{h_i^x h_j^y h_k^z}{8} \sum_{l=1}^{8} \varphi_h(P_l)^2.$$

Let $Q_c(x_c, y_c, z_c)$ be the center of E and

$$\lambda_1 = 2(x - x_c)/h_i^x, \quad \lambda_2 = 2(y - y_c)/h_j^y, \quad \lambda_3 = 2(z - z_c)/h_k^z,$$

then E is transformed to $\widehat{E} = [-1,1]^3$. Constructing trilinear base functions of interpolation over $\widehat{E}$, we have

$$\begin{aligned}
N_1 &= \frac{1}{8}(1-\lambda_1)(1-\lambda_2)(1-\lambda_3), & N_2 &= \frac{1}{8}(1+\lambda_1)(1-\lambda_2)(1-\lambda_3),\\
N_3 &= \frac{1}{8}(1+\lambda_1)(1+\lambda_2)(1-\lambda_3), & N_4 &= \frac{1}{8}(1-\lambda_1)(1+\lambda_2)(1-\lambda_3),\\
N_5 &= \frac{1}{8}(1-\lambda_1)(1-\lambda_2)(1+\lambda_3), & N_6 &= \frac{1}{8}(1+\lambda_1)(1-\lambda_2)(1+\lambda_3),\\
N_7 &= \frac{1}{8}(1+\lambda_1)(1+\lambda_2)(1+\lambda_3), & N_8 &= \frac{1}{8}(1-\lambda_1)(1+\lambda_2)(1+\lambda_3).
\end{aligned}$$

Lemma 3.1. *For $\forall \varphi_h \in U_h$, $|\varphi_h|_{1,h}$ is equivalent to $|\varphi_h|_1$ and $\|\varphi_h\|_{0,h}$ is equivalent to $\|\varphi_h\|_0$, that is, the following inequalities hold*

$$\frac{1}{3}|\varphi_h|_{1,h} \le |\varphi_h|_1 \le |\varphi_h|_{1,h}, \quad \frac{\sqrt{3}}{9}\|\varphi_h\|_{0,h} \le \|\varphi_h\|_0 \le \|\varphi_h\|_{0,h}. \tag{3.4}$$

Proof. Depicted as in Fig. 1, we have

$$\varphi_h = \sum_{l=1}^{8} \varphi_h(P_l) N_l(\lambda_1, \lambda_2, \lambda_3)$$

in element E and

$$\int_E \left(\frac{\partial \varphi_h}{\partial x}\right)^2 dV = \frac{1}{36}\frac{h_j^y h_k^z}{h_i^x}\Phi^{\mathrm{T}} M \Phi,$$

where

$$\Phi = \begin{pmatrix} \varphi_h(P_2) - \varphi_h(P_1) \\ \varphi_h(P_3) - \varphi_h(P_4) \\ \varphi_h(P_6) - \varphi_h(P_5) \\ \varphi_h(P_7) - \varphi_h(P_8) \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 4 & 2 & 2 & 1 \\ 2 & 4 & 1 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 2 & 2 & 4 \end{pmatrix}.$$

The eigenvalues of matrix M are $1,3,3,9$. Similar results can be obtained for $\int_E(\frac{\partial \varphi_h}{\partial y})^2 dV$ and $\int_E(\frac{\partial \varphi_h}{\partial z})^2 dV$. According to the definitions of $|\varphi_h|_{1,E}$ and $|\varphi_h|_{1,h,E}$, the first inequality of (3.4) is proved. Analogously, a straightforward computation can prove the second one of (3.4). $\square$

Lemma 3.2. *For $\forall \varphi_h, \psi_h \in U_h$, denote by $\varphi_{ijk} = \varphi_h(x_i, y_j, z_k)$, $\psi_{ijk} = \psi_h(x_i, y_j, z_k)$,*

$$|||\varphi_h|||_{0,h}^2 = \sum_{i,j,k} \varphi_{ijk} \int_{V_{ijk}} \varphi_h dV,$$

where the summation (i,j,k) is for $1 \le i < N_x$, $1 \le j < N_y$ and $1 \le k < N_z$. Then

$$\sum_{i,j,k}\left[\psi_{ijk}\int_{V_{ijk}}\varphi_h dV - \varphi_{ijk}\int_{V_{ijk}}\psi_h dV\right] = 0, \quad |||\varphi_h|||_{0,h} \ge \frac{1}{2\sqrt{2}}\|\varphi_h\|_{0,h}. \tag{3.5}$$

In addition, for $\forall\ \omega \in L^2(\Omega)$, we have

$$\left|\sum_{i,j,k}\varphi_{ijk}\int_{V_{ijk}}\omega dV\right| \le \|\omega\|_0\|\varphi_h\|_{0,h}. \tag{3.6}$$

Proof. Depicted as in Fig. 1, we convert the integrals in control volumes to the relevant elements. A straightforward calculation shows

$$\sum_{i,j,k}\psi_{ijk}\int_{V_{ijk}}\varphi_h dV = \sum_{E\in Q_h}\frac{h_i^x h_j^y h_k^z}{8}\frac{1}{64}\Psi^{\mathrm{T}}M\Phi,$$

where

$$\Psi = \begin{pmatrix}\psi_h(P_1)\\ \psi_h(P_2)\\ \psi_h(P_3)\\ \psi_h(P_4)\\ \psi_h(P_5)\\ \psi_h(P_6)\\ \psi_h(P_7)\\ \psi_h(P_8)\end{pmatrix}, \quad \Phi = \begin{pmatrix}\varphi_h(P_1)\\ \varphi_h(P_2)\\ \varphi_h(P_3)\\ \varphi_h(P_4)\\ \varphi_h(P_5)\\ \varphi_h(P_6)\\ \varphi_h(P_7)\\ \varphi_h(P_8)\end{pmatrix}, \quad M = \begin{pmatrix}27 & 9 & 3 & 9 & 9 & 3 & 1 & 3\\ 9 & 27 & 9 & 3 & 3 & 9 & 3 & 1\\ 3 & 9 & 27 & 9 & 1 & 3 & 9 & 3\\ 9 & 3 & 9 & 27 & 3 & 1 & 3 & 9\\ 9 & 3 & 1 & 3 & 27 & 9 & 3 & 9\\ 3 & 9 & 3 & 1 & 9 & 27 & 9 & 3\\ 1 & 3 & 9 & 3 & 3 & 9 & 27 & 9\\ 3 & 1 & 3 & 9 & 9 & 3 & 9 & 27\end{pmatrix}.$$

We note that the matrix M is symmetric and positive definite and the eigenvalues of the matrix are $8, 16, 16, 16, 32, 32, 32, 64$, from which we can get (3.5).

As for (3.6), denote $S_{V_{ijk}}$ by the volume of the dual element V_{ijk}, then

$$\begin{aligned}\left|\sum_{i,j,k}\varphi_{ijk}\int_{V_{ijk}}\omega dV\right| &\le \left[\sum_{i,j,k}\varphi_{ijk}^2 S_{V_{ijk}}\right]^{\frac{1}{2}}\left[\sum_{i,j,k}\frac{1}{S_{V_{ijk}}}\left(\int_{V_{ijk}}\omega dV\right)^2\right]^{\frac{1}{2}}\\ &\le \|\omega\|_0\|\varphi_h\|_{0,h}.\end{aligned}$$

Then (3.6) holds. □

Lemma 3.3. *For $\forall\varphi_h, \psi_h \in U_h$, denote by*

$$a_h(\varphi_h, \psi_h) = \sum_{i,j,k}\psi_{ijk}B_{ijk}(\varphi_h),$$

where the summation (i,j,k) is for $1 \le i < N_x$, $1 \le j < N_y$ and $1 \le k < N_z$. Then

$$a_h(\varphi_h, \psi_h) = a_h(\psi_h, \varphi_h), \quad \frac{1}{4}|\varphi_h|_{1,h}^2 \le a_h(\varphi_h, \varphi_h) \le |\varphi_h|_{1,h}^2. \tag{3.7}$$

Proof. Depicted as in Fig. 1, we have

$$
\begin{aligned}
&\sum_{i,j,k}\psi_{ijk}\int_{\partial V_{jk}}\left[\frac{\partial\varphi_h}{\partial x}(x_{i-\frac{1}{2}})-\frac{\partial\varphi_h}{\partial x}(x_{i+\frac{1}{2}})\right]dydz\\
=&\sum_{E\in Q_h}\left\{\left[\psi_h(P_2)-\psi_h(P_1)\right]\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{z_{k-1}}^{z_{k-\frac{1}{2}}}\frac{\partial\varphi_h}{\partial x}(x_{i-\frac{1}{2}})dydz\right.\\
&+\left[\psi_h(P_3)-\psi_h(P_4)\right]\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{z_{k-1}}^{z_{k-\frac{1}{2}}}\frac{\partial\varphi_h}{\partial x}(x_{i-\frac{1}{2}})dydz\\
&+\left[\psi_h(P_6)-\psi_h(P_5)\right]\int_{y_{j-1}}^{y_{j-\frac{1}{2}}}\int_{z_{k-\frac{1}{2}}}^{z_k}\frac{\partial\varphi_h}{\partial x}(x_{i-\frac{1}{2}})dydz\\
&\left.+\left[\psi_h(P_7)-\psi_h(P_8)\right]\int_{y_{j-\frac{1}{2}}}^{y_j}\int_{z_{k-\frac{1}{2}}}^{z_k}\frac{\partial\varphi_h}{\partial x}(x_{i-\frac{1}{2}})dydz\right\}\\
=&\sum_{E\in Q_h}\frac{h_j^y h_k^z}{64h_i^x}\Psi^{\mathrm{T}}M\Phi,
\end{aligned}
$$

where

$$
\Psi=\begin{pmatrix}\psi_h(P_2)-\psi_h(P_1)\\ \psi_h(P_3)-\psi_h(P_4)\\ \psi_h(P_6)-\psi_h(P_5)\\ \psi_h(P_7)-\psi_h(P_8)\end{pmatrix},\quad
\Phi=\begin{pmatrix}\varphi_h(P_2)-\varphi_h(P_1)\\ \varphi_h(P_3)-\varphi_h(P_4)\\ \varphi_h(P_6)-\varphi_h(P_5)\\ \varphi_h(P_7)-\varphi_h(P_8)\end{pmatrix},\quad
M=\begin{pmatrix}9&3&3&1\\3&9&1&3\\3&1&9&3\\1&3&3&9\end{pmatrix}.
$$

Similar results can be obtained for the last two terms of $a_h(\varphi_h,\psi_h)$. Because the matrix M is symmetric and positive definite and the eigenvalues of the matrix are $4,8,8,16$, (3.7) is proved. $\square$

Lemma 3.4. *For* $\forall\varphi_h,\psi_h\in U_h$, *we have*

$$
\begin{aligned}
&\sum_{i,j,k}\psi_{ijk}D_{ijk}(\varphi_h)-\sum_{i,j,k}\varphi_{ijk}D_{ijk}(\psi_h)=0,\\
&\sum_{i,j,k}\varphi_{ijk}D_{ijk}(\varphi_h)\geq\frac{1}{2}\left[\left\|\frac{\partial^2\varphi_h}{\partial x\partial y}\right\|_{0,h}^2+\left\|\frac{\partial^2\varphi_h}{\partial y\partial z}\right\|_{0,h}^2+\left\|\frac{\partial^2\varphi_h}{\partial z\partial x}\right\|_{0,h}^2\right].
\end{aligned}
\tag{3.8}
$$

Proof. Depicted as in Fig. 1, for $\forall \psi_h \in U_h$, we have in element E

$$\frac{\partial^2 \psi_h}{\partial x \partial y}(z_{k-1}) = \frac{1}{h_i^x h_j^y}[\psi_h(P_1) - \psi_h(P_2) + \psi_h(P_3) - \psi_h(P_4)],$$
$$\frac{\partial^2 \psi_h}{\partial x \partial y}(z_k) = \frac{1}{h_i^x h_j^y}[\psi_h(P_5) - \psi_h(P_6) + \psi_h(P_7) - \psi_h(P_8)].$$

Transforming the summations of the left hand side of (3.8) in each element and using the above formula, we have

$$\begin{aligned}&\sum_{i,j,k} \psi_{ijk} \int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}} \left[\frac{\partial^2 \varphi_h}{\partial x \partial y}(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z) - \frac{\partial^2 \varphi_h}{\partial x \partial y}(x_{i-\frac{1}{2}}, y_{j+\frac{1}{2}}, z) \right.\\ &\qquad \left. - \frac{\partial^2 \varphi_h}{\partial x \partial y}(x_{i+\frac{1}{2}}, y_{j-\frac{1}{2}}, z) + \frac{\partial^2 \varphi_h}{\partial x \partial y}(x_{i-\frac{1}{2}}, y_{j-\frac{1}{2}}, z) \right] dz \\ =& \sum_{E \in Q_h} \frac{h_i^x h_j^y h_k^z}{8} \left(\frac{\partial^2 \psi_h}{\partial x \partial y}(z_{k-1}), \quad \frac{\partial^2 \psi_h}{\partial x \partial y}(z_k) \right) \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 \varphi_h}{\partial x \partial y}(z_{k-1}) \\ \frac{\partial^2 \varphi_h}{\partial x \partial y}(z_k) \end{pmatrix},\end{aligned}$$

from which (3.8) can be proved. □

Lemma 3.5. *For $\forall \varphi_h, \psi_h \in U_h$, we have*

$$\sum_{i,j,k} \psi_{ijk} W_{ijk}(\varphi_h) = \sum_{E \in Q_h} \frac{\partial^3 \varphi_h}{\partial x \partial y \partial z} \frac{\partial^3 \psi_h}{\partial x \partial y \partial z} h_i^x h_j^y h_k^z. \tag{3.9}$$

Proof. Depicted as in Fig. 1, for $\forall \psi_h \in U_h$, we have in element E

$$\begin{aligned}\frac{\partial^3 \psi_h}{\partial x \partial y \partial z} = \frac{1}{h_i^x h_j^y h_k^z} \Big[&\psi_h(P_2) - \psi_h(P_1) + \psi_h(P_4) - \psi_h(P_3) \\ &+ \psi_h(P_5) - \psi_h(P_6) + \psi_h(P_7) - \psi_h(P_8) \Big].\end{aligned}$$

Transforming the summations of the left hand side of (3.9) in each element and using the above formula, we can get (3.9). □

Lemma 3.6. *For $\forall u \in H^3(\Omega) \cap H_0^1(\Omega)$ and $\forall \psi_h \in U_h$, we have*

$$\left| \sum_{i,j,k} \psi_{ijk} B_{ijk}(u - \pi u) \right| \le C h^2 |u|_3 |\psi_h|_{1,h}, \tag{3.10}$$

where C is independent of mesh size h.

Proof. Noting that the proof of Lemma 3.3, using imbedding theorem and Bramble-Hilbert Lemma [20], we can prove the lemma. □

Lemma 3.7. *For* $\forall u \in H^4(\Omega)$*, we have*

$$\left|\sum_{i,j,k}\psi_{ijk}D_{ijk}(u-\pi u)\right| \leq C\|\psi\|_{0,h}|u|_{4,\Omega}, \tag{3.11}$$

where C is independent of mesh size h.

Proof. Noting that the proof of Lemma 3.4, we have

$$\begin{aligned}
D_1(u) :=& \sum_{i,j,k}\psi_{ijk}\int_{z_{k-\frac{1}{2}}}^{z_{k+\frac{1}{2}}}\left[\frac{\partial^2(u-\pi u)}{\partial x\partial y}(x_{i+\frac{1}{2}},y_{j+\frac{1}{2}},z)-\frac{\partial^2(u-\pi u)}{\partial x\partial y}(x_{i-\frac{1}{2}},y_{j+\frac{1}{2}},z)\right.\\
&\left.-\frac{\partial^2(u-\pi u)}{\partial x\partial y}(x_{i+\frac{1}{2}},y_{j-\frac{1}{2}},z)+\frac{\partial^2(u-\pi u)}{\partial x\partial y}(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z)\right]dz\\
=&\sum_{E\in Q_h}\frac{h_i^x h_j^y h_k^z}{2}\left(\frac{\partial^2\psi_h}{\partial x\partial y}(z_{k-1}),\ \frac{\partial^2\psi_h}{\partial x\partial y}(z_k)\right)\begin{pmatrix}\dfrac{4}{h_i^x h_j^y}\displaystyle\int_{-1}^{0}\frac{\partial^2(u-\pi u)}{\partial\lambda_1\partial\lambda_2}(0,0,\lambda_3)d\lambda_3\\ \dfrac{4}{h_i^x h_j^y}\displaystyle\int_{0}^{1}\frac{\partial^2(u-\pi u)}{\partial\lambda_1\partial\lambda_2}(0,0,\lambda_3)d\lambda_3\end{pmatrix},
\end{aligned}$$

As in Fig. 1, we have by computation

$$\begin{aligned}
I_2(u) :=& \int_{-1}^{0}\frac{\partial^2(u-\pi u)}{\partial\lambda_1\partial\lambda_2}(0,0,\lambda_3)d\lambda_3\\
=&\int_{-1}^{0}\frac{\partial^2 u}{\partial\lambda_1\partial\lambda_2}(0,0,\lambda_3)d\lambda_3-\frac{1}{16}\Big[3(u(P_1)-u(P_2))+3(u(P_3)-u(P_4))\\
&+(u(P_5)-u(P_6))+(u(P_7)-u(P_8))\Big].
\end{aligned}\tag{3.12}$$

As a linear functional of $u \in H^4(\Omega)$, $I_2(u)$ satisfies $|I_2(u)| \leq C\|u\|_{2,\infty,\widehat{E}}$. In addition, $H^4(\widehat{E}) \hookrightarrow C^2(\widehat{E})$. Hence,

$$|I_2(u)| \leq C\|u\|_{4,\widehat{E}}.$$

A straightforward calculation shows $I_2(u) \equiv 0$ for $u = \lambda_1^{\gamma_1}\lambda_2^{\gamma_2}\lambda_3^{\gamma_3}$, $\gamma_1+\gamma_2+\gamma_3 \leq 3$. By Bramble-Hilbert Lemma, $|I_2(u)| \leq C|u|_{4,\widehat{E}}$. By an integral transformation, noting that E is a quasi-uniformly regular element, we have

$$|I_2(u)| \leq Ch_i^x h_j^y h^{\frac{1}{2}}|u|_{4,E}.$$

Thus, we have

$$|D_1(u)| \leq Ch^2\left\{\sum_{E\in Q_h}\left[\frac{\partial^2\psi_h}{\partial x\partial y}(z_{k-1})^2+\frac{\partial^2\psi_h}{\partial x\partial y}(z_k)^2\right]h_i^x h_j^y h_k^z\right\}^{\frac{1}{2}}|u|_{4,\Omega}.$$

Analogously, the other two terms of $\sum \psi_{ijk} D_{ijk}(u-\pi u)$ have similar estimates. Hence, by inverse estimate, we have

$$\Big|\sum_{i,j,k} \psi_{ijk} D_{ijk}(u-\pi u)\Big| \le Ch^2 |\psi|_{2,h} |u|_{4,\Omega} \le C\|\psi\|_{0,h} |u|_{4,\Omega}.$$

Lemma 3.7 is proved. □

Lemma 3.8. *For an arbitrary element $E=[x_{i-1},x_i]\times[y_{j-1},y_j]\times[z_{k-1},z_k]$ and for $\forall u\in H^5(\Omega)$, we have*

$$\left|\frac{\partial^3(u-\pi u)}{\partial x\partial y\partial z}\left(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}}\right)\right| \le Ch^{\frac{1}{2}}|u|_{5,E}. \tag{3.13}$$

Proof. Carry out the coordinate transformations, we have

$$\begin{aligned}
&\frac{\partial^3(u-\pi u)}{\partial x\partial y\partial z}\left(x_{i-\frac{1}{2}},y_{j-\frac{1}{2}},z_{k-\frac{1}{2}}\right)\\
=&\frac{6}{h_i^x h_j^y h_k^z}\Big\{\frac{\partial^3 u}{\partial\lambda_1\partial\lambda_2\lambda_3}(0,0,0)-\frac{1}{8}\Big[-u(P_1)+u(P_2)-u(P_3)\\
&\quad+u(P_4)+u(P_5)-u(P_6)+u(P_7)-u(P_8)\Big]\Big\}\\
:=&\frac{6}{h_i^x h_j^y h_k^z} I_3(u).
\end{aligned} \tag{3.14}$$

As a linear functional of $u\in H^5(\Omega)$, $I_3(u)$ satisfies $|I_3(u)|\le C\|u\|_{3,\infty,\widehat{E}}$. In addition, $H^5(\widehat{E})\hookrightarrow C^3(\widehat{E})$. Hence, $|I_3(u)|\le C\|u\|_{5,\widehat{E}}$. A straightforward calculation shows $I_3(u)\equiv 0$ for $u=\lambda_1^{\gamma_1}\lambda_2^{\gamma_2}\lambda_3^{\gamma_3}$, $\gamma_1+\gamma_2+\gamma_3\le 4$. By Bramble-Hilbert Lemma, $|I_3(u)|\le C|u|_{5,\widehat{E}}$. By an integral transformation, noting that E is a quasi-uniformly regular element, we have

$$|I_3(u)|\le Ch_i^x h_j^y h_k^z h^{\frac{1}{2}}|u|_{5,E}.$$

Substitute the estimate into (3.14), then (3.13) holds. □

By Lemmas 3.2-3.5, it is easy to obtain the estimates of L_1, L_2, L_3 and L_4, which are

$$L_1=\frac{1}{2\Delta t}\left[|||e^n|||_{0,h}^2-|||e^{n-1}|||_{0,h}^2\right],\quad L_2\ge\frac{1}{4}\left|\frac{e^n+e^{n-1}}{2}\right|_{1,h}^2, \tag{3.15}$$

$$L_3=\frac{\Delta t}{8}\sum_{i,j,k}\left[e_{ijk}^n D_{ijk}(e^n)-e_{ijk}^{n-1}D_{ijk}(e^{n-1})\right], \tag{3.16}$$

$$L_4=\frac{\Delta t^2}{16}\sum_{i,j,k}\left[e_{ijk}^n W_{ijk}(e^n)-e_{ijk}^{n-1}W_{ijk}(e^{n-1})\right]. \tag{3.17}$$

Now we estimate T_1, T_2 and T_3. For T_1, by (3.6), we have

$$|T_1| \le C \left\| \frac{e^n + e^{n-1}}{2} \right\|_{0,h}^2 + C \int_\Omega \left(R_1^{n-\frac{1}{2}} \right)^2 dV.$$

Noting that (2.4), further using Hölder inequality for the above inequality, we obtain

$$|T_1| \le C \left\| \frac{e^n + e^{n-1}}{2} \right\|_{0,h}^2 + C\Delta t^3 \int_{t^{n-1}}^{t^n} \left| \frac{\partial^2 u}{\partial t^2} \right|_{2,\Omega}^2 dt. \tag{3.18}$$

By the formula

$$\frac{u^n + u^{n-1}}{2} = \frac{1}{\Delta t} \left[\int_{t^{n-1}}^{t^n} u dt + \int_{t^{n-1}}^{t^n} \frac{\partial u}{\partial t} \left(t - \frac{t^n + t^{n-1}}{2} \right) dt \right],$$

$R_2^{n-\frac{1}{2}}$ can be rewritten as

$$\begin{aligned} R_2^{n-\frac{1}{2}} = & -\frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} \int_{V_{ijk}} \frac{\partial(u-\pi u)}{\partial t} dV dt - \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} B_{ijk}(u-\pi u) dt \\ & - \frac{1}{\Delta t} \int_{t^{n-1}}^{t^n} B_{ijk} \left(\frac{\partial(u-\pi u)}{\partial t} \right) \left(t - \frac{t^n + t^{n-1}}{2} \right) dt. \end{aligned}$$

Thus, by (3.6) and Lemma 3.6, using Cauchy-ϵ inequality, we have

$$\begin{aligned} |T_2| \le & C \left\| \frac{e^n + e^{n-1}}{2} \right\|_{0,h}^2 + \frac{Ch^4}{\Delta t} \int_{t^{n-1}}^{t^n} \left| \frac{\partial u}{\partial t} \right|_{2,\Omega}^2 dt + \epsilon \left| \frac{e^n + e^{n-1}}{2} \right|_{1,h}^2 \\ & + \frac{Ch^4}{\Delta t} \int_{t^{n-1}}^{t^n} |u|_{3,\Omega}^2 dt + Ch^4 \Delta t \int_{t^{n-1}}^{t^n} \left| \frac{\partial u}{\partial t} \right|_{3,\Omega}^2 dt. \end{aligned} \tag{3.19}$$

From (2.8), we can rewrite $T_3 = T_{31} + T_{32} + T_{33} + T_{34}$, where

$$\begin{aligned} T_{31} &= \frac{\Delta t}{4} \sum_{i,j,k} \frac{e_{ijk}^n + e_{ijk}^{n-1}}{2} \int_{t^{n-1}}^{t^n} D_{ijk} \left(\frac{\partial(\pi u - u)}{\partial t} \right) dt, \\ T_{32} &= \frac{\Delta t}{4} \sum_{i,j,k} \frac{e_{ijk}^n + e_{ijk}^{n-1}}{2} \int_{V_{ijk}} \int_{t^{n-1}}^{t^n} \left(\frac{\partial^5 u}{\partial x^2 \partial y^2 \partial t} + \frac{\partial^5 u}{\partial y^2 \partial z^2 \partial t} + \frac{\partial^5 u}{\partial z^2 \partial x^2 \partial t} \right) dt dV, \\ T_{33} &= \frac{\Delta t^2}{8} \sum_{i,j,k} \frac{e_{ijk}^n + e_{ijk}^{n-1}}{2} \int_{t^{n-1}}^{t^n} W_{ijk} \left(\frac{\partial(\pi u - u)}{\partial t} \right) dt, \\ T_{34} &= \frac{\Delta t^2}{8} \sum_{i,j,k} \frac{e_{ijk}^n + e_{ijk}^{n-1}}{2} \int_{V_{ijk}} \int_{t^{n-1}}^{t^n} \frac{\partial^7 u}{\partial x^2 \partial y^2 \partial z^2 \partial t} dt dV. \end{aligned}$$

T_{31} is estimated by Lemma 3.7 and T_{32} is directly estimated, then

$$|T_{31}|+|T_{32}| \leq C\Delta t^3 \int_{t^{n-1}}^{t^n} \left|\frac{\partial u}{\partial t}\right|_{4,\Omega}^2 dt + C\left\|\frac{e^n+e^{n-1}}{2}\right\|_{0,h}^2. \tag{3.20}$$

By Lemma 3.5, we have

$$T_{33}=\frac{\Delta t^2}{8}\int_{t^{n-1}}^{t^n}\sum_{E\in Q_h}\frac{\partial^3}{\partial x\partial y\partial z}\left(\frac{e^n+e^{n-1}}{2}\right)\frac{\partial^4(\pi u-u)}{\partial x\partial y\partial z\partial t}h_i^x h_j^y h_k^z dt.$$

By Lemma 3.8, using inverse estimate and Cauchy-ϵ inequality, we have

$$|T_{33}| \leq C\Delta t^5 \int_{t^{n-1}}^{t^n} \left|\frac{\partial u}{\partial t}\right|_{5,\Omega}^2 dt + \epsilon\left|\frac{e^n+e^{n-1}}{2}\right|_{1,h}^2. \tag{3.21}$$

It is directly estimated for T_{34},

$$|T_{34}| \leq C\Delta t^5 \int_{t^{n-1}}^{t^n}\int_\Omega \left(\frac{\partial^7 u}{\partial x^2\partial y^2\partial z^2\partial t}\right)^2 dV dt + C\left\|\frac{e^n+e^{n-1}}{2}\right\|_{0,h}^2. \tag{3.22}$$

From (3.20)-(3.22), we have

$$\begin{aligned}|T_3| \leq& C\Delta t^3 \int_{t^{n-1}}^{t^n} \left|\frac{\partial u}{\partial t}\right|_{4,\Omega}^2 dt + C\left\|\frac{e^n+e^{n-1}}{2}\right\|_{0,h}^2 + \epsilon\left|\frac{e^n+e^{n-1}}{2}\right|_{1,h}^2 \\ &+ C\Delta t^5\left[\int_{t^{n-1}}^{t^n} \left|\frac{\partial u}{\partial t}\right|_{5,\Omega}^2 dt + \int_{t^{n-1}}^{t^n}\int_\Omega \left(\frac{\partial^7 u}{\partial x^2\partial y^2\partial z^2\partial t}\right)^2 dV dt\right].\end{aligned} \tag{3.23}$$

By the estimates of L_l $(l=1,2,3,4)$ and T_l $(l=1,2,3)$, we have

$$\begin{aligned}&\frac{1}{2\Delta t}\left[|||e^n|||_{0,h}^2 - |||e^{n-1}|||_{0,h}^2\right] + \frac{1}{4}\left|\frac{e^n+e^{n-1}}{2}\right|_{1,h}^2 + L_3 + L_4 \\ \leq& C\left\|\frac{e^n+e^{n-1}}{2}\right\|_{0,h}^2 + \epsilon\left|\frac{e^n+e^{n-1}}{2}\right|_{1,h}^2 + \frac{Ch^4}{\Delta t}\int_{t^{n-1}}^{t^n}\left(\left|\frac{\partial u}{\partial t}\right|_{2,\Omega}^2 + |u|_{3,\Omega}^2\right)dt \\ &+ C\Delta t^3\int_{t^{n-1}}^{t^n}\left(\left|\frac{\partial^2 u}{\partial t^2}\right|_{2,\Omega}^2 + \left|\frac{\partial u}{\partial t}\right|_{4,\Omega}^2\right)dt + Ch^4\Delta t\int_{t^{n-1}}^{t^n}\left|\frac{\partial u}{\partial t}\right|_{3,\Omega}^2 dt \\ &+ C\Delta t^5\left[\int_{t^{n-1}}^{t^n} \left|\frac{\partial u}{\partial t}\right|_{5,\Omega}^2 dt + \int_{t^{n-1}}^{t^n}\int_\Omega \left(\frac{\partial^7 u}{\partial x^2\partial y^2\partial z^2\partial t}\right)^2 dV dt\right].\end{aligned} \tag{3.24}$$

In (3.24), take $\epsilon \le \frac{1}{4}$, multiply by $2\Delta t$ and add for n, note that (3.5), (3.8), (3.9), (3.16) and (3.17), then discrete Gronwall Lemma implies

$$\begin{aligned}
&\|e^n\|_{0,h}^2 + \frac{\sqrt{2}\Delta t^2}{4}\left[\left\|\frac{\partial^2 e^n}{\partial x\partial y}\right\|_{0,h}^2 + \left\|\frac{\partial^2 e^n}{\partial y\partial z}\right\|_{0,h}^2 + \left\|\frac{\partial^2 e^n}{\partial z\partial x}\right\|_{0,h}^2\right] + \frac{\sqrt{2}\Delta t^2}{4}\left\|\frac{\partial^3 e^n}{\partial x\partial y\partial z}\right\|_{0,h}^2 \\
\le\, & Ch^4\Delta t^2\int_0^{t^n}\left|\frac{\partial u}{\partial t}\right|_{3,\Omega}^2 dt + Ch^4\int_0^{t^n}\left(\left|\frac{\partial u}{\partial t}\right|_{2,\Omega}^2 + |u|_{3,\Omega}^2\right)dt \\
& + C\Delta t^4\int_0^{t^n}\left|\frac{\partial^2 u}{\partial t^2}\right|_{2,\Omega}^2 dt + C\Delta t^4\int_0^{t^n}\left|\frac{\partial u}{\partial t}\right|_{4,\Omega}^2 dt \\
& + C\Delta t^6\left[\int_0^{t^n}\left|\frac{\partial u}{\partial t}\right|_{5,\Omega}^2 dt + \int_0^{t^n}\int_\Omega\left(\frac{\partial^7 u}{\partial x^2\partial y^2\partial z^2\partial t}\right)^2 dVdt\right].
\end{aligned} \tag{3.25}$$

From the fact that $\|u-\pi u\|_0 \le Ch^2|u|_2$ for piecewise trilinear interpolation, we have by (3.25) and Lemma 3.1

$$\|u^n - u_h^n\|_0 \le C(\Delta t^2 + h^2). \tag{3.26}$$

The above discussion can be summarized as follows.

Theorem 3.1. *Assume that $f(x,y,z,t)$ is sufficiently smooth and $u \in H^2(0,T;H_0^1(\Omega)\cap H^6(\Omega))$ is the solution to Eq.* (2.1) *with* (2.2). *Then the solution u_h of alternating direction finite volume element scheme* (2.9) *converges to u with respect to L^2 norm and the error is estimated by* (3.26).

The LOD scheme (2.18)-(2.20) can be estimated analogously. Now we consider the convergence of scheme (2.13)-(2.15) with (2.16). Let $\xi^n = e^n - 2e^{n-1} + e^{n-2}$. We know the error equation of (2.13)-(2.15) with (2.16) is

$$\begin{aligned}
&\int_{V_{ijk}}\frac{e^n - e^{n-1}}{\Delta t}dV + \frac{1}{2}B_{ijk}(e^n + e^{n-1}) + \frac{\Delta t}{4}D_{ijk}(\xi^n) + \frac{\Delta t^2}{8}W_{ijk}(\xi^n) \\
=& \int_{V_{ijk}} R_1^{n-\frac{1}{2}}dV + R_2^{n-\frac{1}{2}} + R_4^{n-\frac{1}{2}},
\end{aligned} \tag{3.27}$$

where

$$R_4^{n-\frac{1}{2}} = \frac{\Delta t}{4}D_{ijk}\left(\pi u^n - 2\pi u^{n-1} + \pi u^{n-2}\right) + \frac{\Delta t^2}{8}W_{ijk}\left(\pi u^n - 2\pi u^{n-1} + \pi u^{n-2}\right). \tag{3.28}$$

Denote by $d_t e^n = (e^n - e^{n-1})/\Delta t$, multiply $d_t e_{ijk}^n$ on the two sides of (3.27) and add from 1 to $N_x - 1$, $N_y - 1$ and $N_z - 1$ for i,j and k, respectively. Denote the terms of the left hand side of the result by $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$, $\widetilde{L}_4$ and the terms of right hand side by $\widetilde{T}_1$, $\widetilde{T}_2$, $\widetilde{T}_3$, sequentially. From Lemma 3.2 and Lemma 3.3, we have

$$\widetilde{L}_1 = |||d_t e^n|||_{0,h}^2 \ge \frac{1}{8}\|d_t e^n\|_{0,h}^2, \quad \widetilde{L}_2 = \frac{1}{2\Delta t}\left[a_h(e^n,e^n) - a_h(e^{n-1},e^{n-1})\right].$$

It is easy to prove for $\forall\ a_1,\ a_2,\ b_1,\ b_2$

$$
\begin{aligned}
&\begin{pmatrix} a_1, & a_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a_1 - b_1 \\ a_2 - b_2 \end{pmatrix} \\
\geq & \frac{1}{2} \begin{pmatrix} a_1, & a_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} b_1, & b_2 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.
\end{aligned}
\tag{3.29}
$$

By (3.29), from the proof of Lemma 3.4, we have

$$
\widetilde{L}_3 \geq \frac{\Delta t^2}{8} \sum_{i,j,k} \left[(d_t e^n)_{ijk} D_{ijk} (d_t e^n) - (d_t e^{n-1})_{ijk} D_{ijk} (d_t e^{n-1}) \right].
$$

From Lemma 3.5, we have

$$
\widetilde{L}_4 \geq \frac{\Delta t^3}{16} \left[\left\| \frac{\partial^3 d_t e^n}{\partial x \partial y \partial z} \right\|_{0,h}^2 - \left\| \frac{\partial^3 d_t e^{n-1}}{\partial x \partial y \partial z} \right\|_{0,h}^2 \right].
$$

Analogous to the estimates of T_1 and T_2, by inverse estimate, we have

$$
\begin{aligned}
|\widetilde{T}_1| \leq & \epsilon \left\| d_t e^n \right\|_{0,h}^2 + C \Delta t^3 \int_{t^{n-1}}^{t^n} \left| \frac{\partial^2 u}{\partial t^2} \right|_{2,\Omega}^2 dt, \\
|\widetilde{T}_2| \leq & \epsilon \left\| d_t e^n \right\|_{0,h}^2 + \frac{C h^4}{\Delta t} \int_{t^{n-1}}^{t^n} \left| \frac{\partial u}{\partial t} \right|_{2,\Omega}^2 dt \\
& + \frac{C h^2}{\Delta t} \int_{t^{n-1}}^{t^n} |u|_{3,\Omega}^2 dt + C h^2 \Delta t \int_{t^{n-1}}^{t^n} \left| \frac{\partial u}{\partial t} \right|_{3,\Omega}^2 dt.
\end{aligned}
$$

Using the formula

$$
u^n - 2u^{n-1} + u^{n-2} = \int_{t^{n-1}}^{t^n} \frac{\partial^2 u}{\partial t^2} (t^n - t) dt + \int_{t^{n-2}}^{t^{n-1}} \frac{\partial^2 u}{\partial t^2} (t - t^{n-2}) dt,
$$

we can rewrite

$$
\widetilde{T}_3 = \widetilde{T}_{31} + \widetilde{T}_{32} + \widetilde{T}_{33} + \widetilde{T}_{34},
$$

where

$$
\begin{aligned}
\widetilde{T}_{31} = & \frac{\Delta t}{4} \sum_{i,j,k} d_t e_{ijk}^n \left[\int_{t^{n-1}}^{t^n} D_{ijk} \left(\frac{\partial^2 (\pi u - u)}{\partial t^2} \right) (t^n - t) dt \right. \\
& \left. + \int_{t^{n-2}}^{t^{n-1}} D_{ijk} \left(\frac{\partial^2 (\pi u - u)}{\partial t^2} \right) (t - t^{n-2}) dt \right],
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{T}_{32} = \frac{\Delta t}{4}\sum_{i,j,k} d_t e_{ijk}^n \bigg[& \int_{V_{ijk}} \int_{t^{n-1}}^{t^n} \left(\frac{\partial^4 u_{tt}}{\partial x^2 \partial y^2} + \frac{\partial^4 u_{tt}}{\partial y^2 \partial z^2} + \frac{\partial^4 u_{tt}}{\partial z^2 \partial x^2} \right) (t^n - t) dt dV \\
& + \int_{V_{ijk}} \int_{t^{n-2}}^{t^{n-1}} \left(\frac{\partial^4 u_{tt}}{\partial x^2 \partial y^2} + \frac{\partial^4 u_{tt}}{\partial y^2 \partial z^2} + \frac{\partial^4 u_{tt}}{\partial z^2 \partial x^2} \right) (t - t^{n-2}) dt dV \bigg], \\
\widetilde{T}_{33} = \frac{\Delta t^2}{8}\sum_{i,j,k} d_t e_{ijk}^n \bigg[& \int_{t^{n-1}}^{t^n} W_{ijk} \left(\frac{\partial^2 (\pi u - u)}{\partial t^2} \right) (t^n - t) dt \\
& + \int_{t^{n-2}}^{t^{n-1}} W_{ijk} \left(\frac{\partial^2 (\pi u - u)}{\partial t^2} \right) (t - t^{n-2}) dt \bigg], \\
\widetilde{T}_{34} = \frac{\Delta t^2}{8}\sum_{i,j,k} d_t e_{ijk}^n \bigg[& \int_{V_{ijk}} \int_{t^{n-1}}^{t^n} \frac{\partial^8 u}{\partial x^2 \partial y^2 \partial z^2 \partial t^2} (t^n - t) dt dV \\
& + \int_{V_{ijk}} \int_{t^{n-2}}^{t^{n-1}} \frac{\partial^8 u}{\partial x^2 \partial y^2 \partial z^2 \partial t^2} (t - t^{n-2}) dt dV \bigg].
\end{aligned}
$$

By Lemma 3.7 and Hölder inequality, we have

$$
|\widetilde{T}_{31}| + |\widetilde{T}_{32}| \le \epsilon \left\| d_t e^n \right\|_{0,h}^2 + C\Delta t^5 \int_{t^{n-2}}^{t^n} \left| \frac{\partial^2 u}{\partial t^2} \right|_{4,\Omega}^2 dt.
$$

Without loss of generality, assume that

$$
\Delta t = \mathscr{O}(h).
$$

By Lemma 3.8 and inverse estimate, we have

$$
\begin{aligned}
|\widetilde{T}_{33}| &\le C\Delta t^{\frac{7}{2}} \left| d_t e^n \right|_{1,h} \left(\int_{t^{n-2}}^{t^n} \left| \frac{\partial^2 u}{\partial t^2} \right|_{5,\Omega}^2 dt \right)^{\frac{1}{2}} \\
&\le \epsilon \left\| d_t e^n \right\|_{0,h}^2 + C\Delta t^5 \int_{t^{n-2}}^{t^n} \left| \frac{\partial^2 u}{\partial t^2} \right|_{5,\Omega}^2 dt.
\end{aligned}
$$

It is directly estimated for $\widetilde{T}_{34}$, we have

$$
|\widetilde{T}_{34}| \le C\Delta t^7 \int_{t^{n-2}}^{t^n} \int_{\Omega} \left(\frac{\partial^8 u}{\partial x^2 \partial y^2 \partial z^2 \partial t^2} \right)^2 dV dt + \epsilon \left\| d_t e^n \right\|_{0,h}^2.
$$

From the estimates of $\widetilde{L}_1,\ \widetilde{L}_2,\ \widetilde{L}_3,\ \widetilde{L}_4$ and $\widetilde{T}_1,\ \widetilde{T}_2,\ \widetilde{T}_{3l}$ $(l=1,2,3,4)$, we have

$$\begin{aligned}
&\frac{1}{8}\|d_te^n\|_{0,h}^2+\frac{1}{2\Delta t}\left[a_h(e^n,e^n)-a_h(e^{n-1},e^{n-1})\right]\\
&\quad+\frac{\Delta t^3}{16}\left[\left\|\frac{\partial^3 d_te^n}{\partial x\partial y\partial z}\right\|_{0,h}^2-\left\|\frac{\partial^3 d_te^{n-1}}{\partial x\partial y\partial z}\right\|_{0,h}^2\right]\\
&\quad+\frac{\Delta t^2}{8}\sum_{i,j,k}\left[(d_te^n)_{ijk}D_{ijk}(d_te^n)-(d_te^{n-1})_{ijk}D_{ijk}(d_te^{n-1})\right]\\
&\le\epsilon\left\|d_te^n\right\|_{0,h}^2+\frac{Ch^4}{\Delta t}\int_{t^{n-1}}^{t^n}\left|\frac{\partial u}{\partial t}\right|_{2,\Omega}^2dt+\frac{Ch^2}{\Delta t}\int_{t^{n-1}}^{t^n}|u|_{3,\Omega}^2dt\\
&\quad+Ch^2\Delta t\int_{t^{n-1}}^{t^n}\left|\frac{\partial u}{\partial t}\right|_{3,\Omega}^2dt+C\Delta t^3\int_{t^{n-1}}^{t^n}\left|\frac{\partial^2 u}{\partial t^2}\right|_{2,\Omega}^2dt+C\Delta t^5\int_{t^{n-2}}^{t^n}\left|\frac{\partial^2 u}{\partial t^2}\right|_{4,\Omega}^2dt\\
&\quad+C\Delta t^5\int_{t^{n-2}}^{t^n}\left|\frac{\partial^2 u}{\partial t^2}\right|_{5,\Omega}^2dt+C\Delta t^7\int_{t^{n-2}}^{t^n}\int_\Omega\left(\frac{\partial^8 u}{\partial x^2\partial y^2\partial z^2\partial t^2}\right)^2dVdt. \qquad (3.30)
\end{aligned}$$

In (3.30), taking $\epsilon\le\frac{1}{8}$, multiplying by $2\Delta t$ and adding from 2 to n, noting that (3.7) and (3.8), we have

$$\begin{aligned}
&|e^n|_{1,h}^2+\frac{\Delta t^3}{2}\left[\left\|\frac{\partial^2 d_te^n}{\partial x\partial y}\right\|_{0,h}^2+\left\|\frac{\partial^2 d_te^n}{\partial y\partial z}\right\|_{0,h}^2+\left\|\frac{\partial^2 d_te^n}{\partial z\partial x}\right\|_{0,h}^2\right]+\frac{\Delta t^4}{2}\left\|\frac{\partial^3 d_te^n}{\partial x\partial y\partial z}\right\|_{0,h}^2\\
&\le 4|e^1|_{1,h}^2+\Delta t^3\left[\left\|\frac{\partial^2 d_te^1}{\partial x\partial y}\right\|_{0,h}^2+\left\|\frac{\partial^2 d_te^1}{\partial y\partial z}\right\|_{0,h}^2+\left\|\frac{\partial^2 d_te^1}{\partial z\partial x}\right\|_{0,h}^2\right]+\frac{\Delta t^4}{2}\left\|\frac{\partial^3 d_te^1}{\partial x\partial y\partial z}\right\|_{0,h}^2\\
&\quad+Ch^4\int_{t_1}^{t^n}\left|\frac{\partial u}{\partial t}\right|_{2,\Omega}^2dt+Ch^2\int_{t_1}^{t^n}\left[|u|_{3,\Omega}^2+\Delta t^2\left|\frac{\partial u}{\partial t}\right|_{3,\Omega}^2\right]dt+C\Delta t^4\int_{t_1}^{t^n}\left|\frac{\partial^2 u}{\partial t^2}\right|_{2,\Omega}^2dt\\
&\quad+C\Delta t^6\int_0^{t^n}\left[\left|\frac{\partial^2 u}{\partial t^2}\right|_{4,\Omega}^2+\left|\frac{\partial^2 u}{\partial t^2}\right|_{5,\Omega}^2\right]dt+C\Delta t^8\int_0^{t^n}\int_\Omega\left(\frac{\partial^8 u}{\partial x^2\partial y^2\partial z^2\partial t^2}\right)^2dVdt.
\end{aligned}$$
(3.31)

Because u_h^1 is computed by iteration method, we can assume

$$|e^1|_{1,h}\le C(h+\Delta t^2).$$

From the fact that $|u-\pi u|_1\le Ch|u|_2$ for trilinear interpolation, we have by (3.31) and Lemma 3.1

$$|u^n-u_h^n|_{1,\Omega}\le C_1h^2+C_2h+C_3\Delta t^2+C_4\Delta t^3+C_5\Delta t^4. \qquad (3.32)$$

The fourth and fifth terms of the right hand side of inequality (3.32) represent the perturbation errors, from which we know the splitting error of scheme (2.13)-(2.15) with (2.16) are least third-order accurate with respect to Δt. The above argument can be stated as the following theorem.

Theorem 3.2. *Suppose that* $u \in H^2(0,T;H_0^1(\Omega)\cap H^6(\Omega))$ *is the solution of equation* (2.1) *with* (2.2)*. Then the solution* u_h *of scheme* (2.13)-(2.15) *with* (2.16) *converges to* u *with respect to* H^1 *semi-norm on condition that* $\Delta t = \mathscr{O}(h)$ *and the error is estimated by* (3.32).

Because scheme (2.13)-(2.15) with (2.17) is just a simple one of scheme (2.13)-(2.15) with (2.16) by dropping a third-order perturbing term, Theorem 3.2 also holds for scheme (2.13)-(2.15) with (2.17).

4. Numerical examples

In this section, we give two numerical examples to illustrate the effectiveness of the schemes in this paper. For convenience, we denote scheme (2.13)-(2.15) with $G^{n-\frac{1}{2}} = \Gamma^{n-\frac{1}{2}}$ by scheme 1, scheme (2.13)-(2.15) with (2.16) by scheme 2, scheme (2.13)-(2.15) with (2.17) by scheme 3 and scheme (2.18)-(2.20) by scheme 4, respectively.

Example 4.1. In (2.1)-(2.2), let $\Omega = [0,1]^3$ and the exact solution to (2.1)-(2.2) is chosen to be $u = \sin 2\pi t \sin 2\pi(x+y+z)$. The source term $f(x,y,z,t)$ and the initial and boundary conditions are obtained based on the exact solution.

Giving a uniform partition for Ω with $h_x = h_y = h_z = h$ and using $\Delta t = \frac{h}{2}$, we compute the example by schemes 1,2,3,4 until $T = 1$. The maximum discrete L^2 norm errors, defined by $Eu_{0,h} = \max_n \|u^n - u_h^n\|_{0,h}$, and CPU times are shown in Table 1.

Table 1: $Eu_{0,h}$ and CPU times (second) computed by schemes 1,2,3 4 in Example 4.1.

	Scheme 1		Scheme 2		Scheme 3		Scheme 4	
	$Eu_{0,h}$	Time	$Eu_{0,h}$	Time	$Eu_{0,h}$	Time	$Eu_{0,h}$	Time
$h=\frac{1}{40}$	5.55E-3	2.4	1.10E-4	3.15	4.83E-4	3.06	2.82E-2	1.86
$h=\frac{1}{80}$	1.34E-3	40.9	8.30E-5	51.3	1.02E-4	50.3	3.99E-3	35.1
$h=\frac{1}{160}$	3.27E-4	741.6	2.74E-5	915.5	2.85E-5	897.3	8.08E-4	615.5

Table 1 shows that the perturbation errors sometimes may be so large that the two-level alternating direction schemes are not accurate as theoretical analysis. The shortcoming could be overcome in some extent by schemes 2 and 3. As for the CPU times of the four schemes costing, scheme 4 costs least CPU times because it does not compute tensor products. On the other hand, because scheme 4 adds more perturbation terms, it is not accurate as other three schemes.

Example 4.2. In (2.1)-(2.2), let $\Omega = [0,1]^3$ and the exact solution to (2.1)-(2.2) is chosen to be $u = \exp(-3t)\sin(x+y+z)$, hence $f = 0$.

Giving a uniform partition for Ω with $h_x = h_y = h_z = h$ and using $\Delta t = h$, we compute the example by schemes 1,2,3,4 until $T = 1$. The maximum discrete L^2 norm errors $Eu_{0,h}$ and CPU times are shown in Table 2.

Table 2: $Eu_{0,h}$ and CPU times (second) computed by schemes 1,2,3 4 in Example 4.2.

	Scheme 1		Scheme 2		Scheme 3		Scheme 4	
	$Eu_{0,h}$	Time	$Eu_{0,h}$	Time	$Eu_{0,h}$	Time	$Eu_{0,h}$	Time
$h=\frac{1}{25}$	3.16E-6	0.19	8.14E-5	0.27	8.04E-5	0.28	1.07E-5	0.16
$h=\frac{1}{50}$	8.42E-7	3.06	1.82E-5	4.13	1.81E-5	4.06	2.68E-6	2.54
$h=\frac{1}{100}$	2.17E-7	52.9	4.38E-6	68.0	4.37E-6	66.6	6.71E-7	47.6

Table 2 tells us when the perturbation errors are relatively small, scheme 1 and scheme 4 could get satisfactory results and scheme 4 is the fastest one of the four schemes.

Further denote numerical convergence order by $r = \log_2(Eu_{0,h}/Eu_{0,h/2})$. Observing Tables 1 and 2, we know $r \approx 2$ for each scheme, which illustrates all the schemes are second-order accurate with respect to discrete L^2 norm.

References

[1] Z.-Q. Cai and S. McCormick, *On the accuracy of the finite volume element method for diffusion equations on composite grid*, SIAM J. Numer. Anal., vol. 27, no. 3 (1990), pp. 636–655.

[2] E. Süli, *Convergence of finite volume schemes for Poisson's equation on nonuniform meshes*, SIAM J. Numer. Anal., vol. 28, no. 5 (1991), pp. 1419–1430.

[3] W. P. Jones and K. R. Menziest, *Analysis of the cell-centred finite volume method for the diffusion equation*, Journal of Computational Physics, vol. 165 (2000), pp. 45–68.

[4] R. Li, Z. Chen, and W. Wu, *Generalized difference methods for differential equations-Numerical analysis of finite volume methods*, Monographs and Textbooks in Pure and Applied Mathematics 226, Marcel Dekker Inc., New York, 2000.

[5] Y. Li and R. Li, *Generalized difference methods on arbitrary quadrilateral networks,* Journal of Computational Mathematics, vol. 17, no. 6 (1999), pp. 653–672.

[6] S.-H. Chou and X. Ye, *Unified analysis of finite volume methods for second order elliptic problems*, SIAM J. Numer. Anal., vol. 45 (2008), pp. 1639–1653.

[7] H. Wu and R. Li, *Error estimates finite volume element methods for general second-order elliptic problems,* Numerical Methods for Partial Differential Equations, vol. 19 (2003), pp. 693–708.

[8] P. Chatzipantelidis, R. D. Lazarov, and V. Thomée, *Error estimates for a finite volume element method for parabolic equations in convex polygonal domains*, Numerical Methods for Partial Differential Equations, vol. 20 (2004), pp. 650–674.

[9] J. Xu and Q. Zou, *Analysis of linear and quadratic simplicial finite volume methods for elliptic equations*, Numer. Math., vol. 111 (2009), pp. 469–492.

[10] Z.-Q. Cai, J. Douglas Jr., and M. Park, *Development and analysis of higher order finite volume methods over rectangles for elliptic equations*, Advances in Computational Mathematics, vol. 19 (2003), pp. 3–33.

[11] M. Yang, J. Liu, and C. Chen, *Error estimation of a quadratic finite volume method on right quadrangular prism grids*, Journal of Computational and Applied Mathematics, vol. 229 (2009), pp. 274–282.

[12] T. WANG, *A mixed finite volume element method based on rectangular mesh for biharmonic equations*, Journal of Computational and Applied Mathematics, vol. 172 (2004), pp. 117–130.

[13] S. SHU, H. YU, Y.-Q. HUANG, AND C. NIE, *A symmetric finite volume element scheme on quadrilateral grids and superconvergence*, International Journal of Numerical Analysis and Modeling, vol. 3 (2006), pp. 348–360.

[14] T. WANG, *Alternating direction finite volume element methods for 2D parabolic partial differential equations*, Numerical Methods for Partial Differential Equations, vol. 24, no. 1 (2008), pp. 24–40.

[15] J. DOUGLAS JR. AND T. DUPONT, *Alternating-direction Galerkin methods on rectangles*, Proceedings Symposium on Numerical Solution of Partial Differential Equations, II, B.Hubbard ed., Academic Press, New York, 1971, pp. 133–214.

[16] J. DOUGLAS JR. AND S. KIM, *Improved accuracy for locally one-dimensioal methods for parabolic equations*, Mathematical Models and Methods in Applied Sciences, vol. 11 (2001), pp. 1563–1579.

[17] H. WANG AND W.-D. ZHAO, *A modified alternating-direction finite volume method for modeling secondary hydrocarbon migration and accumulation processes*, Numerical Methods for Partial Differential Equations, vol. 19 (2003), pp. 254–270.

[18] R. I.FERNANDES AND G. FAIRWEATHER, *An alternating-direction Galerkin method for a class of second-order hyperbolic equations in two space variables*, SIAM J. Numer. Anal., vol. 28 (1991), pp. 1265–1281.

[19] C. WANG AND T. WANG, *Extended locally one-dimensional finite difference and finite element schemes for nonhomogeneous parabolic differential equations with nonhomogeneous boundary conditions*, Numerical Mathematics, A Journal of Chinese Universities, vol. 28, no. 2 (2006), pp. 138–150 (in Chinese).

[20] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods, 2nd ed.*, Springer-Verlag, New York, 2002.