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Numerical Analysis of a System of Singularly Perturbed Convection-Diffusion Equations Related to Optimal Control

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Abstract. We consider an optimal control problem with an 1D singularly perturbed differential state equation. For solving such problems one uses the enhanced system of the state equation and its adjoint form. Thus, we obtain a system of two convection-diffusion equations. Using linear finite elements on adapted grids we treat the effects of two layers arising at different boundaries of the domain. We proof uniform error estimates for this method on meshes of Shishkin type. We present numerical results supporting our analysis.

AMS subject classifications: 34E05, 34E20, 65L10, 65L11, 65L60, 65L70

Key words: Convection-diffusion, linear finite elements, a priori analysis, layer-adapted meshes, singular perturbed, optimal control.

1. Introduction

Let us consider the following optimal control problem governed by a linear convection-diffusion equation

$$\min_{y,q} J(y,q) := \min_{y,q} \left(\frac{1}{2} \|y - y_0\|_0^2 + \frac{\lambda}{2} \|q\|_0^2 \right) \tag{1.1}$$

subject to

$$Ly := -\varepsilon y'' + ay' + by = f + q, \qquad \text{in } (0,1), \tag{1.2a}$$

$$y(0) = y(1) = 0. \tag{1.2b}$$

We assume

$$0 < \varepsilon \ll 1, \lambda > 0, \quad |a(x)| \geq \alpha > 0, \qquad \text{for } x \in (0,1)$$

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and a, b, f, y_0 to be sufficiently smooth. It is well-known (cf. [10]) that then there is an adjoint state p such that

$$\lambda q + p = 0, \tag{1.3a}$$
$$L^* p = -\varepsilon p'' - a p' + (b - a')p = y - y_0, \tag{1.3b}$$
$$p(0) = p(1) = 0. \tag{1.3c}$$

Consequently, y and p solve the system

$$-\varepsilon y'' + a y' + b y + \frac{1}{\lambda} p = f, \qquad y(0) = y(1) = 0, \tag{1.4a}$$
$$-\varepsilon p'' - a p' + (b - a')p - y = -y_0, \qquad p(0) = p(1) = 0. \tag{1.4b}$$

Discretization methods for system (1.4) (even in the two-dimensional case) are analyzed in [1, 4, 8]. Let us denote the numerical solutions of the given system by (y^N, q^N). Furthermore we denote the numerical solution of the state equation with a given right-hand side by $\tilde{y}^N$ (likewise the numerical solution of the adjoint equation for a given right-hand side by $\tilde{p}^N$). Based on the inequality

$$\|y - y^N\|_0^2 + \lambda \|q - q^N\|_0^2 \le \frac{1}{\lambda} \|p - \tilde{p}^N\|_0^2 + \|y - \tilde{y}^N\|_0^2,$$

the authors are able to estimate first the L_2-norm of $y - y^N$ and $q - q^N$ based on the L_2 errors for the discretization of the primal problem (1.2) and the adjoint problem (1.3) for a given right-hand side. In a second step stability estimates are used to prove error estimates in a stronger norm.

The estimates obtained in these papers contain H^2-norms of y and p which tend, in general, to infinity for $\varepsilon \to 0$. The influence of boundary layer terms is not discussed. But if layers exist, the technique just described is not adequate: in the singularly perturbed case one first estimates in a natural energy norm (and not in the L_2-norm) because optimal L_2 error estimates are more difficult to obtain.

In this paper we present a new technique for analyzing finite element discretizations of problem (1.1) on layer-adapted meshes based on information concerning the layer structure.

System (1.4) is a special case of the following system

$$L(u_1, u_2) := \begin{pmatrix} -\varepsilon u_1'' + a_1 u_1' + b_{11} u_1 + b_{12} u_2 \\ -\varepsilon u_2'' - a_2 u_2' - b_{21} u_1 + b_{22} u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \begin{matrix} u_1(0) = u_1(1) = 0, \\ u_2(0) = u_2(1) = 0. \end{matrix} \tag{1.5}$$

Assuming

$$a_1, a_2 \ge \alpha > 0, \tag{1.6a}$$
$$b_{11}, b_{22} \ge 0, \tag{1.6b}$$
$$b_{12} b_{21} > 0, \quad |b_{12}|, |b_{21}| \ge \beta > 0, \tag{1.6c}$$

we want to discuss (1.5), (1.6) in the paper at hand. For simplification in writing we assume furthermore

$$b_{12},\ b_{21} \geq \beta > 0,$$

but the results can easily be generalized to include the case $b_{12} \leq -\beta < 0, b_{21} \leq -\beta < 0$.

In [2] the assumption

$$\min\{b_{11}+b_{12}, b_{21}+b_{22}\} \geq \gamma > 0, \quad b_{12}, b_{21} < 0, \tag{1.7}$$

is used for analyzing the system (1.5), while the analysis in [5,7] is based on

$$\Gamma^{-1} \geq 0 \ \text{ with } \ \Gamma := \begin{pmatrix} 1 & -\left\|\frac{b_{12}}{b_{11}}\right\|_\infty \\ -\left\|\frac{b_{21}}{b_{22}}\right\|_\infty & 1 \end{pmatrix}. \tag{1.8}$$

At least for constant coefficients our analysis is more general (with exception of the less interesting cases with $b_{12}=0$ or $b_{21}=0$) and does not need any further requirements like (1.7) or (1.8).

Based on the assumptions (1.6) we shall prove bounds for the first and second order derivatives of $u := (u_1, u_2)$ in Section 2. In Section 3 we will use the proven bounds to analyze the error of a finite element discretization on a Shishkin mesh.

We assume that the data $a_i,\ b_{ij},\ f_i,\ (i,j \in \{1,2\})$ are sufficient smooth.

2. Properties of the exact solution

In this paper we denote the H^k-seminorm by $|f|_k^2 = \int_0^1 (f^{(k)})^2\,\mathrm{d}x$, where H^k is the Sobolev space of functions such that weak derivatives of order up to k are in L_2. Likewise we use the H^k-norm defined by $\|f\|_k^2 = \sum_{j=0}^k |f|_j^2$, consequently the L_2-norm is referred to as $\|\cdot\|_0$. The L_∞-norm is denoted by $\|f\|_\infty = \operatorname{ess\,sup}_{x\in(0,1)} |f(x)|$. We also need an ε-weighted H^1-norm which is defined by $\|f\|_\varepsilon^2 := \|f\|_0^2 + \varepsilon\,|f|_1^2$.

Throughout this paper C, $\tilde{C}$ and $\bar{C}$ denote constants that are independent of the perturbation parameter ε and the mesh size N.

First we formulate sufficient conditions for the existence of a weak solution of the system (1.5).

Theorem 2.1. *If the assumptions*

$$2b_{11}b_{21} - (a_1 b_{21} + \varepsilon b_{21}')' \geq 0, \tag{2.1a}$$

$$2b_{22}b_{12} + (a_2 b_{12} - \varepsilon b_{12}')' \geq 0, \tag{2.1b}$$

hold the system (1.5) *has a unique weak solution* $u \in H_0^1(0,1)$.

Proof. First we multiply the first and second equation of the system (1.5) by b_{21} and b_{12}, respectively. This leads to an equivalent system. Using the Lax-Milgram lemma one can show this system has an unique solution if the bilinear form

$$\tilde{a}(u,v) := \int_0^1 \varepsilon b_{21} u_1' v_1' + (a_1 b_{21} + \varepsilon b_{21}') u_1' v_1 + (b_{11} b_{21} u_1 + b_{12} b_{21} u_2) v_1 \, \mathrm{d}x$$
$$+ \int_0^1 \varepsilon b_{12} u_2' v_2' - (a_2 b_{12} - \varepsilon b_{12}') u_2' v_2 + (b_{22} b_{12} u_2 - b_{21} b_{12} u_1) v_2 \, \mathrm{d}x \qquad (2.2)$$

is V-elliptic. This is assured by the condition (2.1). Notice that we use the fact, that the terms $b_{12} b_{21} u_2 v_1$ and $-b_{21} b_{12} u_1 v_2$ cancel each other in case of $u_i = v_i$. □

As a consequence we immediately get the a priori estimate

$$\varepsilon \left(|u_1|_1^2 + |u_2|_1^2 \right) \leq C \left(\|f_1\|_0^2 + \|f_2\|_0^2 \right). \qquad (2.3)$$

As a next step we want to prove precise bounds for the derivatives of u_1 and u_2. While in [5] the inverse monotony of the matrix Γ (cf. (1.8)) is used to prove these bounds, we use an asymptotic expansion.

Let us consider the reduced problem

$$a_1 u_{1,l}' + b_{11} u_{1,l} + b_{12} u_{2,l} = \tilde{f}_{1,l}, \qquad u_{1,l}(0) = g_{1,l}, \qquad (2.4a)$$
$$-a_2 u_{2,l}' + b_{22} u_{2,l} - b_{21} u_{1,l} = \tilde{f}_{2,l}, \qquad u_{2,l}(1) = g_{2,l}. \qquad (2.4b)$$

Theorem 2.2. *Assume*

$$b_{11} b_{22} + b_{12} b_{21} - a_2 b_{12} \left(\frac{b_{11}}{b_{12}} \right)' \geq 0, \qquad (2.5a)$$
$$\text{or} \qquad b_{11} b_{22} + b_{12} b_{21} + a_1 b_{21} \left(\frac{b_{22}}{b_{21}} \right)' \geq 0. \qquad (2.5b)$$

Then the reduced problem (2.4) *has an unique solution u which smoothness depends on the smoothness of $f_{1,l}$ and $f_{2,l}$.*

Proof. The system (2.4) can be transformed to a second order boundary value problem in two ways. Condition (2.5) ensures the unique solvability of one of the resulting boundary value problems. □

For sufficient smooth coefficients this lemma implies the relation

$$\|u_{1,l}\|_{k+2} + \|u_{2,l}\|_{k+2} \leq C \left(\|\tilde{f}_{1,l}\|_{k+1} + \|\tilde{f}_{2,l}\|_{k+1} \right). \qquad (2.6)$$

Remark 2.1. In case of constant coefficients b_{11}, b_{12}, b_{21} and b_{22} the prerequisites (1.6) imply the requirement (2.5). If additionally a_1 and a_2 are constant the prerequisites also imply condition (2.1).

Remark 2.2. The system

$$-\varepsilon u_1'' + a_1 u_1' + b_{11}u_1 + b_{12}u_2 = f_1, \qquad u_1(0) = u_1(1) = 0, \tag{2.7a}$$

$$-\varepsilon u_2'' + a_2 u_2' + b_{22}u_2 + b_{21}u_1 = f_2, \qquad u_2(0) = u_2(1) = 0, \tag{2.7b}$$

with $a_1, a_2 > 0$ is significantly different from our system (1.5). First, the reduced problem leads to an initial value problem which always has a unique solution. Second, the transformation $u_i = e^{\mu x} v_i$, $i \in \{1,2\}$ leads to a system where we can choose μ in such a way that condition (1.8) is satisfied. Therefore the coefficients b_{11}, b_{12}, b_{21}, b_{22} have little influence on the behavior of the solution of system (2.7). These are significant differences to the system (1.5) we study here.

Next we construct an asymptotic expansion for u_1, u_2 and introduce the local variables $\xi := x/\varepsilon$, $\eta := (1-x)/\varepsilon$:

$$u_1 = \sum_{l=0}^{n} \varepsilon^l u_{1,l} + \sum_{l=1}^{n} \varepsilon^l v_l(\xi) + \sum_{l=0}^{n} \varepsilon^l w_l(\eta) + R_{1,n}, \tag{2.8a}$$

$$u_2 = \sum_{l=0}^{n} \varepsilon^l u_{2,l} + \sum_{l=0}^{n} \varepsilon^l r_l(\xi) + \sum_{l=1}^{n} \varepsilon^l s_l(\eta) + R_{2,n}. \tag{2.8b}$$

For details see Appendix A.

Combining the results for the asymptotic expansion we get:

Theorem 2.3. *If the data are sufficient smooth and the assumptions* (2.1) *and* (2.5) *hold the solution of system* (1.5) *can be decomposed in*

$$u_1 = S_1 + E_{10} + E_{11},$$
$$u_2 = S_2 + E_{20} + E_{21}$$

with

$$\left\| S_1^{(k)} \right\|_0 \le C, \qquad \left\| S_2^{(k)} \right\|_0 \le C, \tag{2.9a}$$

$$\left| E_{10}^{(k)}(x) \right| \le C\varepsilon^{1-k} e^{-\alpha \frac{x}{\varepsilon}}, \qquad \left| E_{11}^{(k)}(x) \right| \le C\varepsilon^{-k} e^{-\alpha \frac{1-x}{\varepsilon}}, \tag{2.9b}$$

$$\left| E_{20}^{(k)}(x) \right| \le C\varepsilon^{-k} e^{-\alpha \frac{x}{\varepsilon}}, \qquad \left| E_{21}^{(k)}(x) \right| \le C\varepsilon^{1-k} e^{-\alpha \frac{1-x}{\varepsilon}} \tag{2.9c}$$

for $k \le 2$. Here the generic constant C is independent of ε.

Proof. To show the result we consider the asymptotic expansion for $n = 1$. Thus estimate (A.3) of the Appendix gives

$$\|R_{i,1}\|_2 \le C, \qquad \text{for } i \in \{1,2\}.$$

Furthermore we can use estimate (2.6) to get

$$\|u_{i,0}\|_2 \le C, \qquad \|u_{i,1}\|_2 \le C. \tag{2.10}$$

Combining this results we get for

$$S_i = \sum_{l=0}^{1} \varepsilon^l u_{i,l} + R_{i,1} \tag{2.11}$$

the estimate

$$\|S_i\|_2 \le \|u_{i,0}\|_2 + \varepsilon \|u_{i,1}\|_2 + \|R_{i,1}\|_2 \le C. \tag{2.12}$$

From the Theorem A.2 we know that the boundary term E_{11} of the asymptotic expansion (2.8) has the form $\sum_{l=0}^{1} \varepsilon^l \mathbb{P}_l(x/\varepsilon)\exp(-x/\varepsilon)$. Differentiation proves the estimation. Analogously one can prove the bounds for the other layer terms E_{10}, E_{20}, E_{21}. □

This result also yields estimates for $\|u\|_\infty$ and $\|u'\|_\infty$.

In the next chapter we want to use this estimates to prove an a priori estimate for the error of a finite element method.

3. Error estimates for linear FEM on Shishkin meshes

In this section we discretize the system (1.5)

$$\begin{aligned} -\varepsilon u_1'' + a_1 u_1' + b_{11} u_1 + b_{12} u_2 &= f_1, \quad u_1(0) = u_1(1) = 0, \\ -\varepsilon u_2'' - a_2 u_2' - b_{21} u_1 + b_{22} u_2 &= f_2, \quad u_2(0) = u_2(1) = 0, \end{aligned}$$

with linear finite elements. We start from the weak formulation

$$\begin{aligned} \tilde{a}(u,v) &= \langle f,\, v\rangle, \qquad \text{for all } v \in V = \left(H_0^1(0,1)\right)^2, \\ \langle f,\, v\rangle &:= \int_0^1 f\, v \,\mathrm{d}x, \\ \tilde{a}(u,v) &:= \int_0^1 \varepsilon b_{21} u_1' v_1' + (a_1 b_{21} + \varepsilon b_{21}') u_1' v_1 + (b_{11} b_{21} u_1 + b_{12} b_{21} u_2) v_1 \,\mathrm{d}x \\ &\quad + \int_0^1 \varepsilon b_{12} u_2' v_2' - (a_2 b_{12} - \varepsilon b_{12}') u_2' v_2 + (b_{22} b_{12} u_2 - b_{21} b_{12} u_1) v_2 \,\mathrm{d}x. \end{aligned}$$

Denoting our finite element space by $V^N \subset V$, the finite element method reads: Find $u^N \in V^N$ such that

$$\tilde{a}(u^N, v) = \langle f,\, v\rangle, \qquad \text{for all } v \in V^N.$$

Based on the information from Theorem 2.3 concerning the layer structure we use a Shishkin mesh for the discretization. Because u_1 has a strong layer at $x = 1$ and u_2 at $x = 0$, we use different meshes for the two solution components. We neglect the weak layers for the construction of the mesh. A Shishkin mesh is a piecewise equidistant mesh. To cope with a boundary layer at $x = 0$ one chooses the transition point $\sigma_0 := \min\{1/2, 2\varepsilon \ln N/\beta\}$ and uses for the two subdomains $[0,\sigma_0]$, $[\sigma_1, 1]$ an equidistant mesh with $N/2$ nodes.

Analogously one chooses the transition point $\sigma_1 := \max\{1/2, 1-2\varepsilon\ln N/\beta\}$ to take account for a boundary layer at $x = 1$. This leads to meshes of a form shown in Fig. 1.

Now we will prove a priori estimates for our Galerkin method (stabilized methods will be studied in future work) in the ε-weighted H^1-norm. Denoting the nodal linear interpolant of u by u^I, we first bound the interpolation error $\|u-u^I\|_\varepsilon$ by using the inequalities of formula (2.9).

For some terms we use on the fine part Ω_f of the mesh an other estimation than on the coarse part Ω_c. We will denote the norms for the subdomains by attaching an Ω_f and Ω_c as a subscript, respectively. Analogously we denote the locally constant mesh size by h_{Ω_f} and $h = h_{\Omega_c}$.

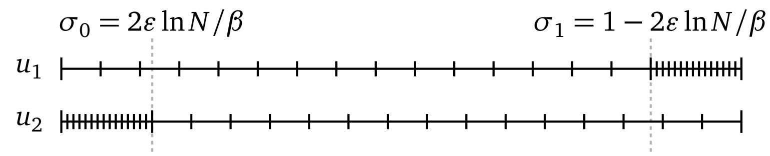

Figure 1: Used Shishkin mesh.

Theorem 3.1. *Provided the solution of the system* (1.5) *has a decomposition that satisfies the bounds of formula* (2.9) *for $k \le 2$ the interpolation error satisfies*

$$\|u-u^I\|_\varepsilon \le CN^{-1}\ln N. \tag{3.1}$$

Proof. By standard interpolation results we can estimate

$$\|S_1 - S_1^I\|_0 \le \tilde{C}h^2|S_1|_2 \le 4CN^{-2}, \tag{3.2a}$$

$$\|E_{11} - E_{11}^I\|_{0,\Omega_f} \le \tilde{C}h_{\Omega_f}^2|E_{11}|_{2,\Omega_f} \le \sqrt{\varepsilon}\,N^{-2}\ln^2 N, \tag{3.2b}$$

$$|S_1 - S_1^I|_1 \le \tilde{C}h|S_1|_2 \le 2CN^{-1}, \tag{3.2c}$$

$$|E_{11} - E_{11}^I|_{1,\Omega_f} \le \tilde{C}h_{\Omega_f}|E_{11}|_{2,\Omega_f} \le C\varepsilon^{-\frac{1}{2}}N^{-1}\ln N, \tag{3.2d}$$

Using the decaying of the boundary terms we can furthermore derive

$$\|E_{11} - E_{11}^I\|_{0,\Omega_c} \le \|E_{11}\|_{0,\Omega_c} + \|E_{11}^I\|_{0,\Omega_c} \le 2\sup_{\Omega_c}|E_{11}| = 2N^{-2}, \tag{3.2e}$$

$$\begin{aligned} |E_{11} - E_{11}^I|_{1,\Omega_c} &\le |E_{11}|_{1,\Omega_c} + |E_{11}^I|_{1,\Omega_c} \\ &\le \sup_{\Omega_c}|E_{11}'| + h_{\Omega_c}^{-1}\sup_{\Omega_c}|E_{11}| \le C\varepsilon^{-\frac{1}{2}}N^{-2} + N^{-1}. \end{aligned} \tag{3.2f}$$

These estimates are all attained by well-known techniques used e.g. in [6]. The interpolation error of the weak boundary layer can be bounded by

$$\begin{aligned} &|E_{10} - E_{10}^I|_1 \le \tilde{C}h|E_{10}|_2 \le C\varepsilon^{-\frac{1}{2}}N^{-1}, \\ &\|E_{10} - E_{10}^I\|_0 \le \tilde{C}h|E_{10}|_1 \le C\varepsilon^{\frac{1}{2}}N^{-1} \\ \text{and}\quad &\|E_{10} - E_{10}^I\|_0 \le \tilde{C}h^2|E_{10}|_2 \le C\varepsilon^{-\frac{1}{2}}N^{-2}. \end{aligned} \tag{3.2g}$$

which follows from the usual interpolation error estimates. This implies

$$\|E_{10} - E_{10}^I\|_0 \le C \min\{\varepsilon^{\frac{1}{2}} N^{-1}, \varepsilon^{-\frac{1}{2}} N^{-2}\} \le CN^{-\frac{3}{2}}. \tag{3.2h}$$

The bounds of the terms S_2, E_{20} and E_{21} can be proved similarly. Combining all these results proves the theorem. □

From the previously attained interpolation error estimates we can deduce an error estimate for $\|u - u^N\|_\varepsilon$:

Theorem 3.2. *If the solution of the system* (1.5) *has a decomposition that satisfies the estimates* (2.9) *for* $k \le 2$ *the finite element error satisfies*

$$\|u - u^N\|_\varepsilon \le CN^{-1} \ln N.$$

Proof. In the following we use the abbreviations $\chi := u^I - u^N$ and $\psi := u^I - u$. The coercivity of $\tilde{a}(\cdot,\cdot)$ and the Galerkin orthogonality of our method provide

$$\begin{aligned} \gamma\|\chi\|_\varepsilon^2 \le& \tilde{a}(\chi,\chi) = \tilde{a}(\psi,\chi) \\ \le& \varepsilon C |\chi|_1 |\psi|_1 + C\|\chi\|_0 \|\psi\|_0 + C\left|\int_\Omega \chi_1 \psi_1' \,\mathrm{d}x\right| + C\left|\int_\Omega \chi_2 \psi_2' \,\mathrm{d}x\right|. \end{aligned} \tag{3.3}$$

To estimate the remaining integral terms we split ψ as we did in (3.2). This way we get for the smooth part S_1 of the solution u using estimate (3.2a)

$$\left|\int_\Omega \chi_1 \left(S_1 - S_1^I\right)' \mathrm{d}x\right| \le \|\chi_1\|_0 \left\|\left(S_1 - S_1^I\right)'\right\|_0 \le CN^{-1}\|\chi_1\|_0 \le CN^{-1}\|\chi_1\|_\varepsilon. \tag{3.4a}$$

For the estimation of the boundary layer terms we transform the integral via integration by parts

$$\left|\int_\Omega \chi_1 \left(E - E^I\right)' \mathrm{d}x\right| = \left|\int_\Omega -\chi_1' \left(E - E^I\right) \mathrm{d}x\right| \le \|\chi_1'\|_0 \left\|E - E^I\right\|_0.$$

Using this transformation we can estimate analogous to estimate (3.2h)

$$\begin{aligned} \left|\int_\Omega \chi_1 \left(E_{10} - E_{10}^I\right)' \mathrm{d}x\right| \le& \|\chi_1'\|_0 \left\|E_{10} - E_{10}^I\right\|_0 \\ \le& \varepsilon^{1/2} CN^{-1}|\chi_1|_1 \le CN^{-1}\|\chi_1\|_\varepsilon. \end{aligned} \tag{3.4b}$$

For the strong boundary layer term we split the integral at the mesh transition point and use an inverse inequality on the coarse part of the mesh domain. This leads to the following formulas

$$\left|\int_{\Omega_f} \chi_1 \left(E_{11} - E_{11}^I\right)' \mathrm{d}x\right| \le C\sqrt{\varepsilon} N^{-2} \ln N \|\chi_1'\|_{0,\Omega_f} \le CN^{-1}\|\chi_1\|_{\varepsilon,\Omega_f}, \tag{3.4c}$$

$$\left|\int_{\Omega_c} \chi_1 \left(E_{11} - E_{11}^I\right)' \mathrm{d}x\right| \le \tilde{C} N^{-2}\|\chi_1'\|_{0,\Omega_c} \le CN^{-1}\|\chi_1\|_{\varepsilon,\Omega_c}. \tag{3.4d}$$

The integral containing χ_2 can be estimated analogously. Combining (3.3), (3.4), (3.1) and their equivalents for χ_2 we get the result

$$\|u^I - u^N\|_\varepsilon = \|\chi\|_\varepsilon \leq \tilde{C}\|\psi\|_\varepsilon + \bar{C}N^{-1} \leq CN^{-1}\ln N.$$

A triangle inequality completes the proof. □

For a single convection-diffusion equation it is well-known that supercloseness of the type

$$\|u^I - u^N\|_\varepsilon \leq C(N^{-1}\ln N)^2 \tag{3.5}$$

leads to the optimal estimate

$$\|u - u^N\|_0 \leq C(N^{-1}\ln N)^2. \tag{3.6}$$

Remark that in the singularly perturbed case it is not possible to use the Aubin-Nitsche-trick to attain optimal L_2-estimates that are independent of ε.

For our system the interpolation error estimate (3.2h) indicates that we do not have optimal L_2-error bounds if we ignore the weak layers for the mesh construction (cf. numerical experiments in Section 4).

If we, however, use two equal meshes for u_1 and u_2 with a refinement at each side of the domain we can adopt the proofs for a single equation (cf. [6]). The estimates of the weak boundary layer do not longer pose a problem; due to the refinement of the grid they can be handled the same way the strong layers are. Consequently we obtain

$$\begin{aligned} &\|u^I - u^N\|_\varepsilon \leq C\left(N^{-1}\ln N\right)^2 \\ \text{and}\qquad &\|u - u^N\|_0 \leq C\left(N^{-1}\ln N\right)^2. \end{aligned}$$

4. Computational results

In the following we solve the test problem

$$-\varepsilon u_1'' + \sqrt{2}u_1' + u_2 = 2, \qquad u_1(0) = u_1(1) = 0, \tag{4.1a}$$

$$-\varepsilon u_2'' - \sqrt{2}u_2' - u_1 = 1, \qquad u_2(0) = u_2(1) = 0 \tag{4.1b}$$

numerically. Because this problem has constant coefficients obviously our theory from the previous chapters applies.

An explicit solution of the system (4.1) is given by

$$u_1 = -1 + \sum_{i=1}^{4} \bar{u}_i\, e^{\lambda_i x}, \qquad u_2 = 2 + \sum_{i=1}^{4} \bar{u}_i\, p_i\, e^{\lambda_i x}$$

with

$$\lambda_i := \pm\frac{\sqrt{1 \pm \sqrt{1-\varepsilon^2}}}{\varepsilon}, \qquad p_i := (\varepsilon\lambda_i - \sqrt{2})\lambda_i.$$

Here $\bar{u}_i \in \mathbb{R}$ are the solution of the linear equation system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} & e^{\lambda_3} & e^{\lambda_4} \\ p_1 & p_2 & p_3 & p_4 \\ p_1 e^{\lambda_1} & p_2 e^{\lambda_2} & p_3 e^{\lambda_3} & p_4 e^{\lambda_4} \end{pmatrix} \bar{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

derived from the boundary conditions of the problem (4.1). Having this exact solution we can compute the discrepancy of the numerical to the explicit solution in various norms.

As in the previous analysis of Theorem 3.2 we first use a Shishkin mesh which only accounts for the strong boundary layers for the computations. The number of mesh intervals is denoted by N which gives us $N-1$ degrees of freedom. From this computations we attained the results shown in Fig. 2.

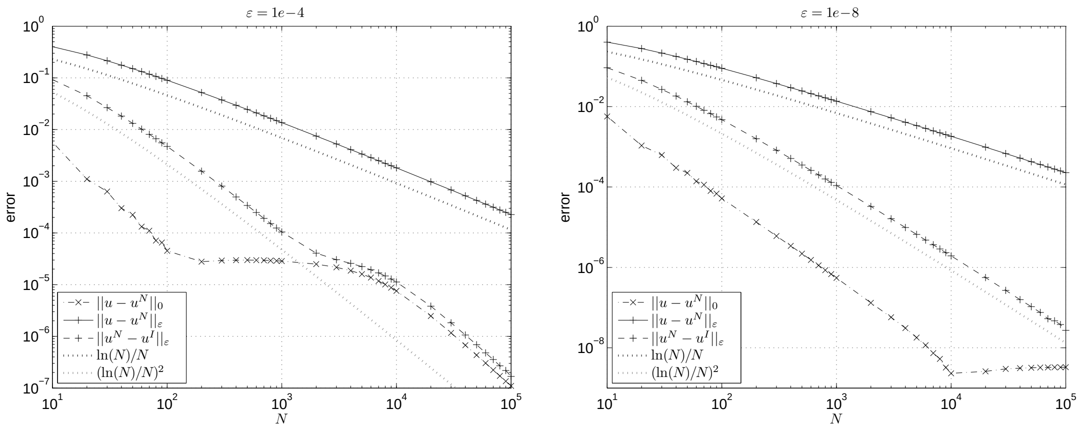

Figure 2: Error of the linear FEM on a one-sided Shishkin mesh (c.f. Fig. 1).

This numerical results confirm the theoretical result of an ε independent convergence in the ε-weighted H^1-norm. However they do not show the almost second order convergence measured in the L_2-norm one could expect knowing the superconvergence results for a single equation. The L_2-error rather exhibits a range of stagnating convergence in the order of magnitude of the perturbation parameter.

Finally we compare the results from the previous calculations with the error attained using a version of the Shishkin mesh where we refine in the region of the weak boundary layers as well as in the region of the strong ones. Thus the mesh has a form shown in Fig. 3.

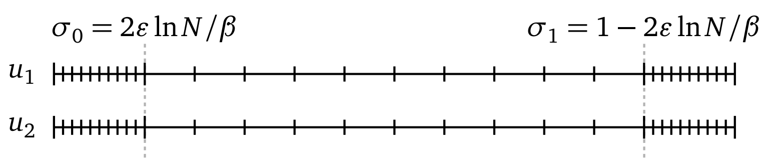

Figure 3: Used two-sided Shishkin mesh.

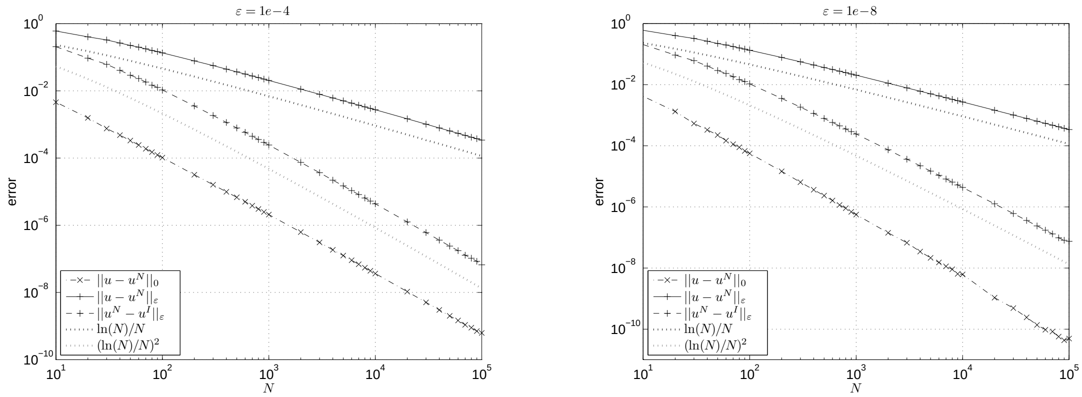

Figure 4: Error of the linear FEM on a two-sided Shishkin mesh.

The results of this computations are presented in Fig. 4. As predicted we now get almost second order convergence in the L_2-norm. Furthermore the range of stagnating convergence does not exist. But the absolute error measured in the ε-weighted H^1-norm is larger compared to the previous calculations. This is not surprising because in the first calculations we have more nodes of the grid to resolve the strong layer and the smooth region of the solution.

5. Summary and perspectives

Based on the asymptotic structure of the solution we solved system (1.5) with linear finite elements on a Shishkin mesh. The one-sided mesh, only adapted to the strong layers of the solution, results in an optimal error behavior in the energy norm but not in L_2. For obtaining optimal errors in L_2 a fine mesh also in the weak layer region seems necessary.

It is possible to extend our results to systems of two equations with two different parameters. For systems of $m > 2$ equations of the form

$$-\varepsilon u'' + \mathrm{diag}(a_i)u' + Bu = f$$

our approach works if there exist numbers $\mu_1, \cdots, \mu_m$ with $\mu_i > 0, i \in \{1, \cdots, m\}$ such that the matrix

$$\begin{pmatrix} \mu_1 B_{1*} \\ \vdots \\ \mu_m B_{m*} \end{pmatrix} \quad (B_{k*} \text{ denotes the } k\text{-th row of } B)$$

is positive definite.

However, our main interest in future is to investigate problem (1.1), (1.2) in the more dimensional case in space and its discretization with stabilized finite element methods on layer adapted meshes.

A. Asymptotic expansion

In the following we construct an asymptotic expansion

$$u_1 = \sum_{l=0}^{n} \varepsilon^l u_{1,l} + \sum_{l=0}^{n} \varepsilon^l v_l(\xi) + \sum_{l=0}^{n} \varepsilon^l w_l(\eta) + R_{1,n}, \tag{A.1a}$$

$$u_2 = \sum_{l=0}^{n} \varepsilon^l u_{2,l} + \sum_{l=0}^{n} \varepsilon^l r_l(\xi) + \sum_{l=0}^{n} \varepsilon^l s_l(\eta) + R_{2,n} \tag{A.1b}$$

of the solution of system (1.5) using the local variables $\xi := x/\varepsilon$, $\eta := (1-x)/\varepsilon$. The construction can mainly be done the same way it is done for a single differential equation (c.f. [9]), but the coupling of the two solutions u_1 and u_2 requires the consideration of a boundary layer on either side of the domain. We claim that the differential equation of (1.5) is fulfilled for $\sum_{l=0}^{n} \varepsilon^l u_{1,l}$ and $\sum_{l=0}^{n} \varepsilon^l u_{2,l}$. Here we skip the boundary condition at the right and left side in the first and second line of the system, respectively. Furthermore we demand that the corresponding homogeneous equation of (1.5) is fulfilled for the boundary terms. Transformation of the resulting system to the local variables ξ and η leads to the following equations:

$$\begin{aligned} a_1 u'_{1,0} + b_{11} u_{1,0} + b_{12} u_{2,0} &= f_1, & u_{1,0}(0) &= 0,\\ -a_2 u'_{2,0} + b_{22} u_{2,0} - b_{21} u_{1,0} &= f_2, & u_{2,0}(1) &= 0, \end{aligned} \tag{A.2a}$$

$l \geq 1$:

$$\begin{aligned} a_1 u'_{1,l} + b_{11} u_{1,l} + b_{12} u_{2,l} &= u''_{1,l-1}, & u_{1,l}(0) &= -v_l(0),\\ -a_2 u'_{2,l} + b_{22} u_{2,l} - b_{21} u_{1,l} &= u''_{2,l-1}, & u_{2,l}(1) &= -s_l(1), \end{aligned} \tag{A.2b}$$

$l \geq 0$:

$$\begin{aligned} & -v''_l + \tilde{a}_{1,0} v'_l = -\sum_{j=1}^{l} \left(\tilde{a}_{1,j} v'_{l-j} + \tilde{b}_{11,j-1} v_{l-j} + \tilde{b}_{12,j-1} r_{l-j} \right),\\ & \lim_{\xi\to\infty} v_l(\xi) = 0,\\ & -r''_l - \tilde{a}_{1,0} r'_l = -\sum_{j=1}^{l} \left(-\tilde{a}_{2,j} r'_{l-j} + \tilde{b}_{22,j-1} r_{l-j} - \tilde{b}_{21,j-1} v_{l-j} \right),\\ & \lim_{\xi\to\infty} r_l(\xi) = 0, \quad r_l(0) = -u_{2,l}(0), \end{aligned} \tag{A.2c}$$

$l \geq 0$:

$$
\begin{aligned}
&-w_l'' - \hat{a}_{1,0} w_l' = -\sum_{j=1}^{l} \left(-\hat{a}_{1,j} w_{l-j}' + \hat{b}_{11,j-1} w_{l-j} + \hat{b}_{12,j-1} s_{l-j} \right), \\
&\lim_{\eta \to \infty} w_l(\eta) = 0, \quad w_l(0) = -u_{1,l}(1), \\
&-s_l'' + \hat{a}_{2,0} s_l' = -\sum_{j=1}^{l} \left(\hat{a}_{2,j} s_{l-j}' + \hat{b}_{22,j-1} s_{l-j} - \hat{b}_{21,j-1} w_{l-j} \right), \\
&\lim_{\eta \to \infty} s_l(\eta) = 0,
\end{aligned} \tag{A.2d}
$$

where $\tilde{z}_i$ and $\hat{z}_i$ denotes the i-th coefficient of the Taylor expansion of $z(\varepsilon\xi)$ and $z(1-\varepsilon\eta)$ at $\xi = 0$ and $\eta = 0$, respectively. Note the difference $u_{1,l}(0) = -v_l(0)$, $u_{2,l}(1) = -s_l(1)$ from the standard expansion. This modification is necessary because the limitary condition for $\xi, \eta \to \infty$ determines $v_n(0)$ and $s_n(1)$ completely. To prove this we need the following theorems.

Theorem A.1. *The first terms of boundary layer correction have the form*

$$
\begin{aligned}
v_0(\xi) &= 0, & w_0(\eta) &= -u_{1,0}(1)\, e^{-a_1(1)\eta}, \\
r_0(\xi) &= -u_{2,0}(0)\, e^{-a_2(0)\xi}, & s_0(\eta) &= 0.
\end{aligned}
$$

Therefore the solutions u_1 and u_2 have a strict boundary layer only at the right and left boundary, respectively.

Proof. By solving the explicitly given boundary value problems (A.2c) and (A.2d). □

Theorem A.2. *The terms of boundary layer correction have the form*

$$
\begin{aligned}
v_i(\xi) &\in \mathbb{P}_{i-1}(\xi)\, e^{-a_2(0)\xi}, & w_i(\eta) &\in \mathbb{P}_i(\eta)\, e^{-a_1(1)\eta}, \\
r_i(\xi) &\in \mathbb{P}_i(\xi)\, e^{-a_2(0)\xi}, & s_i(\eta) &\in \mathbb{P}_{i-1}(\eta)\, e^{-a_1(1)\eta},
\end{aligned}
$$

where $\mathbb{P}_n(x)$ denotes the set of polynomials in the unknown x of degree less than $n+1$.

Proof. By inductive solution of the ordinary boundary value problems for v_i, w_i, r_i and s_i. □

Combining the previous results we get

$$
\begin{aligned}
&R_{i,n}(0) \in \mathcal{O}(\varepsilon^{n+1}), \\
&R_{i,n}(1) \in \mathcal{O}(\varepsilon^{n+1}), \\
&\|L(R_{1,n}, R_{2,n})\|_0 \leq C\, \|\varepsilon^{n+1} + \varepsilon^n e^{-\alpha\xi} + \varepsilon^n e^{-\alpha\eta}\|_0 \in \mathcal{O}(\varepsilon^{n+1/2}).
\end{aligned}
$$

Thus we get by our a priori estimate (2.3) the information

$$
\|R_{i,n}\|_1 \in \mathcal{O}(\varepsilon^n)
$$

for $i \in \{1, 2\}$. For the H^2-norm we get

$$
\|R_{i,n}\|_2 \leq \varepsilon^{-1} C \left(\|L(R_{1,n}, R_{2,n})\|_0 + \|R_{i,1}\|_1 \right) \in \mathcal{O}(\varepsilon^{n-1}). \tag{A.3}
$$

References

[1] R. Becker and B. Vexler, *Optimal Control of the Convection-Diffusion Equation Using Stabilized Finite Element Methods,* Numerische Mathematik, 106 (2007), pp. 349–367.

[2] Z. Cen, *Parameter-uniform finite difference scheme for a system of coupled singularly perturbed convection-diffusion equations,* International Journal of Computer Mathematics, 82 (2005), pp. 177–192.

[3] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* Springer-Verlag, 2001.

[4] M. Hinze, N. Yan and Z. Zhou, *Variational Discretization for Optimal Control Governed by Convection Dominated Diffusion Equations,* Journal of Computational Mathematics, 27 (2009), pp. 237–253.

[5] T. Linss *Analysis of an upwind finite-difference scheme for a system of coupled singularly perturbed convection-diffusion equations,* Computing, 79 (2007), pp. 23–32.

[6] T. Linss *Layer-Adapted Meshes for Reaction-Convection-Diffusion Problems,* Springer-Verlag, 2010.

[7] T. Linss and M. Stynes, *Numerical Solution of Systems of Singularly Perturbed Differential Equations,* Computational Methods in Applied Mathematics, 9 (2009), pp. 165–191.

[8] G. Lube and B. Tews, *Optimal Control of Singularly Perturbed Advection-Diffusion-Reaction Problems,* Mathematical Models and Methods in Applied Sciences, 20 (2010), pp. 375–395.

[9] H.-G. Roos, M. Stynes and L. Tobiska, *Robust Numerical Methods for Singularly Perturbed Differential Equations, 2nd Edition,* Springer-Verlag, 2008.

[10] F. Tröltzsch, *Optimal Control of Partial Differential Equations,* American Mathematical Society, 2010.