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Finite Element Approximation of Semilinear Parabolic Optimal Control Problems

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Abstract. In this paper, the finite element approximation of a class of semilinear parabolic optimal control problems with pointwise control constraint is studied. We discretize the state and co-state variables by piecewise linear continuous functions, and the control variable is approximated by piecewise constant functions or piecewise linear discontinuous functions. Some *a priori* error estimates are derived for both the control and state approximations. The convergence orders are also obtained.

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1. Introduction

Optimal control problems have been widely studied and applied in science and engineering numerical simulation. The finite element method seems to be the most widely used numerical methods in computing optimal control problems. More recently, there have been extensively studies in the finite element approximation of the general optimal control problems, see, for example, [3–5,11–18] and the references cited therein. However, it is impossible to give even a very brief review here. Systematic introductions of the finite element method for PDEs and optimal control problems can be found in, for example, [1,2,7–10].

In this work, we focus our attention on the finite element approximation of the following semilinear parabolic optimal control problems:

$$\min_{u\in K}\left\{\int_0^T (g(y)+h(u))\,dt\right\}, \tag{1.1}$$

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subject to the state equation

$$\begin{cases} y_t - \mathrm{div}(A\nabla y) + \phi(y) = f + Bu, & \text{in } \Omega \times (0,T], \\ y(x,t) = 0, & \text{on } \partial\Omega \times (0,T], \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

where $g(\cdot)$ and $h(\cdot)$ are two given convex functionals, K denotes the admissible set of the control variable u, and B is a linear continuous operator. The details will be specified later on. Problems (1.1)-(1.2) appears, for example, in temperature control problems, see [6].

In this paper, we aim to derive a L^2-norm error estimates for both the control and state approximations in space variables. Either piecewise constant elements ($m = 0$) or piecewise linear discontinuous elements ($m = 1$) for the control approximation is adopted. It is proved that these approximations have convergence order $\mathcal{O}(h_U^{1+m/2} + h^2 + \Delta t)$, where h_U and h are the spatial mesh-sizes for the control and state, respectively, and Δt is the time increment.

The remainder of this paper is organized as follows. In Section 2, we shall briefly discuss the finite element approximation for the semilinear parabolic control problems. In Section 3 some *a priori* error estimates are derived for both the control and state approximations. The paper ends with results from some numerical experiments in Section 4.

Throughout this work, we employ the usual notion for Lebesgue and Sobolev spaces, see [1,2] for details. In addition, c or C denotes a generic positive constant independent of the discrete parameters.

2. Finite element approximation of optimal control problems

In this section, we study the finite element approximation of problems (1.1)–(1.2). To describe it, let Ω and Ω_U be bounded open convex polygons in $\mathbb{R}^n$ ($n \le 3$), with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega_U$. Let $I = (0,T]$ be the time interval, and partition it by $T = N_T \Delta t, N_T \in \mathbb{Z}$, with $t_i = i\Delta t$ for $1 \le i \le N_T$. Let $f^i = f(x,t_i)$. We define, for $1 \le q < \infty$, the discrete time-dependent norms

$$\|f\|_{l^q(I;W^{m,p}(\Omega))} = \left(\sum_{i=1}^{N_T} \Delta t \|f^i\|_{m,p}^q\right)^{\frac{1}{q}},$$

and the standard modification for $q = \infty$. Let

$$l^q(I;W^{m,p}(\Omega)) := \left\{ f : \|f\|_{l^q(I;W^{m,p}(\Omega))} < \infty \right\}, \qquad 1 \le q \le \infty.$$

We shall take the state space $W = L^2(I;V)$ with $V = H_0^1(\Omega)$, the control space $X = L^2(I;U)$ with $U = L^2(\Omega_U)$, and $H = L^2(\Omega)$ to fix the idea. Let B be a linear continuous operator from X to $L^2(I;V')$, and K be a closed convex set in X. Let $g(\cdot)$ be a convex functional which is continuous differentiable on the observation space $H = L^2(\Omega)$, and

$h(\cdot)$ be a strictly convex continuous differentiable functional on U. We further assume that $h(u) \to +\infty$ as $\|u\|_U \to \infty$, and that the functional $g(\cdot)$ is bounded below. Let $A(x) = (a_{i,j}(\cdot))_{n\times n} \in (W^{1,\infty}(\Omega))^{n\times n}$ be symmetric and positive definite. Besides, for any $\mathbb{R} > 0$, the function $\phi(\cdot) \in W^{1,\infty}(-\mathbb{R},\mathbb{R})$, $\phi'(y) \in L^2(\Omega)$ for any $y \in L^2(I;H^1(\Omega))$, and $\phi'(y) \geq 0$.

We recast the state equation (1.2) as the following weak formula: For given f, u and y_0, find $y(u) \in H^1(I;L^2(\Omega)) \cap W$ such that

$$\big(y_t(u), w\big) + a(y(u), w) + (\phi(y(u)), w) = (f + Bu, w), \qquad \forall w \in V, \quad t \in I, \tag{2.1a}$$
$$y(u)(x,0) = y_0(x), \qquad x \in \Omega, \tag{2.1b}$$

where $a(v,w) = (A\nabla v, \nabla w)$. It is clear that under the above assumptions problem (2.1) has a unique weak solution for any $u \in K$.

Let

$$K = \Big\{ v \in X : v(x,t) \geq 0, \ a.e. \text{ in } \Omega_{\mathrm{U}} \times \mathrm{I} \Big\}.$$

Then the above convex optimal control problems can be restated as follows, which we shall label (QCP):

$$\min_{u\in K} \left\{ \int_0^T J(u)\, dt \right\}, \tag{2.2}$$

where $J(u) = g(y(u)) + h(u)$, and $y(u) \in W$ subject to

$$\begin{cases} \big(y_t(u), w\big) + a(y(u), w) + (\phi(y(u)), w) = (f + Bu, w), \quad \forall w \in V, \\ y(u)(x,0) = y_0(x). \end{cases}$$

Hereafter, we assume that

$$h(u) = \int_{\Omega_U} j(u),$$

where $j(\cdot)$ is a convex continuous differential function on $\mathbb{R}$. Then it is easy to see that

$$(h'(u), v)_U = (j'(u), v)_U = \int_{\Omega_U} j'(u)v.$$

It is well known (see, e.g., [7, 9]) that the control problems (QCP) has a solution (y,u), and that a pair (y,u) is the solution of (QCP) if there is a co-state $p \in W$ such that the triplet (y,p,u) satisfies the following optimality conditions: $(QCP - OPT)$

$$\begin{cases} \big(y_t, w\big) + a(y, w) + (\phi(y), w) = (f + Bu, w), \quad \forall w \in V, \\ y(0) = y_0, \end{cases} \tag{2.3}$$

$$\begin{cases} -\big(p_t, q\big) + a(q, p) + (\phi'(y)p, q) = (g'(y), q), \quad \forall q \in V, \\ p(T) = 0, \end{cases} \tag{2.4}$$

$$\int_0^T (j'(u) + B^*p, v - u)_U \, dt \geq 0, \quad \forall v \in K \subset X = L^2(I;U), \tag{2.5}$$

where B^* is the adjoint operator of B, $g'(\cdot)$ and $h'(\cdot)$ are the derivatives of $g(\cdot)$ and $h(\cdot)$, which have been viewed as functions in $H = L^2(\Omega)$ and $U = L^2(\Omega_U)$, respectively, and $(\cdot,\cdot)_U$ is the inner product of U.

Let $\mathcal{T}^h$ and $\mathcal{T}_U^h$ be regular triangulations of Ω and Ω_U, respectively, so that $\bar{\Omega} = \cup_{\tau\in\mathcal{T}^h}\bar{\tau}$, $\bar{\Omega}_U = \cup_{\tau_U\in\mathcal{T}_U^h}\bar{\tau}_U$. Let $h = \max_{\tau\in\mathcal{T}^h} h_\tau$, $h_U = \max_{\tau_U\in\mathcal{T}_U^h} h_{\tau_U}$, where h_τ and h_{τ_U} denote the diameter of the element τ and τ_U, respectively.

Let $V^h \subset V = H_0^1(\Omega)$ consist of continuous, piecewise linear functions on $\mathcal{T}^h$ of Ω, and $U^h \subset U = L^2(\Omega_U)$ consist of piecewise constant functions $(m=0)$ or piecewise linear discontinuous functions $(m=1)$ on $\mathcal{T}_U^h$ of Ω_U. Here there is no requirement for the continuity of the optimal control. Let K^h be a closed convex set in U^h such that $K^h = K \cap U^h$.

Now we are in a position to consider the fully discrete approximation for the control problems (QCP) by using backward Euler scheme.

Let $d_t\varphi^i = (\varphi^i - \varphi^{i-1})/\Delta t$, then a fully discrete approximate scheme of (QCP), which will be labeled as $(QCP)^{hk}$, is to find $(y_h^i, u_h^i) \in V^h \times K^h, i = 1, 2, \cdots, N_T$, such that

$$\min_{u_h^i\in K^h} \sum_{i=1}^{N_T} \Delta t J_h(u_h^i), \tag{2.6}$$

where $J_h(u_h^i) = g(y_h^i) + h(u_h^i)$, and $y_h^i \in V^h$ subject to

$$\begin{aligned}
&\left(d_t y_h^i, w_h\right) + a\left(y_h^i, w_h\right) + (\phi(y_h^i), w_h) = (f(x,t_i) + Bu_h^i, w_h), && \forall w_h \in V^h,\\
&y_h^0(x) = y_0^h(x), && x \in \Omega,
\end{aligned}$$

where $y_0^h \in V^h$ is an approximation of y_0 which is determined by the following elliptic projection (3.26).

The control problems $(QCP)^{hk}$ again has a solution (Y_h^i, U_h^i), and that a pair $(Y_h^i, U_h^i) \in V^h \times K^h$, is the solution of $(QCP)^{hk}$ if there is a co-state $P_h^{i-1} \in V^h$, such that the triplet $(Y_h^i, P_h^{i-1}, U_h^i) \in V^h \times V^h \times K^h$, satisfies the following optimality conditions: $(QCP-OPT)^{hk}$

$$\begin{aligned}
&(d_t Y_h^i, w_h) + a\left(Y_h^i, w_h\right) + (\phi(Y_h^i), w_h) = (f^i + BU_h^i, w_h), \quad \forall w_h \in V^h,\ i = 1, \cdots, N_T,\\
&Y_h^0(x) = y_0^h(x), \quad x \in \Omega, && (2.7)\\
&-(d_t P_h^i, q_h) + a\left(q_h, P_h^{i-1}\right) + (\phi'(Y_h^i)P_h^{i-1}, q_h) = (g'(Y_h^i), q_h), \quad \forall q_h \in V^h,\ i = N_T, \cdots, 1,\\
&P_h^{N_T}(x) = 0, \quad x \in \Omega, && (2.8)\\
&(j'(U_h^i) + B^* P_h^{i-1}, v_h - U_h^i)_U \ge 0, \quad \forall v_h \in K^h = K \cap U^h,\ i = 1, \cdots, N_T. && (2.9)
\end{aligned}$$

3. Convergence analysis and error estimates

In this section, we are able to derive some *a priori* error estimates for the finite element approximation of the optimal control problems (QCP). We show that the convergence order is optimal in $l^2(I; L^2(\Omega_U))$-norm for the control approximation error and in $l^\infty(I; L^2(\Omega))$-norm for the state and co-state approximation errors.

In many applications, $J(\cdot)$ is uniform convex near the solution u. The convexity of $J(\cdot)$ is closely related to the second order sufficient conditions of the control problems, which are assumed in many studies on numerical methods of the problem. For instance, in many applications, $g(\cdot)$ is convex, see [8] for some examples. Thus if $h(\cdot)$ is uniformly convex (e.g., $h(u)=\int_{\Omega_U} u^2$, which is frequently met), then there is a constat $c>0$, independent of $h(\cdot)$, such that

$$(J'(u)-J'(v),u-v)\geq c\|u-v\|_U^2,$$

where $u,v\in X$. Then for sufficiently small h, we have

$$(J_h'(u)-J_h'(v),u-v)\geq c\|u-v\|_U^2. \tag{3.1}$$

Throughout this work we shall assume the above inequality.

Let

$$\begin{aligned}
&\Omega_U^*(t)=\big\{\cup\tau_U:\tau_U\subset\Omega_U,u(\cdot,t)|_{\tau_U}>0\big\},\\
&\Omega_U^c(t)=\big\{\cup\tau_U:\tau_U\subset\Omega_U,u(\cdot,t)|_{\tau_U}\equiv 0\big\},\\
&\Omega_U^b(t)=\Omega_U\setminus(\Omega_U^*(t)\cup\Omega_U^c(t)).
\end{aligned}$$

It is easy to check that the three parts do not intersect on each other, and $\Omega_U=\Omega_U^*(t)\cup\Omega_U^c(t)\cup\Omega_U^b(t)$. In this paper we assume that u and $\mathscr{T}_U^h$ are regular such that meas$(\Omega_U^b(t))\leq Ch_U$ (see [16, 17]). Moreover, set

$$\Omega_U^{**}(t)=\big\{x\in\Omega_U,u(x,t)>0\big\}.$$

Then it is easy to see that $\Omega_U^*(t)\subset\Omega_U^{**}(t)$.

Define $J(\cdot)$ and $J_h(\cdot)$ as before. It is a matter of calculation to show that

$$\begin{aligned}
&(J'(u),v)_U=(j'(u)+B^*p,v)_U,\\
&(J_h'(U_h^i),v)_U=(j'(U_h^i)+B^*P_h^{i-1},v)_U,\\
&(J_h'(u^i),v)_U=(j'(u^i)+B^*P_h^{i-1}(u),v)_U,
\end{aligned}$$

where $P_h^{i-1}(u)\in V^h,\ i=1,2,\cdots,N_T$ is the solution of the following auxiliary problems:

$$\begin{aligned}
&\left(d_tY_h^i(u),w_h\right)+a\left(Y_h^i(u),w_h\right)+(\phi(Y_h^i(u)),w_h)=(f^i+Bu^i,w_h),\quad\forall w_h\in V^h,\\
&Y_h^0(u)=y_0^h(x),\quad x\in\Omega, && (3.2)\\
&-\left(d_tP_h^i(u),q_h\right)+a\left(q_h,P_h^{i-1}(u)\right)+(\phi'(Y_h^i(u))P_h^{i-1}(u),q_h)=(g'(Y_h^i(u)),q_h),\quad\forall q_h\in V^h,\\
&P_h^{N_T}(u)=0,\quad x\in\Omega. && (3.3)
\end{aligned}$$

For simplicity of illustration, in the rest of the paper we use the symbols

$$\begin{aligned}
&\theta^i=Y_h^i-Y_h^i(u),\quad \eta^i=y^i-Y_h^i(u),\qquad i=0,1,\cdots,N_T,\\
&\zeta^i=P_h^i-P_h^i(u),\quad \xi^i=p^i-P_h^i(u),\qquad i=N_T,\cdots,1,0.
\end{aligned}$$

It is clear that $\theta^0=0$ and $\zeta^{N_T}=0$.

Lemma 3.1. *Let (Y_h, P_h) and $(Y_h(u), P_h(u))$ be the solutions of (2.7)–(2.8) and (3.2)–(3.3), respectively. Assume that $g'(\cdot)$ and $\phi'(\cdot)$ are Lipschitz continuous in a neighborhood of y. Then*

$$\|Y_h - Y_h(u)\|_{l^\infty(I;L^2(\Omega))} + \|Y_h - Y_h(u)\|_{l^2(I;H^1(\Omega))} \le C\|u - U_h\|_{l^2(I;L^2(\Omega_U))}, \tag{3.4}$$

$$\|P_h - P_h(u)\|_{l^\infty(I;L^2(\Omega))} + \|P_h - P_h(u)\|_{l^2(I;H^1(\Omega))} \le C\|u - U_h\|_{l^2(I;L^2(\Omega_U))}. \tag{3.5}$$

Proof. We first prove (3.4). We subtract (3.2) from (2.7) to obtain that

$$(d_t\theta^i, w_h) + a\left(\theta^i, w_h\right) = (B(U_h^i - u^i), w_h) + (\phi(Y_h^i(u)) - \phi(Y_h^i), w_h), \qquad \forall w_h \in V^h. \tag{3.6}$$

Select $w_h = \theta^i$ as a test function. The inequality $a(a-b) \ge \frac{1}{2}(a^2 - b^2)$ shows that

$$(d_t\theta^i, \theta^i) \ge \frac{1}{2\Delta t}\left(\|\theta^i\|^2 - \|\theta^{i-1}\|^2\right). \tag{3.7}$$

Incorporate (3.7) into (3.6) and multiply both sides of (3.6) by $2\Delta t$ and sum over i from 1 to N $(1 \le N \le N_T)$, we then derive from the continuous property of B and $\phi(\cdot)$ that

$$\|\theta^N\|^2 + 2\sum_{i=1}^{N}\Delta t\|\theta^i\|_a^2 \le C\sum_{i=1}^{N}\Delta t\|\theta^i\|^2 + C\sum_{i=1}^{N}\Delta t\|u^i - U_h^i\|_U^2, \tag{3.8}$$

where we denote $\|v\|_a^2 = a(v,v)$. Thus (3.4) follows immediately from (3.8), Poincáre's inequality and the discrete Gronwall's lemma for sufficiently small Δt.

Then for (3.5). It follows from the co-state equations (2.8) and (3.3) that

$$\begin{aligned}-(d_t\zeta^i, q_h) + a(q_h, \zeta^{i-1}) =& \left(g'(Y_h^i) - g'(Y_h^i(u)), q_h\right) + \left(\phi'(Y_h^i(u))(P_h^{i-1}(u) - P_h^{i-1}), q_h\right) \\ &+ \left((\phi'(Y_h^i(u)) - \phi'(Y_h^i))P_h^{i-1}, q_h\right), \qquad \forall q_h \in V^h.\end{aligned} \tag{3.9}$$

Similarly, select $q_h = \zeta^{i-1}$ as a test function. We first see from the Lipschitz continuous of $g'(\cdot)$ and Cauchy-Schwarz inequality that

$$\left|\left(g'(Y_h^i) - g'(Y_h^i(u)), \zeta^{i-1}\right)\right| \le C\|\theta^i\|^2 + C\|\zeta^{i-1}\|^2. \tag{3.10}$$

Moreover, note that $\phi(\cdot) \in l^\infty(I; W^{1,\infty}(-\mathbb{R}, \mathbb{R}))$ and $\phi'(\cdot)$ is Lipschitz continuous in a neighborhood of y, thus

$$\begin{aligned}&\left|\left(\phi'(Y_h^i(u))(P_h^{i-1}(u) - P_h^{i-1}), \zeta^{i-1}\right)\right| \\ \le&\|\phi'(Y_h^i(u))\|_{0,\infty}\|P_h^{i-1}(u) - P_h^{i-1}\|\|\zeta^{i-1}\| \le C\|\zeta^{i-1}\|^2, \end{aligned}\tag{3.11}$$

$$\begin{aligned}&\left|\left((\phi'(Y_h^i(u)) - \phi'(Y_h^i))P_h^{i-1}, \zeta^{i-1}\right)\right| \\ \le&\|\phi'(Y_h^i(u)) - \phi'(Y_h^i)\|_{0,4}\|P_h^{i-1}\|\|\zeta^{i-1}\|_{0,4} \\ \le&C\|P_h^{i-1}\|^2\|\phi'(Y_h^i(u)) - \phi'(Y_h^i)\|_1^2 + C\delta\|\zeta^{i-1}\|_1^2 \\ \le&C\|\theta^i\|_1^2 + C\delta\|\zeta^{i-1}\|_a^2,\end{aligned} \tag{3.12}$$

where δ is an arbitrary small positive number and $C(\delta)$ depends on $1/\delta$. In the estimate of (3.12), we have used the embedding $\|v\|_{0,4} \leq C\|v\|_1$ and the property $\|P_h^{i-1}\| \leq C$ (see [1], for example).

Insert the above estimates (3.10)–(3.12) into (3.9), multiply both sides of (3.9) by $2\Delta t$ and sum over i from N_T to $M+1$ $(0 \leq M \leq N_T - 1)$, we have

$$\|\zeta^M\|^2 + \sum_{i=M+1}^{N_T} \Delta t\|\zeta^{i-1}\|_a^2 \leq C \sum_{i=M+1}^{N_T} \Delta t\|\zeta^{i-1}\|^2 + C \sum_{i=M+1}^{N_T} \Delta t\|\theta^i\|_1^2. \tag{3.13}$$

Thus we obtain from (3.13), Poincáre's inequality and the discrete Gronwall's lemma that for sufficiently small Δt

$$\|\zeta\|_{l^\infty(I;L^2(\Omega))} + \|\zeta\|_{l^2(I;H^1(\Omega))} \leq C\|Y_h - Y_h(u)\|_{l^2(I;H^1(\Omega))}. \tag{3.14}$$

Then (3.5) follows from (3.14) and the proved result (3.4). Therefore we complete the proof of Lemma 3.1. □

Lemma 3.2. *Let (y,p,u) and (Y_h,P_h,U_h) be the solutions of $(QCP-OPT)$ and $(QCP-OPT)^{hk}$, respectively. Assume that $u \in l^2(I;H^1(\Omega_U))$, $p \in l^2(I;H^1(\Omega))$, $K^h \subset K$ and $j'(\cdot)$ is locally Lipschitz continuous. Let U^h be the piecewise constant element space $(m=0)$. Then we have*

$$\|u - U_h\|_{l^2(I;L^2(\Omega_U))} \leq C\left(h_U + \Delta t + \|p - P_h(u)\|_{l^2(I;L^2(\Omega))}\right), \tag{3.15}$$

where $P_h(u)$ is defined in (3.3).

Furthermore, let U^h be the piecewise linear element space $(m=1)$. Assume that $u \in l^2(I;W^{1,\infty}(\Omega_U))$, $p \in l^2(I;W^{1,\infty}(\Omega))$, $u(t) \in H^2(\Omega_U^{**}(t))$, where $\Omega_U^{**}(t)$ is defined above. Then we have

$$\|u - U_h\|_{l^2(I;L^2(\Omega_U))} \leq C\left(h_U^{\frac{3}{2}} + \Delta t + \|p - P_h(u)\|_{l^2(I;L^2(\Omega))}\right). \tag{3.16}$$

Proof. From the definitions of $J(\cdot)$ and $J_h(\cdot)$, we have

$$\begin{aligned}
&(J'(u), u - v)_U = (j'(u) + B^*p, u - v)_U \leq 0, && \forall v \in K,\\
&(J_h'(U_h^i), U_h^i - v_h)_U = (j'(U_h^i) + B^*P_h^{i-1}, U_h^i - v_h)_U \leq 0, && \forall v_h \in K^h = K \cap U^h.
\end{aligned} \tag{3.17}$$

Let $\Pi_h u^i \in K^h$ be an approximation of $u(t_i)$, then we obtain from inequalities (3.1) and

(3.17) that

$$\begin{aligned}
c\|u-U_h\|^2_{l^2(I;L^2(\Omega_U))} &\le \sum_{i=1}^{N_T}\Delta t(J'_h(u^i)-J'_h(U_h^i),u^i-U_h^i)_U\\
=&\sum_{i=1}^{N_T}\Delta t(j'(u^i)+B^*P_h^{i-1}(u),u^i-U_h^i)_U+\sum_{i=1}^{N_T}\Delta t(j'(U_h^i)+B^*P_h^{i-1},U_h^i-u^i)_U\\
=&\sum_{i=1}^{N_T}\Delta t(j'(u^i)+B^*p^i,u^i-U_h^i)_U+\sum_{i=1}^{N_T}\Delta t(j'(U_h^i)+B^*P_h^{i-1},U_h^i-\Pi_h u^i)_U\\
&+\sum_{i=1}^{N_T}\Delta t(j'(U_h^i)+B^*P_h^{i-1},\Pi_h u^i-u^i)_U+\sum_{i=1}^{N_T}\Delta t(B^*(P_h^{i-1}(u)-p^i),u^i-U_h^i)_U\\
\le&\sum_{i=1}^{N_T}\Delta t(j'(U_h^i)+B^*P_h^{i-1},\Pi_h u^i-u^i)_U+\sum_{i=1}^{N_T}\Delta t(B^*(P_h^{i-1}(u)-p^i),u^i-U_h^i)_U. \qquad (3.18)
\end{aligned}$$

Note that the first term on the right-hand side of (3.18) can be expressed as

$$\begin{aligned}
&\sum_{i=1}^{N_T}\Delta t(j'(U_h^i)+B^*P_h^{i-1},\Pi_h u^i-u^i)_U\\
=&\sum_{i=1}^{N_T}\Delta t(j'(u^i)+B^*p^i,\Pi_h u^i-u^i)_U+\sum_{i=1}^{N_T}\Delta t(B^*p^i-B^*p^{i-1},\Pi_h u^i-u^i)_U\\
&+\sum_{i=1}^{N_T}\Delta t(j'(U_h^i)-j'(u^i),\Pi_h u^i-u^i)_U+\sum_{i=1}^{N_T}\Delta t(B^*P_h^{i-1}-B^*P_h^{i-1}(u),\Pi_h u^i-u^i)_U\\
&+\sum_{i=1}^{N_T}\Delta t(B^*P_h^{i-1}(u)-B^*p^{i-1},\Pi_h u^i-u^i)_U. \qquad (3.19)
\end{aligned}$$

Thus we conclude by Lemma 3.1, Cauchy-Schwarz inequality and the above equations (3.18)–(3.19) that

$$\begin{aligned}
c\|u-U_h\|^2_{l^2(I;L^2(\Omega_U))} \le& \sum_{i=1}^{N_T}\Delta t(j'(u^i)+B^*p^i,\Pi_h u^i-u^i)_U+C\|u-\Pi_h u\|^2_{l^2(I;L^2(\Omega_U))}\\
&+C\|p-P_h(u)\|^2_{l^2(I;L^2(\Omega))}+C\Delta t^2\left\|p_t\right\|^2_{L^2(I;L^2(\Omega))}+\frac{c}{2}\|u-U_h\|^2_{l^2(I;L^2(\Omega_U))}. \qquad (3.20)
\end{aligned}$$

First let us consider the case that U^h is the piecewise constant element space. Let Π_h be the L^2-projection from $U=L^2(\Omega_U)$ to U^h such that for any $v\in U$

$$(v-\Pi_h v,\phi)=0,\quad \forall\phi\in U^h.$$

It is easy to prove that $\Pi_h u^i\in K^h$, and it follows from [1,2] that for $u\in l^2(I;H^1(\Omega_U))$

$$\|u-\Pi_h u\|_{l^2(I;L^2(\Omega_U))}\le Ch_U\|u\|_{l^2(I;H^1(\Omega_U))}. \qquad (3.21)$$

Moreover, if $u \in l^2(I;H^1(\Omega_U))$ and $p \in l^2(I;H^1(\Omega))$, we have

$$
\begin{aligned}
&\sum_{i=1}^{N_T} \Delta t (j'(u^i)+B^*p^i, \Pi_h u^i - u^i)_U \\
=&\sum_{i=1}^{N_T} \Delta t \sum_{\tau_U \in \mathcal{T}_U^h} \int_{\tau_U} \Big(j'(u^i)+B^*p^i - \Pi_h(j'(u^i)+B^*p^i)\Big)(\Pi_h u^i - u^i) \\
\leq& \|j'(u)+B^*p - \Pi_h(j'(u)+B^*p)\|_{l^2(I;L^2(\Omega_U))} \|\Pi_h u - u\|_{l^2(I;L^2(\Omega_U))} \\
\leq& Ch_U^2 \left(\|u\|^2_{l^2(I;H^1(\Omega_U))} + \|p\|^2_{l^2(I;H^1(\Omega))} \right) \leq Ch_U^2. \qquad (3.22)
\end{aligned}
$$

Then the conclusion (3.15) follows from (3.20)–(3.22).

Next we consider the case that U^h is the piecewise linear element space. Let $\Pi_h u^i \in U^h$ be the standard Lagrange interpolation of u such that $\Pi_h u^i(z) = u(z,t_i)$ for any vertex z. It is clear that $\Pi_h u^i \in K^h$, and for $u \in l^2(I;W^{1,\infty}(\Omega_U))$, $u(t) \in H^2(\Omega_U^{**}(t))$ we have

$$
\|u^i - \Pi_h u^i\|_{0,\Omega_U^*(t_i)} \leq Ch_U^2 \|u^i\|_{2,\Omega_U^*(t_i)}, \qquad \|u^i - \Pi_h u^i\|_{0,\infty,\Omega_U^b(t_i)} \leq Ch_U \|u^i\|_{1,\infty,\Omega_U^b(t_i)}.
$$

Note that $\Pi_h u = u$ on $\Omega_U^c(t)$, then it follows that

$$
\begin{aligned}
&\|u - \Pi_h u\|^2_{l^2(I;L^2(\Omega_U))} = \sum_{i=1}^{N_T} \Delta t \int_{\Omega_U} (u^i - \Pi_h u^i)^2 \\
=&\sum_{i=1}^{N_T} \Delta t \left(\int_{\Omega_U^*(t_i)} (u^i - \Pi_h u^i)^2 + \int_{\Omega_U^c(t_i)} (u^i - \Pi_h u^i)^2 + \int_{\Omega_U^b(t_i)} (u^i - \Pi_h u^i)^2 \right) \\
\leq& Ch_U^4 \sum_{i=1}^{N_T} \Delta t \|u^i\|^2_{2,\Omega_U^*(t_i)} + 0 + Ch_U^2 \sum_{i=1}^{N_T} \Delta t \|u^i\|^2_{1,\infty,\Omega_U^b(t_i)} \mathrm{meas}(\Omega_U^b(t_i)) \\
\leq& Ch_U^4 \sum_{i=1}^{N_T} \Delta t \|u^i\|^2_{2,\Omega_U^*(t_i)} + Ch_U^3 \sum_{i=1}^{N_T} \Delta t \|u^i\|^2_{1,\infty,\Omega_U^b(t_i)} \\
\leq& Ch_U^3 \left(\|u\|^2_{l^2(I;H^2(\Omega_U^{**}(t)))} + \|u\|^2_{l^2(I;W^{1,\infty}(\Omega_U))} \right) \leq Ch_U^3. \qquad (3.23)
\end{aligned}
$$

Moreover, it follows from (2.5) that $j'(u)+B^*p = 0$ on $\Omega_U^*(t)$. Besides, we conclude from the definition of $\Omega_U^b(t)$ that for any element $\tau_U \subset \Omega_U^b(t)$, there is a x_0 such that $u(x_0,t) > 0$, and hence $(j'(u)+B^*p)(x_0) = 0$. Therefore for any $\tau_U \subset \Omega_U^b(t)$ we have

$$
\begin{aligned}
\|j'(u)+B^*p\|_{0,\infty,\tau_U} =& \|j'(u)+B^*p - (j'(u)+B^*p)(x_0)\|_{0,\infty,\tau_U} \\
\leq& Ch_U \|j'(u)+B^*p\|_{1,\infty,\tau_U}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{N_T}\Delta t(j'(u^i)+B^*p^i,\Pi_h u^i-u^i)_U\\
=&\sum_{i=1}^{N_T}\Delta t\int_{\Omega_U^*(t_i)}(j'(u^i)+B^*p^i)(\Pi_h u^i-u^i)+\sum_{i=1}^{N_T}\Delta t\int_{\Omega_U^c(t_i)}(j'(u^i)+B^*p^i)(\Pi_h u^i-u^i)\\
&\quad+\sum_{i=1}^{N_T}\Delta t\int_{\Omega_U^b(t_i)}(j'(u^i)+B^*p^i)(\Pi_h u^i-u^i)\\
=&0+0+\sum_{i=1}^{N_T}\Delta t\int_{\Omega_U^b(t_i)}(j'(u^i)+B^*p^i)(\Pi_h u^i-u^i)\\
\le&\sum_{i=1}^{N_T}\Delta t\|j'(u^i)+B^*p^i\|_{0,\infty,\Omega_U^b(t_i)}\|\Pi_h u^i-u^i\|_{0,\infty,\Omega_U^b(t_i)}\mathrm{meas}(\Omega_U^b(t_i))\\
\le&Ch_U^3\left(\|u\|^2_{l^2(I;W^{1,\infty}(\Omega_U))}+\|p\|^2_{l^2(I;W^{1,\infty}(\Omega))}\right)\le Ch_U^3. \qquad (3.24)
\end{aligned}
$$

Thus the conclusion (3.16) is proved by inserting (3.23)–(3.24) into (3.20). □

Before obtaining the final main error estimates, the following lemma is also needed.

Lemma 3.3. *Let* (y,p) *and* $(Y_h(u),P_h(u))$ *be the solutions of* (2.3)–(2.4) *and* (3.2)–(3.3)*, respectively. Assume that the conditions in Lemmas* 3.1–3.2 *are valid. Besides, we assume* $y,p\in l^\infty(I;H_0^1(\Omega)\cap H^2(\Omega))\cap H^1(I;H^2(\Omega))\cap H^2(I;L^2(\Omega))$. *Then the following estimate holds*

$$\|y-Y_h(u)\|_{l^\infty(I;L^2(\Omega))}+\|p-P_h(u)\|_{l^\infty(I;L^2(\Omega))}\le C(h^2+\Delta t), \qquad (3.25)$$

where C *depends on some spatial and temporal derivatives of* y *and* p.

Proof. These estimates are basically similar to those of Lemma 3.1, thus here we only give a rough description.

We decompose the error $\eta=y-Y_h(u)$ as $\eta=(y-\Theta y)+(\Theta y-Y_h(u))=\mu+\vartheta$, where $\Theta y(t)\in V^h$ is defined to be the elliptic projection of $y(t)\in V$ which satisfies

$$a(y(t)-\Theta y(t),w_h)=0,\quad \forall w_h\in V^h,\ t\in I. \qquad (3.26)$$

Similarly, the error $\xi=p-P_h(u)$ can be split as $\xi=(p-\Theta p)+(\Theta p-P_h(u))=\rho+\pi$, where $\Theta p(t)\in V^h$ is the elliptic projection of $p(t)\in V$ which satisfies

$$a(q_h,p(t)-\Theta p(t))=0,\quad \forall q_h\in V^h,\ t\in I. \qquad (3.27)$$

As in [19], the following estimates can be proved for $v=y$ or p that

$$
\begin{aligned}
&\|v-\Theta v\|_{l^\infty(I;L^2(\Omega))}+h\|v-\Theta v\|_{l^\infty(I;H^1(\Omega))}\le Ch^2\|v\|_{l^\infty(I;H^2(\Omega))},\\
&\|(v-\Theta v)_t\|_{L^2(I;L^2(\Omega))}+h\|(v-\Theta v)_t\|_{L^2(I;H^1(\Omega))}\le Ch^2\|v\|_{H^1(I;H^2(\Omega))}. \qquad (3.28)
\end{aligned}
$$

Since the estimates for μ and ρ are known, we need only to derive estimates for ϑ and π. Thus, we subtract (3.2) from (3.26) and choose $w_h = \vartheta^i$ to obtain an error equation on $\vartheta = \Theta y - Y_h(u)$:

$$(d_t\vartheta^i, \vartheta^i) + a(\vartheta^i, \vartheta^i) = -(d_t\mu^i, \vartheta^i) - (y_t^i - d_t y^i, \vartheta^i) + (\phi(Y_h^i(u)) - \phi(y^i), \vartheta^i). \quad (3.29)$$

Besides, Eqs. (3.3) and (3.27) can be differenced with $q_h = \pi^{i-1}$ to obtain an error equation on $\pi = \Theta p - P_h(u)$:

$$\begin{aligned} &-(d_t\pi^i, \pi^{i-1}) + a(\pi^{i-1}, \pi^{i-1}) \\ =&(d_t\rho^i, \pi^{i-1}) + (p_t^{i-1} - d_t p^i, \pi^{i-1}) + (g'(y^i) - g'(Y_h^i(u)), \pi^{i-1}) \\ &+ (g'(y^{i-1}) - g'(y^i), \pi^{i-1}) + (\phi'(Y_h^i(u))(P_h^{i-1}(u) - p^{i-1}), \pi^{i-1}) \\ &+ ((\phi'(Y_h^i(u)) - \phi'(y^i))p^{i-1}, \pi^{i-1}) + ((\phi'(y^i) - \phi'(y^{i-1}))p^{i-1}, \pi^{i-1}). \end{aligned} \quad (3.30)$$

Similar to the proof in Lemma 3.1, we derive from (3.29) and (3.30) that

$$\begin{aligned} &\|\vartheta\|_{l^\infty(I;L^2(\Omega))} + \|\vartheta\|_{l^2(I;H^1(\Omega))} \\ \le& C\Delta t\|y_{tt}\|_{L^2(I;L^2(\Omega))} + Ch^2\left(\|y\|_{H^1(I;H^2(\Omega))} + \|y\|_{l^2(I;H^2(\Omega))}\right), \end{aligned} \quad (3.31)$$

and

$$\begin{aligned} \|\pi\|_{l^\infty(I;L^2(\Omega))} + \|\pi\|_{l^2(I;H^1(\Omega))} \le& C\Delta t\left(\sum_{v=y,p}\|v_{tt}\|_{L^2(I;L^2(\Omega))} + \|y_t\|_{L^2(I;L^2(\Omega))}\right) \\ &+ Ch^2\sum_{v=y,p}\left(\|v\|_{H^1(I;H^2(\Omega))} + \|v\|_{l^2(I;H^2(\Omega))}\right). \end{aligned} \quad (3.32)$$

We then gather the results (3.31)–(3.32) with the well-known estimates for μ and ρ to finish the assertion of Lemma 3.3. □

Combing the bounds given by Lemmas 3.1–3.3 together, we can easily establish the following main result.

Theorem 3.1. *Suppose that* $\{y, p, u\}$ *and* $\{Y_h, P_h, U_h\}$ *are the solutions of* (2.3)–(2.5) *and* (2.7)–(2.9)*, respectively. Assume that all conditions of Lemmas* 3.1–3.3 *are valid. Then for* m*=0, 1, we have*

$$\begin{aligned} &\|y - Y_h\|_{l^\infty(I;L^2(\Omega))} + \|p - P_h\|_{l^\infty(I;L^2(\Omega))} + \|u - U_h\|_{l^2(I;L^2(\Omega_U))} \\ \le& C(h_U^{1+\frac{m}{2}} + h^2 + \Delta t), \end{aligned} \quad (3.33)$$

where C *depends on some spatial and temporal derivatives of* y, p *and* u.

Proof. It follows from (3.15)–(3.16) and (3.25) that

$$\begin{aligned} \|u - U_h\|_{l^2(I;L^2(\Omega_U))} \le& C\left(h_U^{1+\frac{m}{2}} + \Delta t + \|p - P_h(u)\|_{l^2(I;L^2(\Omega))}\right) \\ \le& C(h_U^{1+\frac{m}{2}} + h^2 + \Delta t). \end{aligned} \quad (3.34)$$

Moreover, it follows from Lemma 3.1, Lemma 3.3 and (3.34) that

$$\begin{aligned}
&\|y-Y_h\|_{l^\infty(I;L^2(\Omega))}+\|p-P_h\|_{l^\infty(I;L^2(\Omega))}\\
\le&\|Y_h-Y_h(u)\|_{l^\infty(I;L^2(\Omega))}+\|P_h-P_h(u)\|_{l^\infty(I;L^2(\Omega))}+\|y-Y_h(u)\|_{l^\infty(I;L^2(\Omega))}\\
&\quad+\|p-P_h(u)\|_{l^\infty(I;L^2(\Omega))}\\
\le&C\|u-U_h\|_{L^2(I;L^2(\Omega_U))}+C(h^2+\Delta t)\\
\le&C(h_U^{1+\frac{m}{2}}+h^2+\Delta t).
\end{aligned}\tag{3.35}$$

Thus we finish the proof of Theorem 3.1. □

4. Numerical experiments

In this section, we carry out two numerical examples to validate the *a priori* error estimates for the control, state and co-state. To solve the optimal control problems numerically, we use the C++ software package: AFEPack. It is freely available at `http://dsec.pku.edu.cn` **/~rli/**.

In our numerical test, we consider the following optimal control problems:

$$\min_{u(t)\in K}\frac{1}{2}\int_0^T\left(\|y-y_d\|^2+\|u-u_d\|^2\right)\,dt,\tag{4.1}$$

subject to a well-posed semilinear parabolic equation:

$$\begin{cases} y_t-\Delta y+y^3=f+u, & \text{in } \Omega\times I,\\ y(x,0)=y_0(x), & \text{in } \Omega,\end{cases}\tag{4.2}$$

and the co-state equation is

$$\begin{cases} -p_t-\Delta p+3y^2p=y-y_d, & \text{in } \Omega\times I,\\ p(x,T)=0, & \text{in } \Omega.\end{cases}\tag{4.3}$$

Both Eqs. (4.2) and (4.3) are combined with homogeneous Dirichlet boundary conditions.

In this work, we choose the domain $\Omega=[0,1]\times[0,1]$ and $T=1$. We adopt the same mesh partition for the state and control such that $\Delta t=h$ all along. The convergence order is computed by the following formula:

$$\text{Rate}\simeq\frac{\log(E_i)-\log(E_{i+1})}{\log(h_i)-\log(h_{i+1})},$$

where i responds to the spatial partition, and E_i denote the $l^\infty(I;L^2(\Omega))$-norm for the state and co-state approximations and $l^2(I;L^2(\Omega))$-norm for the control approximation.

Example 4.1. For the first example, the control is approximated by piecewise constant elements. The data and solutions under testing are as follows:

$$
\begin{aligned}
&p(x,t)=0,\\
&u_d(x,t)=\max(4\pi^2\sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi t),0),\\
&u(x,t)=u_d,\\
&y(x,t)=\sin(2\pi x_1)\sin(2\pi x_2)\sin(2\pi t),
\end{aligned}
$$

where the functions $f(x,t)$ and $y_d(x,t)$ are determined by inserting the known functions $y(x,t)$, $p(x,t)$, and $u(x,t)$ into (4.2)–(4.3).

In Table 1 numerical results are presented on a series of uniformly triangular meshes with $h=\frac{1}{10},\frac{1}{20},\frac{1}{40},\frac{1}{80}$. Fig. 1 shows the approximate solution and the contour-line for the

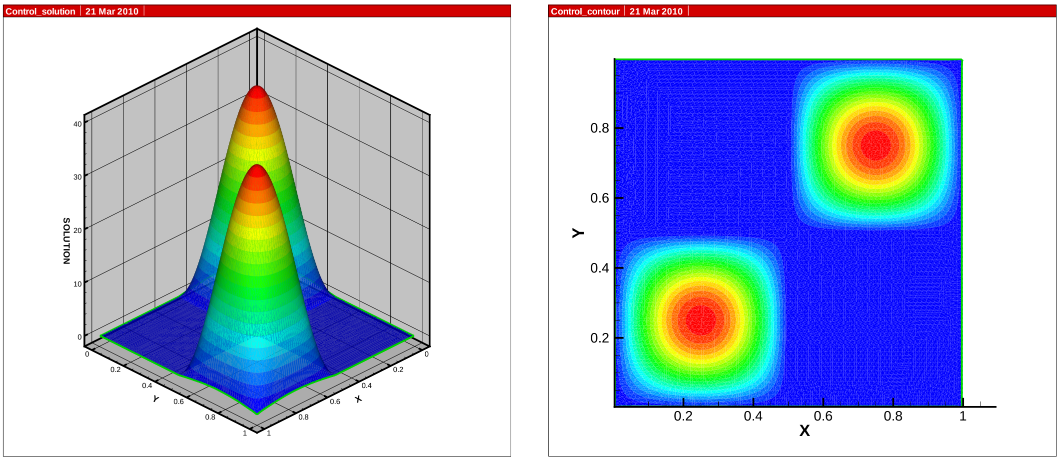

Figure 1: The control solution and its contour-line at $t=0.25$.

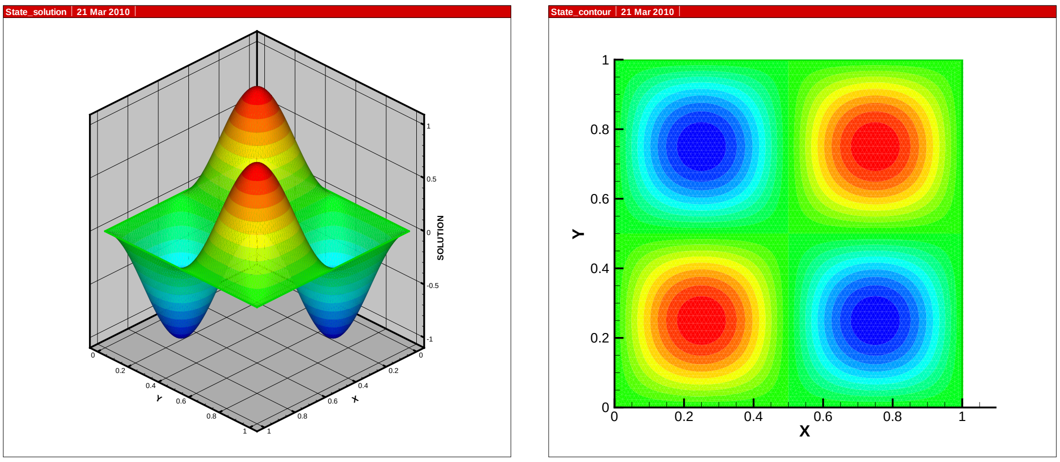

Figure 2: The state solution and its contour-line at $t=0.25$.

Table 1: Example 4.1 with piecewise constant approximation for the control.

h	$\|y-y_h\|_E$	Rate	$\|p-p_h\|_E$	Rate	$\|u-u_h\|_E$	Rate
$\frac{1}{10}$	4.8944e-02	——	3.3402e-03	——	1.7702e+00	——
$\frac{1}{20}$	1.5918e-02	1.6204	1.8669e-03	0.8393	8.8903e-01	0.9936
$\frac{1}{40}$	5.6093e-03	1.5048	9.6802e-04	0.9475	4.4870e-01	0.9865
$\frac{1}{80}$	2.1603e-03	1.3766	4.9053e-04	0.9807	2.2342e-01	1.0060

control at $t=0.25$ with $h=\frac{1}{80}$. Fig. 2 is the surface of the approximate state solution and its contour-line at $t=0.25$ with $h=\frac{1}{80}$.

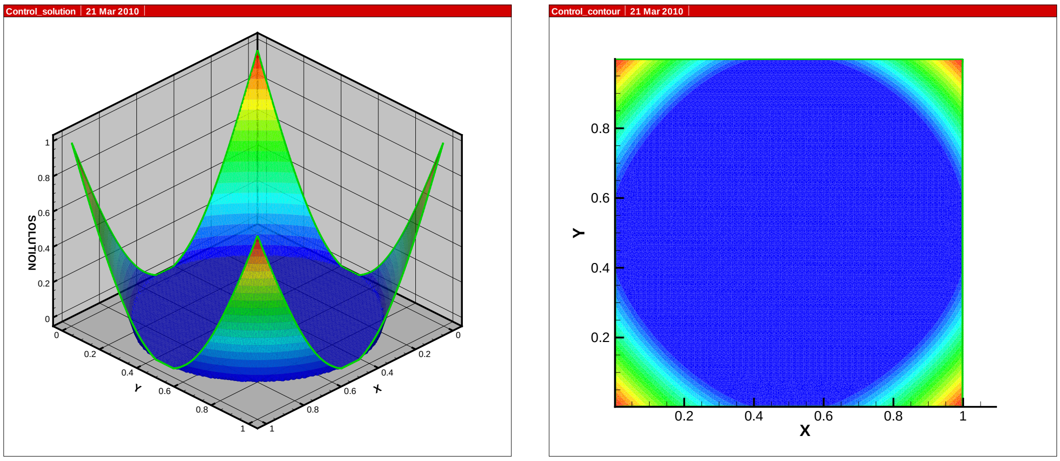

Figure 3: The control solution and its contour-line at $t=0.5$

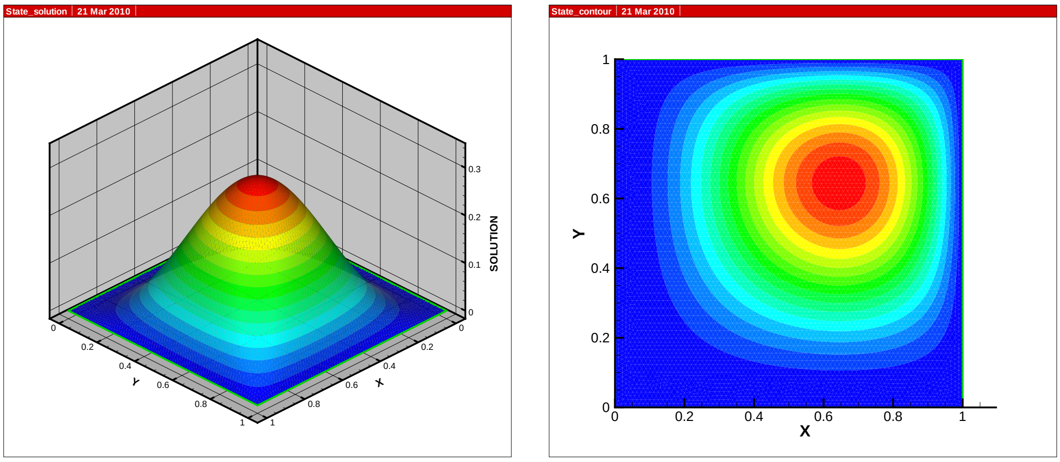

Figure 4: The state solution and its contour-line at $t=0.5$.

Example 4.2. For the second example, the control is approximated by piecewise linear elements. Let the desired functions $f(x,t)$ and $y_d(x,t)$ be chosen such that the corresponding analytical solutions for problems (4.1)–(4.3) are:

$$
\begin{aligned}
&y(x,t) = x_1 x_2 \sin(\pi x_1)\sin(\pi x_2)\sin(\pi t),\\
&p(x,t) = 0.5 x_1 x_2 \sin(\pi x_1)\sin(\pi x_2)\sin(\pi t),\\
&u_d(x,t) = 1.0 - \sin(\pi x_1) - \sin(\pi x_2),\\
&u(x,t) = \max(u_d - p, 0).
\end{aligned}
$$

Table 2 contains the same data as in Example 4.1 documenting the convergence rate versus the spatial mesh-sizes h, and in Figs. 3–4 we also display the profiles of the solutions for the control and state at $t = 0.5$ with $h = \frac{1}{80}$, respectively.

Table 2: Example 4.2 with piecewise linear approximation for the control.

h	$\|y - y_h\|_E$	Rate	$\|p - p_h\|_E$	Rate	$\|u - u_h\|_E$	Rate
$\frac{1}{10}$	5.2366e-03	——	1.6768e-02	——	1.0496e-02	——
$\frac{1}{20}$	2.0882e-03	1.3264	9.4314e-03	0.8302	3.9105e-03	1.4244
$\frac{1}{40}$	9.0622e-04	1.2043	4.9122e-03	0.9411	1.4739e-03	1.4077
$\frac{1}{80}$	4.2094e-04	1.1062	2.4889e-03	0.9809	5.4954e-04	1.4233

From the above numerical results, we can see that the convergence order obtained agrees very well with the *a priori* error estimates displayed in Theorem 3.1. The finite element method for the approximation of semilinear parabolic optimal control is effective and reliable.

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