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Conjugate Symmetric Complex Tight Wavelet Frames with Two Generators

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Abstract. Two algorithms for constructing a class of compactly supported conjugate symmetric complex tight wavelet frames $\Psi = \{\psi_1, \psi_2\}$ are derived. Firstly, a necessary and sufficient condition for constructing the conjugate symmetric complex tight wavelet frames is established. Secondly, based on a given conjugate symmetric low pass filter, a description of a family of complex wavelet frame solutions is provided when the low pass filter is of even length. When one wavelet is conjugate symmetric and the other is conjugate antisymmetric, the two wavelet filters can be obtained by matching the roots of associated polynomials. Finally, two examples are given to illustrate how to use our method to construct conjugate symmetric complex tight wavelet frames which have some vanishing moments.

AMS subject classifications: 42C15, 94A12

Key words: Complex tight wavelet frame, conjugate symmetry, vanishing moments.

1. Introduction

Recently, research on wavelets mainly concerns conventional real-valued wavelet bases and filter banks (see [1–8]). However, complex-valued wavelet bases also have been successfully studied and they offer a number of potential advantageous properties (see [9–14]). For example, in Radar and Sonar applications, the complex I/Q orthogonal signals can be processed with complex filter banks rather than processing the I/Q channels separately. Additionally, it is shown in [9–11] that the complex Daubechies wavelet can be symmetric and orthogonal, whereas the real-valued wavelet cannot. Another advantage of

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complex wavelets compared to real-valued wavelets is that they provide both magnitude and phase information. It is well known that linear phase is desirable for reasons of both computational complexity and image quality, which is usually important in image and audio sinal analysis (see [14]). As we know, real-valued symmetric wavelets and complex conjugate symmetric wavelets can have the linear phase (see [9]). The (conjugate) symmetry of linear phase filters leads to a lower complexity hardware implementation because the multiplies involving origin (conjugate) symmetric coefficient pairs can be combined. In the real-valued wavelets case, in order to achieve symmetry in a wavelet system, many generalizations of wavelet frames have been proposed and investigated in the literature (see [1–7]).

In this paper, we are interested in complex conjugate symmetric wavelet frames which have a linear phase. As a generalization of an orthonormal wavelet basis, a tight wavelet frame is an overcomplete wavelet system that preserves many desirable properties of an orthonormal wavelet basis. Based on the real wavelet frames, researchers study a class of complex tight wavelet frames with three conjugate symmetric generators and give an explicit parametric formula for the construction in [15]. Though by increasing the number of generators in a tight wavelet frame one has a great deal of freedom to construct them from refinable functions, in many applications, for various purposes such as computational and storage costs, one prefers a tight wavelet frame with as small as possible number of generators. This motivates us to consider the construction of complex tight wavelet frames with two conjugate (anti) symmetric generators. Furthermore, we give a criterion for the existence of the two generators and provide a description of a family of solutions when the low pass filter is of even length.

2. Construction of conjugate symmetric complex wavelet frames

Let us recall that a frame in a Hibert space $\mathscr{H}$ is a family of its elements $\{f_k\}_{k\in\mathbb{Z}}$ such that, for any $f \in \mathscr{H}$, $\exists\ 0 < A \leq B < \infty$,

$$A \parallel f \parallel^2 \leq \sum_{k\in\mathbb{Z}} | \langle f, f_k \rangle |^2 \leq B \parallel f \parallel^2,$$

where optimal A and B are called frame constants. If $A = B$, the frame is called *a tight frame*.

The frame $\{\psi_{s;j,k}\}_{s=1}^n$, where $\psi_{s;j,k}(x) = 2^{j/2}\psi_s(2^j x - k)$, $j, k \in \mathbb{Z}$, generated by translations and dilations of finite number of functions, is called *an affine* or *wavelet frame*. In this case, $\psi_1, \cdots, \psi_n$ are called the *generators* or *framelets*.

As in [1–3], following the multiresolution framework, we suppose that the refinable function and wavelets satisfy the refinement equation

$$\phi = 2\sum_{k\in\mathbb{Z}} h_0(k)\phi(2\cdot -k), \quad h_0(k) \in \mathbb{C}, \tag{2.1a}$$

$$\psi_s = 2\sum_{k\in\mathbb{Z}} h_s(k)\phi(2\cdot -k), \quad s = 1, 2, \cdots, n, \ \ h_s(k) \in \mathbb{C}. \tag{2.1b}$$

By taking the Fourier transform on both sides of (2.1a) and (2.1b), respectively, we get

$$\hat{\phi}(\omega)=m_0(\omega/2)\hat{\phi}(\omega/2),\quad \hat{\psi}_s(\omega)=m_s(\omega/2)\hat{\phi}(\omega/2),$$

where $m_0(\omega)=\sum_{k\in\mathbb{Z}}h_0(k)e^{-ik\omega}$ and $m_s(\omega)=\sum_{k\in\mathbb{Z}}h_s(k)e^{-ik\omega}$ are called the *mask symbols* of the refinable function ϕ and the wavelet function ψ_s, respectively. In addition, $m_0(\omega)$ and $m_s(\omega)$ are also called the *low-pass filter* and *high-pass filter* in engineering, respectively.

Definition 2.1. *We say that the refinable function $\phi(x)$ is* even *after an appropriate whole integer (l is even) or half-integer (l is odd) shift if its mask symbol satisfies*

$$\overline{m_0(\omega)}=e^{il\omega}m_0(\omega),\quad l\in\mathbb{Z}. \tag{2.2}$$

We also say that $\phi(x)$ is conjugate symmetric. *Similarly, we say that the framelet $\psi_s(x)$ is* conjugate symmetric (antisymmetric) *if its mask symbol satisfies*

$$\overline{m_s(\omega)}=\pm e^{il\omega}m_s(\omega),\quad s=1,2,\cdots,n,\ \ l\in\mathbb{Z}. \tag{2.3}$$

Remark 2.1. The refinable function $\phi(x)$ satisfying the condition (2.2) is generally called *symmetric* when $h_0(k)\in\mathbb{R}$. In such case, there is no difference between symmetric and conjugate symmetric since $h_0(k)=\overline{h_0(k)}$.

Denote

$$M(\omega)=\begin{pmatrix} m_0(\omega) & m_1(\omega) & \cdots & m_n(\omega)\\ m_0(\omega+\pi) & m_1(\omega+\pi) & \cdots & m_n(\omega+\pi)\end{pmatrix}.$$

It is shown in [1] that if

$$M(\omega)M^*(\omega)=I, \tag{2.4}$$

then ψ_s, $s=1,\cdots,n$, generate a tight wavelet frame of $L^2(\mathbb{R})$.

Lemma 2.1 ([1]). *Let a 2π-periodic function $m_0(\omega)$ satisfy $|m_0(\omega)|^2+|m_0(\omega+\pi)|^2\le 1$. Then there exists a pair of 2π-periodic measurable functions $m_1(\omega),m_2(\omega)$ which satisfy* (2.4) *for $n=2$. Furthermore, any solution of* (2.4)*, which is $\{m_1,m_2\}$ can be represented in the form of the first row of the matrix*

$$M_\psi(\omega)=P(\omega)D(\omega)Q(\omega), \tag{2.5}$$

where

$$P(\omega)=\begin{pmatrix} \overline{(\frac{e^{i\omega}m_0(\omega+\pi)}{\beta(\omega)})} & \frac{m_0(\omega)}{\beta(\omega)}\\ -\overline{(\frac{e^{i\omega}m_0(\omega)}{\beta(\omega)})} & \frac{m_0(\omega+\pi)}{\beta(\omega)}\end{pmatrix}, \tag{2.6a}$$

$$\Lambda(\omega)=\begin{pmatrix} 1 & 0\\ 0 & 1-|m_0(\omega)|^2-|m_0(\omega+\pi)|^2\end{pmatrix}, \tag{2.6b}$$

$|\beta(\omega)|^2=|m_0(\omega)|^2+|m_0(\omega+\pi)|^2$*, $D(\omega)$ is a diagonal matrix, $D(\omega)\overline{D(\omega)}=\Lambda(\omega)$, and $Q(\omega)$ is an arbitrary unitary (a.e.) matrix with π-periodic measurable components.*

Lemma 2.2 (Riesz Lemma [16]). *Let $a(0),\cdots,a(N)\in\mathbb{R}$ and $a(N)\neq 0$, such that*

$$\mathscr{A}(\omega):=\frac{a(0)}{2}+\sum_{k=1}^{N}a(k)\cos k\omega\geq 0,\quad \omega\in\mathbb{R}.$$

Then there exists a polynomial $\alpha(z)=\sum_{k=1}^{N}b(k)z^k$ with real coefficients that satisfies

$$|\alpha(z)|^2=\mathscr{A}(\omega),\quad z=e^{-i\omega}. \tag{2.7}$$

In what follows, the Laurent polynomial $p_s(z)$ is specified by the z-transform of the symbol $m_s(\omega)$, i.e., $p_s(e^{-i\omega}):=m_s(\omega)$, $s=0,1,\cdots,n$.

Theorem 2.1. *Let $p_0(z)$ be a conjugate symmetric complex (non-real) coefficients Laurent polynomial. Define $p(z)=|p_0(z)|^2+|p_0(-z)|^2$, such that $p(z)$ satisfies:*

$$1-p(z)\geq 0,\quad |z|=1. \tag{2.8}$$

Then there exist two conjugate (anti) symmetric complex (non-real) coefficients polynomials solutions to (2.4) if and only if all roots of the Laurent polynomial $1-p(z)$ have even multiplicity.

Proof. The proof can be shown by referring to the proof of Theorem 1 in [2]. The main distinction from the proof in [2] is the two solutions to (2.4) depending on the conjugate (anti) symmetric complex polynomials. Then in the proof of the sufficiency, we start by substituting $\lambda B(\omega)$ for $B(\omega)$ in [2] to get the complex (non-real) coefficients polynomials solutions to (2.4), where $\lambda\in\mathbb{C}\backslash\mathbb{R}$, $|B(\omega)|^2=p(z)=|m_0(\omega)|^2+|m_0(\omega+\pi)|^2$. Next we take the way which is the same as the proof in [2] but not in detail. Compared to the proof in [2], we need to substitute $\overline{f(z)}$ and conjugate (anti) symmetric for $f(1/z)$ and (anti) symmetric, respectively, where $f(z)$ is an arbitrary complex polynomial. Finally, the proof of the sufficiency and necessity can be established. ☐

Corollary 2.1. *Let $p_0(z)$ be a conjugate symmetric complex (non-real) coefficients Laurent polynomial. The conjugate symmetric complex (non-real) refinable function ϕ with the symbol $p_0(z)$ does not exist if $p_0(z)$ satisfies $deg(p_0)=1,2,3$.*

Proof. It is easy to check that the case $\deg(p_0)=1$ corresponds to the Haar wavelets which have no complex (non-real) coefficients. If $\deg(p_0)=2$, we assume that $p_0(z)=h_0(0)+h_0(1)z+h_0(2)z^2$. From $p_0(1)=1$, $p_0(-1)=0$ and $h_0(0)=\overline{h_0(2)}$, we get that $p_0(z)=\frac{1}{4}+ci+\frac{1}{2}z+(\frac{1}{4}-ci)z^2$, $\forall\, c\in\mathbb{R}$. Then

$$\begin{aligned}1-p(z)&=1-|p_0(z)|^2-|p_0(-z)|^2\\&=-z^{-2}\left[2\Big(\frac{1}{4}-ci\Big)^2z^4-\Big(\frac{1}{4}-4c^2\Big)z^2+2\Big(\frac{1}{4}+ci\Big)^2\right].\end{aligned}$$

We can prove that all roots of $1-p(z)$ have even multiplicity if and only if $c=0$. This implies that the complex (non-real) coefficients polynomial $p_0(z)$ does not exist. Similarly, the case from $\deg(p_0)=3$ can also be proved to have the same result. ☐

3. Solutions of conjugate symmetric complex wavelet frames

In the above argumentation, we provide a necessary and sufficient condition that the refinable function satisfies so that there exist the complex tight wavelet frames with two conjugate (anti) symmetric framelets (generators). Consider the problem: given a refinable function satisfying Theorem 2.1, how to find a simple way to obtain the two framelets associated with the refinable function?

Let ϕ be a complex refinable function, and ψ_1, ψ_2 generate a complex tight wavelet frame corresponding to ϕ; $p_0(z)$, $p_1(z)$ and $p_2(z)$, respectively, be the mask symbols of ϕ, ψ_1 and ψ_2. In this section, we describe how two conjugate (anti) symmetric framelets can be obtained by matching the roots of associated polynomials.

In the following, we deal exclusively with the case of even length filters. Let the filters $\{p_0(z), p_1(z), p_2(z)\}$ satisfy (2.4) and

$$h_0(n) = \overline{h_0(N-1-n)}, \quad h_2(n) = \overline{h_1(N-1-n)}. \tag{3.1}$$

Define

$$h_1^{new}(n) = \frac{1}{\sqrt{2}}\Big(h_1(n) + h_2(n-2d)\Big), \tag{3.2a}$$

$$h_2^{new}(n) = \frac{1}{\sqrt{2}}\Big(h_1(n) - h_2(n-2d)\Big). \tag{3.2b}$$

Write $p_s^{new}(z) = \sum_{k\in\mathbb{Z}} h_s^{new}(k) z^k,\ s = 1,2$. It is easy to check that $\{p_0(z), p_1^{new}(z), p_2^{new}(z)\}$ also satisfy (2.4), and

$$h_1^{new}(n) = \overline{h_1^{new}(N_2-1-n)}, \quad h_2^{new}(n) = -\overline{h_2^{new}(N_2-1-n)},$$

where $N_2 = N + 2d,\ d \in \mathbb{Z}$. Then, to obtain the two conjugate (anti) symmetric high-pass filters $p_1^{new}(z)$ and $p_2^{new}(z)$, we just need to find $p_1(z)$ and $p_2(z)$.

By (2.1a) and (2.1b), we get the filters $p_s(z) = \sum_{k\in\mathbb{Z}} h_s(k) z^k,\ s = 0,1,2,\ z = e^{-i\omega}$. Write

$$\begin{aligned} p_s(z) &= \sum h_s(2k) z^{2k} + \Big(\sum h_s(2k+1) z^{2k}\Big) z \\ &= \frac{\sqrt{2}}{2} p_{s0}(z^2) + \frac{\sqrt{2}}{2} p_{s1}(z^2) z. \end{aligned}$$

Define

$$S(z) := \begin{pmatrix} p_{00}(z) & p_{10}(z) & p_{20}(z) \\ p_{01}(z) & p_{11}(z) & p_{21}(z) \end{pmatrix}.$$

It is easy to check that (2.4) holds if and only if $S(z)S^*(z) = I$. Equivalently,

$$\begin{pmatrix} p_{10}(z) & p_{20}(z) \\ p_{11}(z) & p_{21}(z) \end{pmatrix} \begin{pmatrix} \overline{p_{10}(z)} & \overline{p_{11}(z)} \\ \overline{p_{20}(z)} & \overline{p_{21}(z)} \end{pmatrix} = \begin{pmatrix} 1 - |p_{00}(z)|^2 & -p_{00}(z)\overline{p_{01}(z)} \\ -p_{01}(z)\overline{p_{00}(z)} & 1 - |p_{01}(z)|^2 \end{pmatrix}. \tag{3.3}$$

According to (3.1), we obtain

$$p_{01}(z)=z^{-(\frac{N}{2}-1)}\overline{p_{00}(z)},\quad p_{20}(z)=z^{-(\frac{N}{2}-1)}\overline{p_{11}(z)},\quad p_{21}(z)=z^{-(\frac{N}{2}-1)}\overline{p_{10}(z)}. \tag{3.4}$$

Substituting (3.4) into (3.3), we get

$$\begin{aligned}&\begin{pmatrix} p_{10}(z) & z^{-(\frac{N}{2}-1)}\overline{p_{11}(z)} \\ p_{11}(z) & z^{-(\frac{N}{2}-1)}\overline{p_{10}(z)} \end{pmatrix}\begin{pmatrix} \overline{p_{10}(z)} & \overline{p_{11}(z)} \\ z^{(\frac{N}{2}-1)}p_{11}(z) & z^{(\frac{N}{2}-1)}p_{10}(z) \end{pmatrix}\\ =&\begin{pmatrix} 1-|p_{00}(z)|^2 & -z^{(\frac{N}{2}-1)}p_{00}^2(z) \\ -z^{-(\frac{N}{2}-1)}\overline{p_{00}^2(z)} & 1-|p_{00}(z)|^2 \end{pmatrix}.\end{aligned} \tag{3.5}$$

Taking the determinant of both sides of (3.5) and simplifying, we obtain

$$U^2(z)=1-2|p_{00}(z)|^2, \tag{3.6}$$

where $U(z)=|p_{10}(z)|^2-|p_{11}(z)|^2$. From (3.6), it is clear that all the roots of $1-2|p_{00}(z)|^2$ must be of even multiplicity. This is consistent with the condition of Theorem 2.1 that all roots of $1-p(z)$ have even multiplicity.

In what follows, to obtain the two high-pass filters $p_1(z)$ and $p_2(z)$ by the low-pass filter $p_0(z)$, some concrete conditions are added to the filters.

Theorem 3.1. *Let the filters* $\{p_0(z),p_1(z),p_2(z)\}$ *satisfy* (3.1) *and their polyphase components be of the form:*

$$p_{00}(z)=z^{-\frac{1}{2}(\frac{N}{2}-1)}\sqrt{2}x(z)\overline{y(z)},\quad p_{10}(z)=x^2(z),\quad p_{11}(z)=-y^2(z), \tag{3.7}$$

where $|x(z)|^2+|y(z)|^2=1$. *Then* $\{p_0(z),p_1(z),p_2(z)\}$ *satisfy* (2.4), *and* $p_1(z)$, $p_2(z)$ *can be determined by* $p_0(z)$.

Proof. By a direct calculation, it is easy to check that (3.7) satisfies (3.5). This means that $\{p_0(z),p_1(z),p_2(z)\}$ satisfy (2.4). Substituting (3.7) into (3.6), we obtain

$$\begin{aligned}U(z)&=|x^2(z)|^2-|y^2(z)|^2\\&=2|x(z)|^2-1\\&=1-2|y(z)|^2.\end{aligned}$$

So

$$|x(z)|^2=0.5+0.5U(z), \tag{3.8a}$$

$$|y(z)|^2=0.5-0.5U(z). \tag{3.8b}$$

It follows from (3.7) and (3.8a) that $x(z)$ is a common factor of both $p_{00}(z)$ and $0.5+0.5U(z)$, so $x(z)$ can be determined by identifying the common roots. Similarly, $y(z)$ can also be determined by the common roots of $\overline{p_{00}(z)}$ and $0.5-0.5U(z)$. Then

$$\begin{aligned} p_1(z) &= \frac{\sqrt{2}}{2}p_{10}(z^2)+\frac{\sqrt{2}}{2}p_{11}(z^2)z \\ &= \frac{\sqrt{2}}{2}x^2(z^2)-\frac{\sqrt{2}}{2}y^2(z^2)z, \end{aligned}$$

and $p_2(z)$ can be derived from $p_1(z)$. $\square$

Remark 3.1. As usual, it is convenient to normalize ϕ in (2.1a) such that $\hat{\phi}(0)=1$. Then $p_0(1)=1$. Since $p_{00}(1)=\overline{p_{01}(1)}$, we can take $p_{00}(1)=p_{01}(1)=\sqrt{2}/2$. Furthermore, it follows from (3.7) that $x(1)y(1)=1/2$. In addition, according to $p_1(1)=0$ and (3.7), it requires that $x^2(1)-y^2(1)=0$. Therefore, $x(z)$ and $y(z)$ can be normalized such that

$$x(1)=y(1)=\frac{1}{\sqrt{2}}. \tag{3.9}$$

Remark 3.2. In this section, we do not impose any sum rules on the filters about the smoothness. In fact, we can add $L-1$ vanishing moments on the low-pass filter $p_0(z)$, which satisfy the following equations (see [17,18]):

$$\sum_n(-1)^n n^k h_0(1-n)=0, \quad k=1,2,\cdots,L-2. \tag{3.10}$$

In Section 4, we will give two examples about the filters which have 2 vanishing moments and 3 vanishing moments by applying (3.10).

4. Examples

Example 4.1. Let

$$\begin{aligned} p_0(z) &= 0.187500+0.000278i+(0.312500-0.000464i)z-0.000742iz^2 \\ &\quad +0.000742iz^3+(0.312500+0.000464i)z^4+(0.187500-0.000278i)z^5. \end{aligned}$$

It is easy to check that $p_0(1)=1$, $p_0(-1)=0$, and $p_0(z)$ satisfies (3.10) for $L=3$, which means the filter $p_0(z)$ has 2 vanishing moments. By a direct calculation, it follows that all roots of $1-p(z)\geq 0$ have even multiplicity. Then according to Theorem 2.1, there exists a complex tight wavelet frame with two conjugate (anti) symmetric framelets. In what follows, we give a concrete algorithm to obtain the two high-pass filters $p_1(z)$ and $p_2(z)$.

First, we note that

$$p_{00}(z)=\sqrt{2}\Big[0.187500+0.000278i-0.000742iz+(0.312500+0.000464i)z^2\Big].$$

It is easy to find the roots of $p_{00}(z)$ and $\overline{p_{00}(z)}$ are approximately $\{1.292975i,-1.289017i\}$ and $\{0.773410i,-0.775785i\}$, respectively. Applying (3.6), it is not difficult to show that the roots of $0.5+0.5U(z)$ and $0.5-0.5U(z)$ are approximately $\{-1.289017i, -0.775785i\}$ and $\{1.292975i, 0.773410i\}$, respectively. Therefore, according to (3.7), (3.8a) and (3.8b), the roots of $x(z)$ and $y(z)$ are $\{-1.289017i\}$ and $\{0.773410i\}$, respectively. Using the normalization (3.9), we obtain

$$
\begin{aligned}
x(z) &= (0.265673-0.342457i)(z+1.289017i),\\
y(z) &= (0.442450+0.342195i)(z-0.773410i).
\end{aligned}
$$

Then we get that

$$
\begin{aligned}
p_1(z) =& \frac{\sqrt{2}}{2}\Big[0.077586+0.302344i+(0.047054+0.181129i)z\\
&+(0.469107-0.120380i)z^2-(0.468390-0.121680i)z^3\\
&-(0.046695+0.181963i)z^4-(0.078665+0.302808i)z^5\Big],\\
p_2(z) =& \frac{\sqrt{2}}{2}\Big[-0.078665+0.302808i-(0.046695-0.181963i)z\\
&-(0.468390+0.121680i)z^2+(0.469107+0.120380i)z^3\\
&+(0.047054-0.181129i)z^4+(0.077586-0.302344i)z^5\Big].
\end{aligned}
$$

Finally, applying (3.2a), (3.2b) and choosing $d=1$, we obtain a wavelet frame with two conjugate (anti) symmetric framelets whose symbols are:

$$
\begin{aligned}
p_1^{new}(z) =& \frac{1}{2}\Big[0.077586+0.302344i+(0.047054+0.181129i)z\\
&+(0.390442+0.182428i)z^2-(0.515085-0.303643i)z^3\\
&-(0.515085+0.303643i)z^4+(0.390442-0.182428i)z^5\\
&+(0.047054-0.181129i)z^6+(0.077586-0.302344i)z^7\Big],\\
p_2^{new}(z) =& \frac{1}{2}\Big[0.077586+0.302344i+(0.047054+0.181129i)z\\
&+(0.547772-0.423188i)z^2-(0.421695+0.060283i)z^3\\
&+(0.421695-0.060283i)z^4-(0.547772+0.423188i)z^5\\
&-(0.047054-0.181129i)z^6-(0.077586-0.302344i)z^7\Big].
\end{aligned}
$$

Then we get the real part figures and imaginary part figures of the refinable function and two framelets (see Figs. 1, 2 and 3, respectively).

Example 4.2. Let

$$
\begin{aligned}
p_0(z) =& -0.06250000+0.00000010i+(0.06250000+0.00000030i)z\\
&+(0.50000000+0.00000020i)z^2+(0.50000000-0.00000020i)z^3\\
&+(0.06250000-0.00000030i)z^4-(0.06250000+0.00000010i)z^5.
\end{aligned}
$$

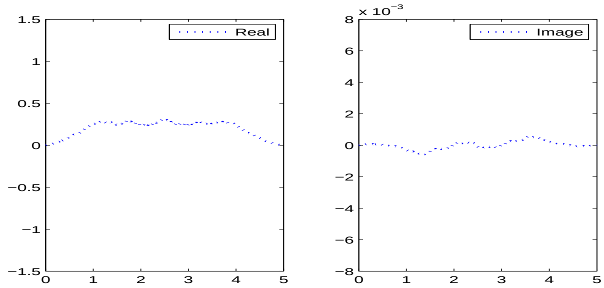

Figure 1: The refinable function ϕ with 2 vanishing moments from Example 4.1.

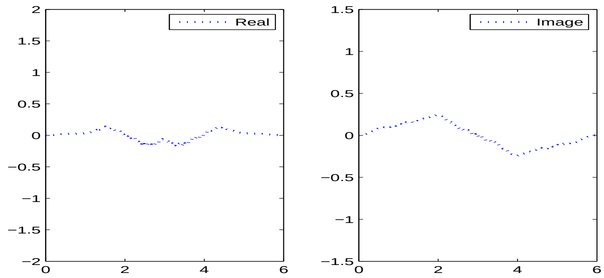

Figure 2: The framelet ψ_1^{new} from Example 4.1.

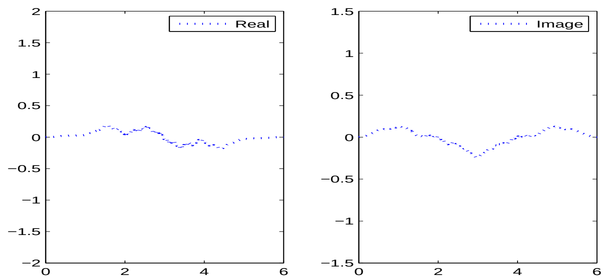

Figure 3: The framelet ψ_2^{new} from Example 4.1.

By a direct calculation, it is easy to check that all the conditions in Theorem 2.1 are satisfied and the filter $p_0(z)$ has 3 vanishing moments. Repeating the steps in Example 4.1, we obtain another wavelet frame with two conjugate (anti) symmetric framelets whose symbols are:

$$\begin{aligned} p_1^{new}(z) = \frac{1}{2}\Big[& 0.00600737 - 0.00000005i + (0.00985442 + 0.00000004i)z \\ & - (0.55264553 + 0.00000006i)z^2 + (0.55649267 - 0.00000004i)z^3 \\ & + (0.55649267 + 0.00000004i)z^4 - (0.55264553 - 0.00000006i)z^5 \\ & + (0.00985442 - 0.00000004i)z^6 + (0.00600737 + 0.00000005i)z^7 \Big], \end{aligned}$$

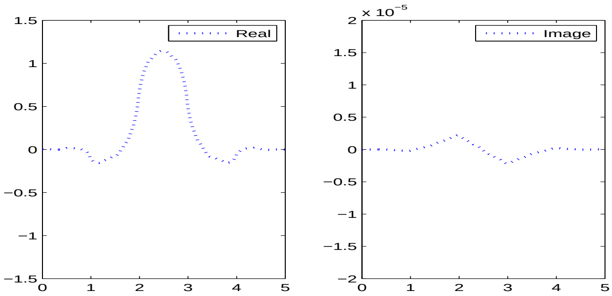

Figure 4: The refinable function ϕ with 3 vanishing moments from Example 4.2.

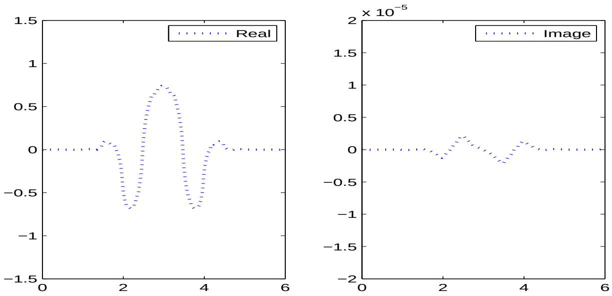

Figure 5: The framelet ψ_1^{new} from Example 4.2.

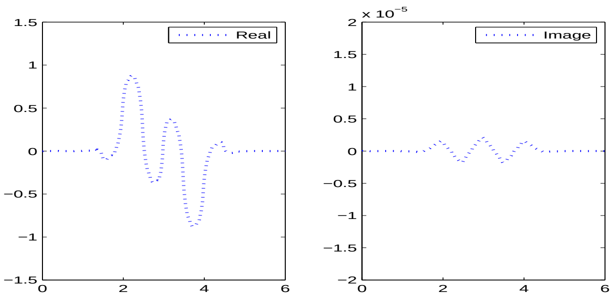

Figure 6: The framelet ψ_2^{new} from Example 4.2.

$$
\begin{aligned}
p_2^{new}(z) =& \frac{1}{2}\Big[0.00600737-0.00000005i+(0.00985442+0.00000004i)z \\
&+(0.7474783963-0.00000007i)z^2-(0.23629867-0.00000072i)z^3 \\
&+(0.23629867+0.00000072i)z^4-(0.7474783963+0.00000007i)z^5 \\
&-(0.00985442-0.00000004i)z^6-(0.00600737+0.00000005i)z^7\Big].
\end{aligned}
$$

Then we get the real part figures and imaginary part figures of the refinable function and two framelets (see Figs. 4, 5 and 6, respectively).

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