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THE UNIQUENESS OF VISCOSITY SOLUTIONS OF THE SECOND ORDER FULLY NONLINEAR ELLIPTIC EQUATIONS

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Abstract

Recently R. Jensen [1] has proved the uniqueness of viscosity solutions in $W^{1,\infty}$ of second order fully nonlinear elliptic equation $F(D^2u, Du, u)=0$. He does not assume F to be convex. In this paper we extend his result [1] to the case that F can be dependent on x, i. e. prove that the viscosity solutions in $W^{1,\infty}$ of the second order fully nonlinear elliptic equation $F(D^2u, Du, u, x)=0$ are unique. We do not assume F to be convex either.

1. Introduction

This paper deals with the problem of uniqueness of viscosity solutions of the fully nonlinear second order elliptic partial differential equation

$$F(D^2u, Du, u, x)=0 \quad \text{in} \quad \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u=g \quad \text{on} \quad \partial\Omega \tag{1.2}$$

For any $\varepsilon>0$ we define

$$F_\varepsilon^+(D^2u, Du, u, x)=F(D^2u, Du, u, x+\varepsilon Du/(1+|Du|^2)^{\frac{1}{2}}) \quad \text{in} \quad \Omega_\varepsilon \tag{1.3}$$

$$F_\varepsilon^-(D^2u, Du, u, x)=F(D^2u, Du, u, x-\varepsilon Du/(1+|Du|^2)^{\frac{1}{2}}) \quad \text{in} \quad \Omega_\varepsilon \tag{1.4}$$

where $\Omega_\varepsilon=\{x\in\Omega \mid \text{dist}(x,\partial\Omega)>\varepsilon\}$

In 1983 the definition of "viscosity solution" was introduced by M. G. Crandall and P. L. Lions [2] as a notion of weak solution of Hamilton-Jacobi equation

$$H(Du, u, x)=0 \quad \text{in} \quad \Omega \tag{1.5}$$

Under some assumptions, they have established global uniqueness and existence of viscosity solutions. In P. L. Lions work[4] the definiton of "viscosity solution" was extended to second order problems, i. e., to (1.1), and under some regularity assumptions on F which include the convexity of F, the uniqueness of viscosity solutions was proved. Finally R. Jensen[1] proved uniqueness of viscosity solutions of (1.1) and (1.2) in 1986. He does not assume F to be convex but only not allow spatial dependence in x. The techniques he used in [1] are new. He constructed two approximation operators $A_\varepsilon^+[u]=u_\varepsilon^+\geq A_\varepsilon^-[u]=u_\varepsilon^-$ and proved his result.

In this paper we prove a maximum principle of viscosity solutions which implies the uniqueness of viscosity solutions of (1.1) and (1.2) in the two cases: (α) F is degenerate elliptic, decreasing and uniformly continuous in x; or (β) F is uniformly elliptic, nonincreasing, Lipschitz continuous in p and uniformly continuous in x. We do not assume F to be convex either. The techniques which we use are similar to that in [1] but with some improvement. First we prove that $A_\varepsilon^+[\cdot]$ takes viscosity subsolutions of (1.1) into viscosity subsolutions of $F_\varepsilon^+[\cdot]=0$ and $A_\varepsilon^-[\cdot]$ takes viscosity supersolutions into viscosity supersolutions of $F_\varepsilon^-[\cdot]=0$. Then we obtain an estimation of semiconvex functions. Lastly we combine these results with results of [1] and give the maximum principle of viscosity solutions.

We implicitly assume throughout this paper that Ω is a bounded domain in $\mathbf{R}^n$, g is continuous on $\partial\Omega$ and solutions of (1. 1) and (1. 2) are always in $C(\overline{\Omega})$.

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2. Viscosity Solutions

We begin by recalling some definitions. The set of $n \times n$ real symmetric matrices will be denoted by $S(n)$. These matrices admit the partial ordering $\succ$ where $M \succ N$ if $M - N$ is positive semidefinite. A fully nonlinear *P. D. O.* $F[\cdot]$ is defined by

$$F[\varphi](x) = F(D^2\varphi,\ D\varphi,\ \varphi,\ \cdot)(x) \qquad \text{for all} \quad \varphi \in C^\infty(\Omega) \tag{2. 1}$$

where $F \in C(S(n) \times \mathbf{R}^n \times \mathbf{R} \times \Omega)$.

Definition 2. 1. *The operator $F[\cdot]$ is degenerate elliptic if*

$$F(M,\ p,\ t,\ x) \geq F(N,\ p,\ t,\ x) \tag{2. 2}$$

for all $M \succ N$ and all $(p, t, x) \in \mathbf{R}^n \times \mathbf{R} \times \Omega$. The operator $F[\cdot]$ is uniformly elliptic if there is a constant $c_1 > 0$ such that

$$F(M,\ p,\ t,\ x) = F(N,\ p,\ t,\ x) \geq c_1 \operatorname{trace}(M - N) \tag{2. 3}$$

for all $M \succ N$ and $(p, t, x) \in \mathbf{R}^n \times \mathbf{R} \times \Omega$

Definition 2. 2. *The operator $F[\cdot]$ is nonincreasing if*

$$F(M,\ p,\ t,\ x) \leq F(M,\ p,\ s,\ x) \tag{2. 4}$$

for all $t \geq s$ and $(M, p, x) \in S(n) \times \mathbf{R}^n \times \Omega$. The operator $F[\cdot]$ is decreasing if there is a constant $c_2 > 0$ such that

$$F(M,\ p,\ t,\ x) - F(M,\ p,\ s,\ x) \leq c_2(s - t) \tag{2. 5}$$

for all $t > s$ and $(M, p, x) \in S(n) \times \mathbf{R}^n \times \Omega$.

Definition 2. 3. *The operator $F[\cdot]$ is Lipschitz in p if there is a constant $c_3 > 0$ such that*

$$F(M,\ p,\ t,\ x) - F(M,\ q,\ t,\ x) \leq c_3|p - q| \tag{2. 6}$$

for all $\quad (M,\ p,\ q,\ t,\ x) \in S(n) \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \times \Omega$

The operator $F[\cdot]$ is uniformly continuous in x if there is a continuous increasing function $\sigma(x)$ such that $\sigma(0) = 0$ and

$$F(M,\ p,\ t,\ x) - (F(M,\ p,\ t,\ y) \leq \sigma(|x - y|) \tag{2. 7}$$

for all $\quad (M,\ p,\ t,\ x,\ y) \in S(n) \times \mathbf{R}^n \times \mathbf{R} \times \Omega \times \Omega$

Definition 2. 4. *$w \in C(\Omega)$ is a viscosity supersolution of (1. 1) if*

$$F(M,\ p,\ w(x),\ x) \leq 0 \qquad \text{for all } (p,\ M) \in D^- w(x) \text{ and all } \quad x \in \Omega \tag{2. 8}$$

$w \in C(\Omega)$ is a viscosity subsolution of (1. 1) if

$$F(M,\ p,\ w(x),\ x) \geq 0 \qquad \text{for all } (p,\ M) \in D^+ w(x) \text{ and all } x \in \Omega \tag{2. 9}$$

$w \in C(\Omega)$ is a viscosity solution of (1. 1) if both (2. 8) and (2. 9) hold, where $D^+ w(x)$ and $D^- w(x)$ denote superdifferential and subdifferential of $w(x)$, respectively (see [1]).

Lemma 2. 5. *Let $w \in C(\Omega)$. The following are equivalent:*

(i) w is a viscosity supersolution of (1. 1);

(ii) $F(D^2\varphi(x_0),\ D\varphi(x_0),\ \varphi(x_0),\ x_0) \leq 0$ for all open set $G \subset \Omega$ and all $(x_0, \varphi) \in G \times C^\infty(G)$ such that $w(x) \geq \varphi(x)$ for all $x \in G$, $w(x_0) = \varphi(x_0)$.

The proof of Lemma 2. 5 is similar to that of Lemma 2. 15 in [1].

Lemma 2. 6. *Let $w \in C(\Omega)$. The following are equivalent:*

(i) w is a viscosity subsolution of (1. 1);

(ii) $F(D^2\varphi(x_0),\ D\varphi(x_0),\ \varphi(x_0),\ x_0) \geq 0$ for all open set $G \subset \Omega$ and all $(x_0, \varphi) \in G \times C^\infty(G)$ such that $w(x) \leq \varphi(x)$ for all $x \in G$, $w(x_0) = \varphi(x_0)$.

Definition 2. 7. *For all $\varepsilon \in (0, \bar{\varepsilon}_0]$ ($\bar{\varepsilon}_0$ is the range in the implicit function Theorem, see [1]), we define*

$$F_\varepsilon^\pm(M,\ p,\ t,\ x) = F(M,\ p,\ t,\ x \pm \varepsilon p/(1 + |p|^2)^{\frac{1}{2}}) \tag{2. 10}$$

for all $(M, p, t, x) \in S(n) \times \mathbf{R}^n \times \mathbf{R} \times \Omega$. It is very clear that $F_\varepsilon^\pm \in C(S(n) \times \mathbf{R}^n \times \mathbf{R}$

$\times \Omega)$.

Definition 2.8. *Given $G \subset \mathbf{R}^n$ and $\varphi \in C^\infty(G)$ define the normal to graph (φ) at $(x, \varphi(x))$ by*

$$\nu(x) = (1 + |D\varphi(x)|^2)^{-\frac{1}{2}}(D\varphi(x), -1) \tag{2.11}$$

and define

$$\eta_\varphi(G) = \sup\{\eta \geq 0 \mid B((x, \varphi(x)) + \eta\nu(x), \eta) \cap \text{graph}(\varphi) = \varnothing, \text{ for all } x \in G\} \tag{2.12}$$

Lemma 2.9. *Assume $u \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ is a viscosity subsolution of (1.1). If $F[\cdot]$ is degenerate elliptic and nonincreasing then $u_\varepsilon^+ - \varepsilon$ is a viscosity subsolution of $F_\varepsilon^+[\cdot] = 0$ for any $\varepsilon \in (0, \bar{\varepsilon}_0]$, where u_ε^+ was defined in [1].*

Proof. We shall prove this Lemma by showing that (ii) of Lemma 2.6 holds for $u_\varepsilon^+ - \varepsilon$ on Ω_ε. Let $G \subset \Omega_\varepsilon$ be the open set such that $(x_0, \varphi) \in G \times C^\infty(G)$, $u_\varepsilon^+(x) - \varepsilon \leq \varphi(x)$ for all $x \in G$ and $u_\varepsilon^+(x_0) - \varepsilon = \varphi(x_0)$. We define that $\varphi_\delta(x) = \varphi(x) + \delta|x - x_0|^2$ and $\hat{\varphi}(x) = \varphi_\delta(x) + \varepsilon - \delta|x - x_0|^2$, then $u_\varepsilon^+(x) \leq \hat{\varphi}(x)$ for all $x \in G$ and $u_\varepsilon^+(x_0) = \hat{\varphi}(x_0)$. Let $\hat{\nu}(x)$ be defined by (2.11) with $\varphi = \hat{\varphi}$ and $\nu(x)$ defined by (2.11) with $\varphi = \varphi_\delta + \varepsilon$, we have

$$(x_0, \hat{\varphi}(x_0)) + \varepsilon\hat{\nu}(x_0) \in \text{graph}(u) \tag{2.13}$$

by the proof of Theorem 2.21 of [1]. Note that $\nu(x_0) = \hat{\nu}(x_0)$ and $\hat{\varphi}(x_0) = \varphi_\delta(x_0) + \varepsilon$, thus we see

$$(x_0, \varphi_\delta(x_0) + \varepsilon) + \varepsilon\nu(x) \in \text{graph}(u) \tag{2.14}$$

We claim that $\eta_{\varphi_\delta+\varepsilon}(G_\delta) > \varepsilon$ for some open set $G_\delta \subset G$. Indeed, let $(x', u(x')) = (x_0, \hat{\varphi}(x_0)) + \varepsilon\hat{\nu}(x_0)$ then

$$\begin{aligned} &|(x' - x_0, u(x') - \hat{\varphi}(x_0))| = \varepsilon \\ &|(x' - x, u(x') - \hat{\varphi}(x))| \geq \varepsilon \quad \text{for all } x \in G \end{aligned} \tag{2.15}$$

Thus, we have

$$B((x_0, \hat{\varphi}(x_0)) + \varepsilon\hat{\nu}(x_0), \varepsilon) \cap \text{graph}(\hat{\varphi}) = \varnothing$$

It follows by the definition of $\hat{\varphi}$ that for some $\eta(\delta) > \varepsilon$

$$B((x_0, \varphi_\delta(x_0) + \varepsilon) + \eta\nu(x), \eta) \cap \text{graph}(\varphi_\delta + \varepsilon) = \varnothing$$

Thus $\eta_{\varphi_\delta+\varepsilon}(x_0) \geq \eta > \varepsilon$ and by the continuity of $D\varphi$ and $D^2\varphi$ we conclude that there is an open set $G_\delta \subset G$ which is a small neighborhood of x_0 such that $\eta_{\varphi_\delta+\varepsilon}(G_\delta) > \varepsilon$.

Apply Lemma 1.29 of [1] to G_δ and $\varphi_\delta + \varepsilon \in C^\infty(G_\delta)$, the conclusion is that there is an open set $G_{\delta,\varepsilon}$ and a function $\varphi_{\delta,\varepsilon} \in C(G_{\delta,\varepsilon})$ such that

$$\begin{aligned} &\varphi_{\delta,\varepsilon}(x + \varepsilon p \circ \nu(x)) = \varphi_\delta(x) + \varepsilon + \varepsilon q \circ \nu(x) \\ &D\varphi_{\delta,\varepsilon}(x + \varepsilon p \circ \nu(x)) = D\varphi_\delta(x) \qquad \text{for all } x \in G_\delta \\ &D^2\varphi_{\delta,\varepsilon}(x + \varepsilon p \circ \nu(x)) < D^2\varphi_\delta(x) \end{aligned} \tag{2.16}$$

Since $u_\varepsilon^+(x) \leq \varphi_\delta(x) + \varepsilon$ for all $x \in G$ and dist (graph $(\varphi_\delta + \varepsilon)$, graph $(\varphi_{\delta,\varepsilon})) = \varepsilon$ we have $u(x) \leq \varphi_{\delta,\varepsilon}(x)$ for all $x \in G_{\delta,\varepsilon}$. By (2.14) we see that

$$u(x_0 + \varepsilon p \circ \nu(x_0)) = \varphi_{\delta,\varepsilon}(x_0 + \varepsilon p \circ \nu(x_0))$$

Set $x_0' = x_0 + \varepsilon p \circ \nu(x_0)$, note that u is a viscosity subsolution of (1.1) and by Lemma 2.6 we conclude

$$F(D^2\varphi_{\delta,\varepsilon}(x_0'), D\varphi_{\delta,\varepsilon}(x_0'), \varphi_{\delta,\varepsilon}(x_0'), x_0') \geq 0 \tag{2.17}$$

By (2.16) we have

$$\begin{aligned} &\varphi_{\delta,\varepsilon}(x_0') \geq \varphi_\delta(x_0) = \varphi(x_0) \\ &D\varphi_{\delta,\varepsilon}(x_0') = D\varphi_\delta(x_0) = D\varphi(x_0) \\ &D^2\varphi_{\delta,\varepsilon}(x_0') < D^2\varphi_\delta(x_0) = D^2\varphi(x_0) + 2\delta I \end{aligned}$$

Since $F[\cdot]$ is degenerate elliptic and nonincreasing, we see

$$F(D^2\varphi(x_0) + 2\delta I, D\varphi(x_0), \varphi(x_0), x_0 + \varepsilon D\varphi(x_0)(1 + |D\varphi(x_0)|^2)^{-1/2}) \geq 0$$

By the continuity of $F[\cdot]$ and the definition of $F_\varepsilon^+[\cdot]$

$$F_\varepsilon^+(D^2\varphi(x_0),\ D\varphi(x_0),\ \varphi(x_0),\ x_0) \geq 0$$

Lemma 2.6 now shows that $u_\varepsilon^+ - \varepsilon$ is viscosity subsolution of $F_\varepsilon^+[\cdot]=0$. This concludes the proof of this Lemma.

Lemma 2.10. *Assume that $u \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ is a viscosity supersolution of (1.1). If $F[\cdot]$ is degenerate elliptic and nonincreasing then $u_\varepsilon^- + \varepsilon$ is a viscosity supersolution of $F_\varepsilon^-[\cdot]=0$ for all $\varepsilon \in (0, \bar{\varepsilon}_0]$.*

3. The Maximum Principle

Definition 3.1. *Let $w \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ and define*

$$g_\delta = \{x \in \Omega \mid \text{for some } p \in \overline{B(0,\ \delta)},\ w(z) \leq w(x) + p(z-x) \text{ for all } z \in \Omega\} \tag{3.1}$$

Remark. It is clear that if $x \in g_\delta$ and w is differentiable at x then $Dw(x) \in \overline{B(0,\ \delta)}$ and $w(z) \leq w(x) + Dw(x)(z-x)$ for all $z \in \Omega$. Furthermore if $x \in g_\delta$ and w is twice differentiable at x then $D^2w(x) \prec 0$.

The next two Lemmas are from [1].

Lemma 3.2. (Lemma 3.3 of [1]). *Let $w \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ and assume*

$$D_\lambda^2 w \geq -K_0 \qquad \text{in } \Omega \quad (\text{in the sense of distributions}) \tag{3.2}$$

for any directions λ. Then there is a function $M \in L^1(\Omega, S(n))$ and a matrix valued measure $\Gamma \in M(\Omega, S(n))$ such that

(i) $D^2w = M + \Gamma$ (in the sense of distributions),

(ii) Γ is singular with respect to Lebesque measure,

(iii) $\Gamma(s)$ is positive semidefinite for all Borel subsets, S, of Ω,

(iV) $(M(x)\xi, \xi) \geq -K_0|\xi|^2$ for all $\xi \in \mathbf{R}^n$, for a. e. $x \in \Omega$

Lemma 3.3. (Lemma 3.10 of [1]) *Assume $w \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ and that (3.2) holds. If w has an interior maximum then there are constants $c_4 > 0$ and $\delta_0 > 0$ such that c_4 is dependent on K_0, δ_0 is independent of K_0 and*

$$\text{means } (g_\delta) \geq c_4\delta^n \qquad \text{for all } \delta < \delta_0 \tag{3.3}$$

Furthermore, let $w_\eta(x)$ be the regularization of $w(x)$ and let g_δ^η be the analogs for g_δ, then there is a constat $\eta_0 > 0$ such that

$$Dw_\eta(g_\delta^\eta) = \overline{B(0,\ \delta)} \qquad \text{if } \delta < \delta_0 \text{ and } \eta < \eta_0 \tag{3.4}$$

Now let us estimate the ratio of trace $(M(x))$ and $|Dw(x)|$ on appropriate subsets of Ω

Lemma 3.4. *Let $w \in C(\bar{\Omega}) \cap W^{1,\infty}$ and assume (3.2) valid. If w has an interior maximum then there is a constant $c_5 > 0$ which is independent of K_0 such that*

$$\int_{g_\delta \setminus g_{2^{-k}\delta}} [(\operatorname{tr}(M(x)))^- / |Dw(x)|]^n dx \geq C_5 k$$
$$\text{for all } \delta < \delta_0 \text{ and all } k \in \mathbf{Z}^+ \tag{3.5}$$

Proof. By (3.4) we have

$$Dw_\eta(g_\delta^\eta \setminus g_{\delta/2}^\eta) = \overline{B(0,\ \delta)} \setminus \overline{B(0,\ \delta/2)} \qquad \text{if } \delta < \delta_0 \text{ and } \eta < \eta_0$$

Thus, we conclude that

$$\int_{g_\delta^\eta \setminus g_{\delta/2}^\eta} |\operatorname{tr} D^2 w_\eta(x)|^n dx \geq n^n \int_{g_\delta^\eta \setminus g_{\delta/2}^\eta} |\det D^2 w_\eta(x)|\, dx \geq n^n \int_{Dw_\eta(g_\delta^\eta \setminus g_{\delta/2}^\eta)} dp$$
$$= n^n \int_{B(0,\delta) \setminus B(0,\delta/2)} dp = c_5\delta^n \qquad \text{for all } \delta < \delta_0 \text{ and } \eta < \eta_0$$

where $c_5 = n^n w_n(1-2^{-n})$, and w_n is the volume of unit sphere. Because $D^2w_\eta(x) \prec 0$ for all $x \in g_\delta^\eta$ we have

$$\int_{g_\delta^\eta \setminus g_{\delta/2}^\eta} |(\operatorname{tr} D^2 w_\eta(x))^-|^n dx \geq c_\delta \delta^n$$

For any sequence $\eta_i \searrow 0$, by the definition of g_δ, g_δ^η and w_η it is not difficult to see that

$$\operatorname{meas}\{(\limsup_{i\to\infty} g_\delta^{\eta_i}) \setminus g_\delta\} = 0 \qquad \text{for all } \delta \in (0, \delta_0) \tag{3.6}$$

$$\operatorname{meas}\{\limsup_{i\to\infty} [(g_{\delta'} \setminus g_\delta^{\eta_i}) \cap g_{2\delta}^{\eta_i}]\} = 0 \qquad \text{for } 0 < \delta' < \delta < \delta_0 \tag{3.7}$$

$$\operatorname{meas}\{g_\delta \setminus \bigcup_{\delta' < \delta} g_{\delta'}\} = 0 \qquad \text{for all } \delta \in I \tag{3.8}$$

where $I \subset (0, \delta_0)$ and meas $\{(0, \delta_0) \setminus I\} = 0$. We prove (3.7) only. Indeed, $w \in W^{1,\infty}$ and so we know that there is a set $E \subset \Omega$ such that meas $(E) = 0$ and $Dw(x)$ exist for all $x \in \Omega \setminus E$ and

$$w_{\eta_i}(x) \to w(x), \qquad Dw_{\eta_i}(x) \to Dw(x) \qquad \text{as} \quad i \to \infty$$

Let $x \in \{\limsup_{i\to\infty} [(g_{\delta'} \setminus g_\delta^{\eta_i}) \cap g_{2\delta}^{\eta_i}]\} \setminus E$, there is a subsequence $\{\eta_{i_k}\} \subset \{\eta_i\}$ such that $x \in (g_{\delta'} \setminus g_\delta^{\eta_{i_k}}) \cap g_{2\delta}^{\eta_{i_k}}$ for all k. Now by the Remark we see

$$Dw(x) \in \overline{B(0, \delta')} \tag{3.9}$$

$$w_{\eta_{i_k}}(z) \leq w_{\eta_{i_k}}(x) + Dw_{\eta_{i_k}}(x)(z - x) \qquad \text{for all } z \in \Omega_{\eta_{i_k}}$$

But $x \overline{\in} g_\delta^{\eta_{i_k}}$ so $|Dw_{\eta_{i_k}}(x)| > \delta$ for all k, let $k \to \infty$ we have $|Dw(x)| \geq \delta > \delta'$. This contradicts (3.9) and so

$$\{\limsup_{i\to\infty} [(g_{\delta'} \setminus g_\delta^{\eta_i}) \cap g_{2\delta}^{\eta_i}]\} \subset E \qquad \text{for } 0 < \delta' < \delta < \delta_0 \qquad Q.\ E.\ D$$

Let $\delta \in I$, we have

$$\operatorname{meas}[(g_\delta \setminus g_\delta^\eta) \cap g_{2\delta}^\eta] \leq \operatorname{meas}[(g_{\delta'} \setminus g_\delta^\eta) \cap g_{2\delta}^\eta] + \operatorname{meas}(g_\delta \setminus g_{\delta'})$$

First let $\eta \to 0$, and then let $\delta' \to \delta^-$, by (3.7) and (3.8) we obtain $\limsup_{\eta\to 0}$ meas $[(g_\delta \setminus g_\delta^\eta) \cap g_{2\delta}^\eta] \leq 0$, i. e. $\lim_{\eta \to 0}$ meas $[(g_\delta \setminus g_\delta^\eta) \cap g_{2\delta}^\eta] = 0$. Thus, $a_\eta(\delta) = \operatorname{meas}(g_\delta^\eta \setminus g_\delta) + \operatorname{meas}[(g_\delta \setminus g_\delta^\eta) \cap g_{2\delta}^\eta] \to 0$ as $\eta \to 0$, for a. e. $\delta \in (0, \delta_0)$. Let $V_j = \{\delta \in (0, \delta_0) \mid a_\eta(2^{-j}\delta) \to 0$ as $\eta \to 0\}$. Each set V_j is of measure zero so $V = \bigcup_{j=1}^{\infty} V_j$ is also of measure zero. For each $\delta \in (0, \delta_0) \setminus V$ we have

$$\operatorname{meas}\{g_{2^{-j}\delta}^\eta \setminus g_{2^{-j}\delta}\} + \operatorname{meas}\{(g_{2^{-j}\delta} \setminus g_{2^{-j}\delta}^\eta) \cap g_{2^{-j+1}\delta}^\eta\} \to 0 \qquad \text{as} \quad \eta \to 0 \tag{3.10}$$

for all $j \in \mathbf{Z}^+$. By the definition of w_η, for any direction λ

$$D_\lambda^2 w_\eta(x) = \int_{R^n} \psi_\eta(x - \xi) D_\lambda^2 w(\xi) d\xi$$

$$= \int_{R^n} \psi_\eta(x - \xi)(M(\xi)\lambda, \lambda) d\xi + \int_{R^n} \psi_\eta(x - \xi)(d\Gamma(\xi)\lambda, \lambda)$$

By lemma 3. 2 we see

$$D^2 w_\eta(x) \succ \int_{R^n} \psi_\eta(x - \xi) \cdot M(\xi) d\xi = M_\eta(x) \tag{3.11}$$

$$M_\eta(x) \to M(x) \qquad \text{as} \quad \eta \to 0, \qquad \text{for a. e. } x \in \Omega \tag{3.12}$$

By the definitions of g_δ^η and g_δ and Remark, we have

$$0 \geq \operatorname{trace}(M(x)) \geq -nK_0 \qquad \text{for a. e. } x \in g_\delta \tag{3.13}$$

$$0 \geq \operatorname{trace}(D^2 w_\eta(x)) \geq \operatorname{trace}(M_\eta(x)) \geq -nK_0 \qquad \text{for a. e. } x \in g_\delta^\eta \tag{3.14}$$

Set $A = g_{2^{-j+1}\delta} \setminus g_{2^{-j}\delta}$ and $A_\eta = g_{2^{-j+1}\delta}^\eta \setminus g_{2^{-j}\delta}^\eta$, we have

$$\int_{g_{2^{-j+1}\delta} \setminus g_{2^{-j}\delta}^\eta} |(\operatorname{trace} M(x))^-|^n dx - \int_{g_{2^{-j+1}\delta}^\eta \setminus g_{2^{-j}\delta}^\eta} |\operatorname{trace} D^2 w_\eta(x)|^n dx$$

$$\geq \int_{A\cap A_\eta} |(\operatorname{tr} M(x))^-|^n dx - \int_{A\cap A_\eta} |\operatorname{tr} M(x)|^n dx - \int_{A_\eta\setminus A} |\operatorname{tr} D^2 W_\eta(x)|^n dx$$

where $\delta \in (0, \delta_0)\setminus V$. By dominated convergence Theorem and by (3.11) — (3.14) we have

$$\int_{A\cap A_\eta} ||\operatorname{tr} M|^n - |\operatorname{tr} M_\eta|^n| dx \leq \int_\Omega X_{A\cap A_\eta}(x) ||\operatorname{tr} M|^n - |\operatorname{tr} M_\eta|^n| dx \to 0 \text{ as } \eta \to 0$$

$$\int_{A_\eta\setminus A} |\operatorname{tr} D^2 w_\eta(x)|^n dx \leq (nK_0)^n \operatorname{meas}(A_\eta\setminus A)$$

Since (3.10) holds, we see that

$$\operatorname{meas}(A_\eta\setminus A) = \operatorname{meas}\{(g^\eta_{2^{-j+1}\delta}\setminus g_{2^{-j+1}\delta}) \cup [(g_{2^{-j}\delta}\setminus g^\eta_{2^{-j}\delta}) \cap g^\eta_{2^{-j+1}\delta}]\} \to 0 \quad \text{as } \eta \to 0$$

Thus $\int_{A_\eta\setminus A} |\operatorname{tr} D^2 w_\eta(x)|^n dx \to 0$ as $\eta \to 0$ and we conclude that $\int_{g_{2^{-j+1}\delta}\setminus g_{2^{-j}\delta}} |(\operatorname{tr} M(x))^-|^n dx \geq c_5 (2^{-j+1}\delta)^n$ for all $j \in \mathbf{Z}^+$ and a. e. $\delta \in (0, \delta_0)$. On the set $g_{2^{-j+1}\delta}\setminus g_{2^{-j}\delta}$ we have $|Dw(x)| \leq 2^{-j+1}\delta$ a. e., and so

$$\int_{g_{2^{-j+1}\delta}\setminus g_{2^{-j}\delta}} [|(\operatorname{tr} M(x))^-/|Dw(x)|]^n dx \geq c_5 \text{ for all } j \in \mathbf{Z}^+ \text{ and a. e. } \delta \in (0, \delta_0).$$

By adding these inequalities from $j = 1$ to $j = k$ we obtain

$$\int_{g_\delta\setminus g_{2^{-k}\delta}} [|(\operatorname{tr} M(x))^-/|Dw(x)|]^n dx \geq c_5 k \text{ for all } k \in \mathbf{Z}^+ \text{ and a. e. } \delta \in (0, \delta_0) \tag{3.15}$$

This gives the result claimed by this Lemma.

Next Theorem is the fundamental result of this paper.

Theorem 3.5. *Let $u, v \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$. Assume u, v are viscosity supersolution and subsolution of (1.1), respectively. If either*

(α) $F[\cdot]$ is degenerate elliptic, decreasing and uniformly continuous in x,

or

(β) $F[\cdot]$ is uniformly elliptic, nonincreasing, Lipschitz continuous in p and uniformly continuous in x

then

$$\sup_\Omega (v-u)^+ \leq \sup_{\partial\Omega} (v-u)^+$$

Proof. This will be a proof by contradiction. Assume that the Theorem is false, then

$$m_0 = \sup_\Omega (v-u)^+ - \sup_{\partial\Omega} (v-u)^+ > 0$$

Let $\tilde{v} = v_\varepsilon^+ - \varepsilon$ and $\tilde{u} = u_\varepsilon^- + \varepsilon$. We find that there is a constant $\varepsilon_0 > 0$ such that

$$\sup_{\Omega_\varepsilon} (\tilde{v}-\tilde{u})^+ - \sup_{\partial\Omega_\varepsilon} (\tilde{v}-\tilde{u})^+ \geq m_0/2 \qquad \text{if } \varepsilon \in (0, \varepsilon_0) \tag{3.16}$$

By the Theorem 1.11 of [1], for any direction λ and all $\varepsilon \in (0, \bar{\varepsilon}_0)$ we have

$$D^2_\lambda \tilde{u} \leq K/\varepsilon,\ D^2_\lambda \tilde{v} \geq -K/\varepsilon \qquad \text{(in the sense of distributions)} \tag{3.17}$$

Let $\tilde{w} = \tilde{v} - \tilde{u}$ and we see that $\tilde{w} \in C(\bar{\Omega}) \cap W^{1,\infty}(\Omega)$ and satisfies (3.2) wiht $K_0 = 2K/\varepsilon$. With $\tilde{g}_\delta$ defined by (3.1), $w = \tilde{w}$ and $\Omega = \Omega_\varepsilon$ we see by Lemma 3.3 that

$$\operatorname{meas}(\tilde{g}_\delta) \geq c_4(\varepsilon)\delta^n \qquad \text{if } \delta \in (0, \delta_0)$$

Let $\tilde{M}^+$, $-\tilde{M}^-$, $\tilde{\Gamma}^+$, $-\tilde{\Gamma}^-$ be from the representation given in Lemma 3.2 for $\tilde{v}$ and $-\tilde{u}$, respectively, i. e.,

$$D^2\tilde{v} = \tilde{M}^+ + \tilde{\Gamma}^+ \qquad \text{and} \qquad -D^2\tilde{u} = -\tilde{M}^- - \tilde{\Gamma}^-$$

then $\widetilde{M}=\widetilde{M}^{+}-\widetilde{M}^{-}$ and $\widetilde{\Gamma}=\widetilde{\Gamma}^{+}-\widetilde{\Gamma}^{-}$ give a representation for $\widetilde{w}$. By Lemma 3.15 in [1], for a. e. $x\in\widetilde{g}_{\delta}$, $D\widetilde{u}(x)$ and $D\widetilde{v}(x)$ exist and

$$\widetilde{v}(x+z)-\widetilde{v}(x)-D\widetilde{v}(x)\cdot z-(\widetilde{M}^{+}(x)/2z,\ z)\leq o(|z|^{2})$$
$$\widetilde{u}(x+z)-\widetilde{u}(x)-D\widetilde{u}(x)\cdot z-(\widetilde{M}^{-}(x)/2z,\ z)\geq -o(|z|^{2})$$

Thus $(D\widetilde{v}(x),\ \widetilde{M}^{+}(x))\in D^{+}\widetilde{v}(x)$ and $(D\widetilde{u}(x),\ \widetilde{M}^{-}(x))\in D^{-}\widetilde{u}(x)$. Applying Lemma 2.7, Lemma 2.8 and definitions of viscosity subsolution and supersolution we conclude that if $\delta\in(0,\delta_0)$ and $\varepsilon\in(0,\bar{\varepsilon}_0)$ then

$$\begin{aligned}&F_{\varepsilon}^{+}(\widetilde{M}^{+}(x),\ D\widetilde{v}(x),\ \widetilde{v}(x),\ x)\geq 0\\&F_{\varepsilon}^{-}(\widetilde{M}^{-}(x),\ D\widetilde{u}(x),\ \widetilde{u}(x),\ x)\geq 0\end{aligned}\qquad\text{for a. e. } x\in\widetilde{g}_{\delta}$$

These imply that

$$F(\widetilde{M}^{+}(x),\ D\widetilde{v}(x),\ \widetilde{v}(x),\ x+\varepsilon D\widetilde{v}(x)\cdot(1+|D\widetilde{v}(x)|^{2})^{-\frac{1}{2}})$$
$$\geq F(\widetilde{M}^{-}(x),\ D\widetilde{u}(x),\ \widetilde{u}(x),\ x-\varepsilon D\widetilde{u}(x)\cdot(1+|D\widetilde{u}(x)|^{2})^{-\frac{1}{2}}) \tag{3.18}$$
for a. e. $x\in\widetilde{g}_{\delta}$, for all $\delta\in(0,\delta_0)$ and $\varepsilon\in(0,\bar{\varepsilon}_0)$

Furthermore, by Lemma 3.2, (3.16) and the definition of $\widetilde{g}_{\delta}$ we find that for any $\delta\in(0,\min(\delta_0, m_0/4\mathrm{diam}\Omega))$, any $\varepsilon\in(0,\min(\varepsilon_0,\bar{\varepsilon}_0))$

$$K/\varepsilon\cdot I\succ\widetilde{M}^{-}(x)\succ\widetilde{M}^{+}(x)\succ -K/\varepsilon\cdot I \tag{3.19}$$
$|D\widetilde{v}(x)-D\widetilde{u}(x)|\leq\delta$, $\widetilde{v}(x)-\widetilde{u}(x)\geq m_0/4$ for a. e. $x\in\widetilde{g}_{\delta}$

If (α) holds then for a. e. $x\in\widetilde{g}_{\delta}$

$$F(\widetilde{M}^{+}(x),\ D\widetilde{v}(x),\ \widetilde{v}(x),\ x+\varepsilon D\widetilde{v}(x)\cdot(1+|D\widetilde{v}(x)|^{2})^{-\frac{1}{2}})$$
$$-F(\widetilde{M}^{-}(x),\ D\widetilde{v}(x),\ \widetilde{u}(x),\ x-\varepsilon D\widetilde{u}(x)\cdot(1+|D\widetilde{u}(x)|^{2})^{-\frac{1}{2}})$$
$$\leq -c_2(\widetilde{v}(x)-\widetilde{u}(x))+\sigma(\varepsilon|D\widetilde{v}(x)\cdot(1+|D\widetilde{v}(x)|^{2})^{-\frac{1}{2}}+$$
$$+D\widetilde{u}(x)\cdot(1+|Du(x)|^{2})^{-\frac{1}{2}})$$

By the continuity of F and (3.19) there is a continuous increasing function $\eta(t)$ (which is dependent on ε) such that $\eta(0)=0$ and

$$F(\widetilde{M}^{+}(x),\ D\widetilde{v}(x),\ \widetilde{v}(x),\ x+\varepsilon D\widetilde{v}(x)\cdot(1+|D\widetilde{v}(x)|^{2})^{-\frac{1}{2}})$$
$$-F(\widetilde{M}^{-}(x),\ D\widetilde{u}(x),\ \widetilde{u}(x),\ x-\varepsilon D\widetilde{u}(x)\cdot(1+|D\widetilde{u}(x)|^{2})^{-\frac{1}{2}})$$
$$\leq -c_2/4\cdot m_0+\sigma(2\varepsilon)+\eta(\delta)$$

for a. e. $x\in\widetilde{g}_{\delta}$, all $\delta\in\left(0,\ \min\left(\delta_0,\ \frac{m_0}{4\mathrm{diam}\Omega}\right)\right)$ and all $\varepsilon\in(0,\min(\varepsilon_0,\bar{\varepsilon}_0))$. First choosing ε sufficiently small and then taking δ small enough then yields

$$F(\widetilde{M}^{+}(x),\ D\widetilde{v}(x),\ \widetilde{v}(x),\ x+\varepsilon D\widetilde{v}(x)\cdot(1+|D\widetilde{v}(x)|^{2})^{-\frac{1}{2}})$$
$$<F(\widetilde{M}^{-}(x),\ D\widetilde{u}(x),\ \widetilde{u}(x),\ x-\varepsilon D\widetilde{u}(x)\cdot(1+|D\widetilde{u}(x)|^{2})^{-\frac{1}{2}})$$

for a. e. $x\in\widetilde{g}_{\delta}$.

This contradicts (3.18)

If (β) holds then for a. e. $x\in\widetilde{g}_{\delta}$

$$F(\widetilde{M}^{+}(x),\ D\widetilde{v}(x),\ \widetilde{v}(x),\ x+\varepsilon D\widetilde{v}(x)\cdot(1+|D\widetilde{v}(x)|^{2})^{-\frac{1}{2}})$$
$$-F(\widetilde{M}^{-}(x),\ D\widetilde{u}(x),\ \widetilde{u}(x),\ x-\varepsilon D\widetilde{u}(x)\cdot(1+|D\widetilde{u}(x)|^{2})^{-\frac{1}{2}})$$
$$\leq -c_1\mathrm{trace}(\widetilde{M}^{-}(x)-\widetilde{M}^{+}(x))+c_3|D\widetilde{u}(x)-D\widetilde{v}(x)|+\sigma(2\varepsilon)$$

By Lemma 3.4 for all $\delta\in(0,\delta_0)$ and all $\varepsilon\in(0,\bar{\varepsilon}_0)$

$$\int_{\tilde{g}_\delta \setminus \tilde{g}_{2^{-k_0\delta}}} \left[\frac{\operatorname{tr}(\widetilde{M}^-(x) - \widetilde{M}^+(x))}{|D\tilde{u}(x) - D\tilde{v}(x)|}\right]^n dx \geq c_5 k_0$$

where $k_0 = \left[\operatorname{meas}(\Omega)\left(\frac{c_3}{c_1}\right)c_5^{-1}\right] + 1$, $[x]$ denote integral part of x. Let

$$\widetilde{E}_\delta(\varepsilon) = \left\{x \in \tilde{g}_\delta \setminus \tilde{g}_{2^{-k_0\delta}} \left| \frac{\operatorname{tr}(\widetilde{M}^- - \widetilde{M}^+)}{|D\tilde{u} - D\tilde{v}|} \geq 2 \cdot c_3/c_1 \right.\right\}$$

then meas $(\widetilde{E}_\delta(\varepsilon)) > 0$ and for a. e. $x \in \widetilde{E}_\delta(\varepsilon)$

$$\begin{aligned}
&F(\widetilde{M}^+(x),\ D\tilde{v}(x),\ \tilde{v}(x),\ x + \varepsilon D\tilde{v}(x) \cdot (1 + |D\tilde{v}(x)|^2)^{-\frac{1}{2}}) \\
&\quad - F(\widetilde{M}^-(x),\ D\tilde{u}(x),\ \tilde{u}(x),\ x - \varepsilon D\tilde{u}(x) \cdot (1 + |D\tilde{u}(x)|^2)^{-\frac{1}{2}}) \\
&\leq -c_3|D\tilde{v}(x) - D\tilde{u}(x)| + \sigma(2\varepsilon) \\
&\leq -c_3 2^{-k_0}\delta + \sigma(2\varepsilon)
\end{aligned}$$

Choosing ε sufficiently small then yields

$$\begin{aligned}
&F(\widetilde{M}^+(x),\ D\tilde{v}(x),\ \tilde{v}(x),\ x + \varepsilon D\tilde{v}(x) \cdot (1 + |D\tilde{v}(x)|^2)^{-\frac{1}{2}}) \\
&< F(\widetilde{M}^-(x),\ D\tilde{u}(x),\ \tilde{u}(x),\ x - \varepsilon D\tilde{u}(x) \cdot (1 + |D\tilde{u}(x)|^2)^{-\frac{1}{2}})
\end{aligned}$$

for a. e. $x \in \widetilde{E}_\delta(\varepsilon)$ This also contradicts (3. 18) and so completes the proof of Theorem.

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