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LIFE-SPAN OF CLASSICAL SOLUTIONS TO NONLINEAR WAVE EQUATIONS IN TWO-SPACE-DIMENSIONS II

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Abstract In two-space-dimensional case we get the sharp lower bound of the life-span of classical solutions to the Cauchy problem with small initial data for fully nonlinear wave equations of the form $\Box u = F(u, Du, D_x Du)$ in which $F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha})$ with $\alpha = 2$ in a neighbourhood of $\hat{\lambda} = 0$. The cases $\alpha = 1$ and $\alpha \geq 3$ have been considered respectively in [1] and [2].

Key Words Life-span; classical solution; Cauchy problem; nonlinear wave equation

Classification 35G25, 35L15, 35L70, 35L05

1. Introduction

Consider the Cauchy problem for fully nonlinear wave equations

$$\Box u = F(u, Du, D_x Du) \tag{1.1}$$

$$t = 0 : u = \varepsilon\phi(x), u_t = \varepsilon\psi(x) \tag{1.2}$$

where

$$\Box = \frac{\partial^2}{\partial t^2} - \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{1.3}$$

is the wave operator,

$$D_x = \Big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big), \quad D = \Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\Big) \tag{1.4}$$

$\phi, \psi \in C_0^\infty(R^n)$ and $\varepsilon > 0$ is a small parameter.

Let

$$\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}),\ i, j = 0, 1, \cdots, n, i + j \geq 1) \tag{1.5}$$

Suppose that in a neighbourhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in (1.1) is a sufficiently smooth function satisfying

$$F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\alpha}) \tag{1.6}$$

where α is an integer ≥ 1.

Our aim is to study the life-span of classical solution to (1.1)–(1.2) for $n = 2$ and all integers $\alpha \geq 1$. By definition, the life-span $\tilde{T}(\varepsilon) = \sup \tau$ for all $\tau > 0$ such that there exists a classical solution to (1.1)–(1.2) on $0 \leq t \leq \tau$.

In the previous papers [1] and [2] we have respectively considered the cases $\alpha = 1$ and $\alpha \geq 3$. The result is the following:

$$\tilde{T}(\varepsilon) = +\infty \quad \text{if } n = 2 \text{ and } \alpha \geq 3 \tag{1.7}$$

while if $n = 2$ and $\alpha = 1$,

$$\tilde{T}(\varepsilon) \geq \begin{cases} be(\varepsilon) \\ b\varepsilon^{-1}, & \text{if } \int_{R^2} \psi(x)dx = 0 \\ b\varepsilon^{-2}, & \text{if } \partial_u^2 F(0,0,0) = 0 \end{cases} \tag{1.8}$$

where b is a positive constant and $e(\varepsilon)$ is defined by

$$\varepsilon^2 e^2(\varepsilon) \ln(1 + e(\varepsilon)) = 1 \tag{1.9}$$

In this paper we will consider the remainder case $n = 2$ and $\alpha = 2$ and prove

$$\tilde{T}(\varepsilon) \geq \begin{cases} b\varepsilon^{-6} \\ \exp\{a\varepsilon^{-2}\}, & \text{if } \partial_u^\beta F(0,0,0) = 0 \ (\beta = 3, 4) \end{cases} \tag{1.10}$$

where a, b are positive constants. For this purpose, some refined estimates are needed.

All results mentioned above are sharp due to H.Lindblad [3], Zhou Yi [4]–[5] etc.

In order to prove the desired result, by differentiation, is suffices to consider the Cauchy problem for the following general kind of quasilinear wave equations

$$\Box u = \sum_{i,j=1}^{2} b_{ij}(u, Du) u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(u, Du) u_{t x_j} + F_0(u, Du) \tag{1.11}$$

$$t = 0: \quad u = \varepsilon\phi(x), \quad u_t = \varepsilon\psi(x) \tag{1.12}$$

where $x = (x_1, x_2)$, $\Box u = \dfrac{\partial^2}{\partial t^2} - \dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2}$, $\varepsilon > 0$ is a small parameter,

$$\phi, \ \psi \in C_0^\infty(R^2) \tag{1.13}$$

with

$$\text{supp } \{\phi, \psi\} \subseteq \{ x \mid |x| \leq \rho \} \quad (\rho > 0 \text{ constant}) \tag{1.14}$$

and for $|\tilde{\lambda}| \leq 1$, where $\tilde{\lambda} = (\lambda; (\lambda_i), i = 0, 1, 2)$, $b_{ij}(\tilde{\lambda})$, $a_{0j}(\tilde{\lambda})$ and $F_0(\tilde{\lambda})$ are sufficiently smooth functions satisfying

$$b_{ij}(\tilde{\lambda}) = b_{ji}(\tilde{\lambda}) \quad (i, j = 1, 2) \tag{1.15}$$

$$b_{ij}(\tilde{\lambda}), a_{0j}(\tilde{\lambda}) = O(|\tilde{\lambda}|^2) \quad (i,j=1,2) \tag{1.16}$$

$$F_0(\tilde{\lambda}) = O(|\lambda|^3) \tag{1.17}$$

and

$$\sum_{i,j=1}^{2} a_{ij}(\tilde{\lambda})\xi_i\xi_j \geq m_0|\xi|^2, \quad \forall \xi \in R^2 \tag{1.18}$$

where m_0 is a positive constant and

$$a_{ij}(\tilde{\lambda}) = \delta_{ij} + b_{ij}(\tilde{\lambda}) \tag{1.19}$$

where δ_{ij} is the Kronecker delta.

We point out that the condition

$$\partial_u^\beta F(0,0,0) = 0 \quad (\beta = 3,4) \tag{1.20}$$

implies

$$\partial_u^\beta F_0(0,0) = 0 \quad (\beta = 3,4) \tag{1.21}$$

In Section 2 we cite some estimates from [1]–[2], [6] and prove some new estimates on the solution to two-space-dimensional wave equations. Then we prove in a direct and simple manner the main result (1.10) for the general case and for the special case $\partial_u^\beta F(0,0,0) = 0$ $(\beta = 3,4)$ in Section 3 and Section 4 respectively.

2. Preliminaries

Following S.Klainerman [7], introduce a set of partial differential operators

$$\Gamma = (L_0; (\partial_a), a = 0,1,\cdots,n; (\Omega_{ab}), a,b = 0,1,\cdots,n) \tag{2.1}$$

where

$$L_0 = t\partial_t + x_1\partial_1 + \cdots + x_n\partial_n \tag{2.2}$$

$$\partial_0 = -\frac{\partial}{\partial t}, \quad \partial_i = -\frac{\partial}{\partial x_i} \quad (i = 1,\cdots,n) \tag{2.3}$$

$$\Omega_{ab} = x_a\partial_b - x_b\partial_a \quad (a,b = 0,1,\cdots,n) \tag{2.4}$$

in which

$$x_0 = t \tag{2.5}$$

and for any integer $N \geq 0$, define

$$\|u(t,\cdot)\|_{\Gamma,N,p} = \sum_{|k|\leq N} \|\Gamma^k u(t,\cdot)\|_{L^p(R^n)}, \quad \forall t \geq 0 \tag{2.6}$$

for any function $u = u(t,x)$ such that all norms appearing on the right-hand side are bounded, where $1 \leq p \leq +\infty$, $k = (k_1,\cdots,k_\sigma)$ is a multi-index, $|k| = k_1 + \cdots + k_\sigma$, σ is the number of partial differential operators in $\Gamma : \Gamma = (\Gamma_1,\cdots,\Gamma_\sigma)$ and

$$\Gamma^k = \Gamma_1^{k_1}\cdots\Gamma_\sigma^{k_\sigma} \tag{2.7}$$

In this paper we only consider the case $n = 2$, however, the following Lemmas 2.1–2.3 are still valid for any space dimension $n \geq 2$.

It is easy to prove the following three lemmas.

Lemma 2.1 *For any multi-index $k = (k_1, \cdots, k_\sigma)$ we have*

$$[\Box, \Gamma^k] = \sum_{|i| \leq |k|-1} A_{ki} \Gamma^i \Box \tag{2.8}$$

and

$$[\partial_a, \Gamma^k] = \sum_{|i| \leq |k|-1} B_{ki} \Gamma^i D = \sum_{|i| \leq |k|-1} \tilde{B}_{ki} D \Gamma^i \quad (a = 0, 1, \cdots, n) \tag{2.9}$$

where $[\ ,]$ stands for the Poisson's bracket, $i = (i_1, \cdots, i_\sigma)$ are multi-indices, $\Box$ is the wave operator, D is defined by (1.4) and A_{ki}, B_{ki} and $\tilde{B}_{ki}$ are constants.

Lemma 2.2 *For any non-negative integer N we have*

$$c\|Du(t,\cdot)\|_{\Gamma,N,p} \leq \sum_{|k| \leq N} \|D\Gamma^k u(t,\cdot)\|_{L^p(R^n)} \leq C\|Du(t,\cdot)\|_{\Gamma,N,p}, \quad \forall t \geq 0 \tag{2.10}$$

where $1 \leq p \leq +\infty$, c and C are positive constants independent of t.

Lemma 2.3 *Suppose that $G = G(w)$ is a sufficiently smooth function of $w = (w_1, \cdots, w_M)$ satisfying that if*

$$|w| \leq \nu_0 \tag{2.11}$$

then

$$G(w) = O(|w|^\beta) \tag{2.12}$$

where ν_0 is a positive constant and β is an integer ≥ 1. For any given integer $N \geq 0$, if a vector function $w = w(t,x)$ satisfies

$$\|w(t,\cdot)\|_{\Gamma,[\frac{N}{2}],\infty} \leq \nu_0, \quad \forall t \geq 0 \tag{2.13}$$

where $[\ ]$ stands for the integer part of a real number, then for any multi-index k with $|k| \leq N$, we have

$$|\Gamma^k G(w(t,x))| \leq C(\nu_0) \sum_{\substack{|l_1|+\cdots+|l_\beta| \leq |k| \\ 1 \leq i_j \leq M \\ (j=1,\cdots,\beta)}} \prod_{j=1}^{\beta} |\Gamma^{l_j} w_{i_j}(t,x)| \tag{2.14}$$

where $C(\nu_0)$ is a positive constant depending on ν_0.

Lemma 2.4 *Suppose that $n = 2$. Let $u = u(t,x)$ be a sufficiently smooth function with compact support in the variable x for any fixed $t \geq 0$. Then for any integer $N \geq 0$ we have*

$$\|u(t,\cdot)\|_{\Gamma,N,\infty} \leq C(1+t)^{-\frac{n-1}{p}} \|u(t,\cdot)\|_{\Gamma,N+[\frac{n}{p}]+1,p}, \quad \forall t \geq 0 \tag{2.15}$$

and

$$\|u(t,\cdot)\|_{\Gamma,N,q} \leq C(1+t)^{-\frac{n-1}{p}(1-\frac{p}{q})} \|u(t,\cdot)\|_{\Gamma,N+[\frac{n}{p}]+1,p}, \quad \forall t \geq 0 \tag{2.16}$$

where $1 < p < +\infty$, $p \le q \le +\infty$ and C is a positive constant.

Proof By S.Klainerman [10] we have (2.15). Noting that for any $q \ge p$

$$\|u(t,\cdot)\|_{\Gamma,N,q} \le C\|u(t,\cdot)\|_{\Gamma,N,\infty}^{1-\frac{p}{q}}\|u(t,\cdot)\|_{\Gamma,N,p}^{\frac{p}{q}} \tag{2.17}$$

we get (2.16) immediately.

Lemma 2.5 *Suppose that $n = 2$. Let $w^0 = w^0(t,x)$ be the solution to the Cauchy problem*

$$\Box w^0 = 0 \tag{2.18}$$

$$t = 0 : w^0 = \phi(x),\ w_t^0 = \psi(x) \tag{2.19}$$

where

$$\phi, \psi \in C^\infty(R^2) \tag{2.20}$$

such that

$$\text{supp } \{\phi, \psi\} \subseteq \{ x \mid |x| \le \rho\} \tag{2.21}$$

where ρ is a positive constant. Then

$$\|w^0(t,\cdot)\|_{L^2(R^2)} \le C_\rho\sqrt{\ln(2+t)}\left(\|\phi\|_{W^{2,1}(R^2)} + \|\psi\|_{W^{1,1}(R^2)}\right) \tag{2.22}$$

and for any fixed p with $2 < p < +\infty$,

$$\|w^0(t,\cdot)\|_{L^p(R^2)} \le C_\rho(1+t)^{-\frac{p-2}{2p}}\left(\|\phi\|_{W^{2,1}(R^2)} + \|\psi\|_{W^{1,1}(R^2)}\right) \tag{2.23}$$

moreover,

$$\|w^0(t,\cdot)\|_{L^\infty(R^2)} \le C(1+t)^{-1/2}\left(\|\phi\|_{W^{2,1}(R^2)} + \|\psi\|_{W^{1,1}(R^2)}\right) \tag{2.24}$$

where C_ρ (depending on ρ) and C are positive constants independent of t.

Proof For the proof of (2.22) and (2.23), see [2]. The proof of (2.24) can be found in S. Klainerman [8].

Lemma 2.6 *Suppose that $n = 2$. Let $w = w(t,x)$ be the solution to the wave equation*

$$\Box w = f(t,x) \tag{2.25}$$

with the zero initial data, where $f(t,x)$ has compact support in the variable x for any fixed $t \ge 0$. Then

$$(1+t)^{\frac{1}{2}}\|w(t,\cdot)\|_{L^\infty(R^2)} \le C\sum_{|I|\le 1}\int_0^t (1+\tau)^{-1/2}\|\Gamma^I f(\tau,\cdot)\|_{L^1(R^2)}d\tau, \quad \forall t \ge 0 \tag{2.26}$$

where C is a positive constant.

Proof See [2].

Lemma 2.7 *Suppose that $n = 2$. Let $w = w(t,x)$ be the solution to the wave equation (2.25) with the zero initial data. Then, for any p with $1 \le p < 2$ we have*

$$\|w(t,\cdot)\|_{L^p(R^2)} \le C(1+t)^{\frac{2}{p}-1} \int_0^t \|f(\tau,\cdot)\|_{L^1(R^2)} d\tau, \quad \forall t \ge 0 \tag{2.27}$$

where C is a positive constant.

Proof See [1].

Remark In [1] we only used Lemma 2.7 in the case $p = 1$, however, we do need Lemma 2.7 in the case $1 \le p < 2$ in this paper.

Lemma 2.8 *Suppose that $n = 2$. Let $w = w(t,x)$ be the solution to the wave equation*

$$\Box w = |f_1 f_2 f_3(t,x)| \tag{2.28}$$

with the zero initial data. Suppose furthermore that f_1, f_2 and f_3 have compact support included in $\{\, x \mid |x| \le t + \rho\}$ ($\rho > 0$ constant) in the variable x for any fixed $t \ge 0$. Then for any real number γ we have

$$\begin{aligned}\|w(t,\cdot)\|_{L^3(R^2)} \le {} & C(1+t)^{-1/12} \\ & \cdot \Big(\sum_{|I|+|J| \le 1} \int_0^t (1+\tau)^{\gamma-\frac{1}{2}} \|(f_1 \cdot \Gamma^I f_1 \cdot \Gamma^J f_2)(\tau,\cdot)\|_{L^1(R^2)} d\tau \Big)^{1/2} \\ & \cdot \Big(\int_0^t (1+\tau)^{-\gamma} \|f_2 \cdot f_3^2(\tau,\cdot)\|_{L^1(R^2)} d\tau \Big)^{1/2}\end{aligned} \tag{2.29}$$

where C is a positive constant depending on ρ.

Proof Let

$$\begin{cases} g_1(t,x) = f_1(t,x)(1+t^2+|x|^2)^{\gamma/4} \\ g_3(t,x) = f_3(t,x)(1+t^2+|x|^2)^{-\gamma/4} \end{cases} \tag{2.30}$$

and $E = E(t,x)$ be the forward fundamental solution of the wave operator. Then, by the positivity of E and Hölder's inequality, we have

$$\begin{aligned}|w(t,x)| &= E * |g_1 f_2 g_3|(t,x) \\ &\le (E * (g_1^2|f_2|)(t,x))^{1/2} (E * (g_3^2|f_2|)(t,x))^{1/2}\end{aligned} \tag{2.31}$$

so

$$\|w(t,\cdot)\|_{L^3(R^2)} \le \left\|E * (g_1^2|f_2|)(t,\cdot)\right\|_{L^\infty(R^2)}^{1/2} \left\|E * (g_3^2|f_2|)(t,\cdot)\right\|_{L^{\frac{3}{2}}(R^2)}^{1/2} \tag{2.32}$$

where "$*$" stands for the convolution.

By Lemma 2.7 (in which we take $p = \frac{3}{2}$) and noting the hypothesis on the compact support of $f_i (i = 1, 2, 3)$, we get

$$\left\| E * (g_3^2 |\, f_2|)(t, \cdot)\right\|_{L^{\frac{3}{2}}(\mathbf{R}^2)}$$
$$\leq C(1+t)^{1/3} \int_0^t \|g_3^2 f_2(\tau, \cdot)\|_{L^1(\mathbf{R}^2)} d\tau$$
$$\leq C(1+t)^{1/3} \int_0^t (1+\tau)^{-\gamma} \|f_2 f_3^2(\tau, \cdot)\|_{L^1(\mathbf{R}^2)} d\tau \tag{2.33}$$

On the other hand, noting that

$$|\Gamma(1+t^2+|x|^2)^{\gamma/2}| \leq C(1+t^2+|x|^2)^{\gamma/2} \tag{2.34}$$

and the hypothesis on the compact support of $f_i (i = 1, 2, 3)$, by Lemma 2.6 we have

$$\|E * (g_1^2 |\, f_2|)(t, \cdot)\|_{L^\infty(\mathbf{R}^2)}$$
$$\leq C(1+t)^{-1/2} \sum_{|I| \leq 1} \int_0^t (1+\tau)^{-1/2} \|\Gamma^I (g_1^2 | f_2|)(\tau, \cdot)\|_{L^1(\mathbf{R}^2)} d\tau$$
$$\leq C(1+t)^{-1/2} \sum_{|I| \leq 1} \int_0^t (1+\tau)^{\gamma-1/2} \|\Gamma^I (f_1^2 | f_2|)(\tau, \cdot)\|_{L^1(\mathbf{R}^2)} d\tau \tag{2.35}$$

Noting that

$$\sum_{|I| \leq 1} \|\Gamma^I\ (f_1^2 | f_2|)(\tau, \cdot)\|_{L^1(\mathbf{R}^2)}$$
$$\leq C \sum_{|I|+|J| \leq 1} \|(\Gamma^I f_1^2 \cdot \Gamma^J f_2)(\tau, \cdot)\|_{L^1(\mathbf{R}^2)}$$
$$\leq C \sum_{|I|+|J| \leq 1} \|(f_1 \cdot \Gamma^I f_1 \cdot \Gamma^J f_2)(\tau, \cdot)\|_{L^1(\mathbf{R}^2)} \tag{2.36}$$

(2.29) directly follows from (2.32), (2.33) and (2.35).

Lemma 2.9 *Suppose that $n = 2$. Let $w = w(t, x)$ be the solution to the wave equation*

$$\Box w = |f_1 f_2(t, x)| \tag{2.37}$$

with the zero initial data. Suppose furthermore f_1 and f_2 have compact support included in $\{ x \mid |x| \leq t + \rho\}$ ($\rho > 0$ constant) in the variable x for any fixed $t \geq 0$. Then for any real number γ we have

$$\|w(t, \cdot)\|_{L^2(\mathbf{R}^2)}$$
$$\leq C(1+t)^{1/4} \Big(\sum_{|I| \leq 1} \int_0^t (1+\tau)^{-\gamma-1/2} \|\Gamma^I f_1(\tau, \cdot)\|^2_{L^2(\mathbf{R}^2)} d\tau\Big)^{1/2}$$
$$\cdot \Big(\int_0^t (1+\tau)^{\gamma} \|f_2(\tau, \cdot)\|^2_{L^2(\mathbf{R}^2)} d\tau\Big)^{1/2} \tag{2.38}$$

where C is a positive constant depending on ρ.

Proof Let

$$\begin{cases} g_1(t,x) = f_1(t,x)(1+t^2+|x|^2)^{-\gamma/4} \\ g_2(t,x) = f_2(t,x)(1+t^2+|x|^2)^{\gamma/4} \end{cases} \tag{2.39}$$

As in Lemma 2.8, we have

$$|w(t,x)| \le (E * g_1^2(t,x))^{1/2}(E * g_2^2(t,x))^{1/2} \tag{2.40}$$

hence

$$\|w(t,x)\|_{L^2(R^2)} \le \|E * g_1^2(t,\cdot)\|_{L^\infty(R^2)}^{1/2} \|E * g_2^2(t,\cdot)\|_{L^1(R^2)}^{1/2} \tag{2.41}$$

By Lemma 2.6 and noting (2.34), we have

$$\begin{aligned} &\|E * g_1^2(t,\cdot)\|_{L^\infty(R^2)} \\ &\le C(1+t)^{-1/2} \sum_{|I|\le 1} \int_0^t (1+\tau)^{-1/2} \|\Gamma^I g_1^2(\tau,\cdot)\|_{L^1(R^2)} d\tau \\ &\le C(1+t)^{-1/2} \sum_{|I|\le 1} \int_0^t (1+\tau)^{-\gamma-1/2} \|\Gamma^I f_1(\tau,\cdot)\|_{L^2(R^2)}^2 d\tau \end{aligned} \tag{2.42}$$

On the other hand, by Lemma 2.7 (in which we take $p=1$) we get

$$\begin{aligned} \|E * g_2^2(t,\cdot)\|_{L^1(R^2)} &\le C(1+t) \int_0^t \|g_2^2(\tau,\cdot)\|_{L^1(R^2)} d\tau \\ &\le C(1+t) \int_0^t (1+\tau)^{\gamma} \|f_2(\tau,\cdot)\|_{L^2(R^2)}^2 d\tau \end{aligned} \tag{2.43}$$

(2.38) comes immediately from (2.41)–(2.43).

Lemma 2.10 *Suppose that $n \ge 1$. Let $w = w(t,x)$ be the solution to the wave equation*

$$\Box w = \partial_a f(t,x) \quad (a \in \{0,1,\cdots,n\}) \tag{2.44}$$

with the zero initial data, then

$$\|w(t,\cdot)\|_{L^2(R^n)} \le \int_0^t \|f(\tau,\cdot)\|_{L^2(R^n)} d\tau + \|v_0(\tau,\cdot)\|_{L^2(R^n)} \tag{2.45}$$

where for $a = 1, 2\cdots, n$, $v_0 \equiv 0$; while for $a = 0$, $v_0(t,x)$ is the solution to the following Cauchy problem:

$$\begin{cases} \Box v_0 = 0 \\ t = 0: \ v_0 = 0, \quad v_{0t} = f(0,x) \end{cases} \tag{2.46}$$

Proof See [1].

Lemma 2.11 *Suppose that $n \geq 2$. Let $w = w(t,x)$ be the solution to the wave equation (2.44) with the zero initial data, where $f(t,x)$ has a compact support included in $\{ x \mid |x| \leq t+\rho\}$ in the variable x for any fixed $t \geq 0$. Then*

$$(1+t)^{(n-1)/2}\|w(t,\cdot)\|_{L^\infty(R^n)} \leq C\Big\{\int_0^t (1+\tau)^{(n-1)/2}\|f(\tau,\cdot)\|_{L^\infty(R^n)}d\tau + \int_0^t (1+\tau)^{-(n+1)/2}\|f(\tau,\cdot)\|_{\Gamma,n+1,1}d\tau\Big\} \qquad (2.47)$$

where C is a positive constant depending on ρ.

Proof See [6].

Lemma 2.12 *Suppose that $n \geq 2$. Let $v = v(t,x)$ and $w = w(t,x)$ be functions with compact support included in $\{ x \mid |x| \leq t+\rho\}$ in the variable x for any fixed $t \geq 0$. Then, for any $a = 0, 1, \cdots, n$ we have*

$$\|v(t,\cdot)\partial_a w(t,\cdot)\|_{L^2(R^n)} \leq C\|D_x v(t,\cdot)\|_{L^2(R^n)} \cdot \sum_{|I|\leq 1}\|\Gamma^I w(t,\cdot)\|_{L^\infty(R^n)} \qquad (2.48)$$

where C is a positive constant depending on ρ.

Proof See [6].

3. Life-span of Classical Solutions in the General Case

By the Sobolev embedding theorem, there exists $E_0 > 0$ so small that

$$\|f\|_{L^\infty(R^2)} \leq 1, \quad \forall f \in H^2(R^2), \quad \|f\|_{H^2(R^2)} \leq E_0 \qquad (3.1)$$

For any given integer $S \geq 5$, any given positive real numbers $E(\leq E_0)$ and $T(>0)$, introduce the following set of functions

$$X_{S,E,T} = \{ v(t,x) \mid D_{S,T}(v) \leq E; \partial_t^l v(0,x) = u_l^{(0)}(x)(l = 0, 1, \cdots, S+1)\} \qquad (3.2)$$

where

$$D_{S,T}(v) = \sum_{i=1}^{2} \sup_{0\leq t\leq T}\|D^i v(t,\cdot)\|_{\Gamma,S,2} + \sup_{0\leq t\leq T}(1+t)^{1/6}\|v(t,\cdot)\|_{\Gamma,S,3} \qquad (3.3)$$

and $u_0^{(0)} = \varepsilon\phi(x)$, $u_1^{(0)} = \varepsilon\psi(x)$ and $u_l^{(0)}(x)$ $(l = 2, \cdots, S+1)$ are the values of $\partial_t^l u(t,x)$ at $t = 0$ formally determined from Equation (1.11) and the initial data (1.12). Obviously, $u_l^{(0)}(l = 0, 1, \cdots, S+1)$ are all sufficiently smooth functions with compact support in $\{ x \mid |x| \leq \rho\}$.

It is easy to prove the following

Lemma 3.1 *Endowed with the metric*

$$\rho(\bar{v}, \bar{\bar{v}}) = D_{S,T}(\bar{v} - \bar{\bar{v}}), \quad \forall \bar{v}, \bar{\bar{v}} \in X_{S,E,T} \qquad (3.4)$$

$X_{S,E,T}$ is a nonempty complete metric space, provided that $\varepsilon > 0$ is suitably small.

Let $\tilde{X}_{S,E,T}$ be the subset of $X_{S,E,T}$ composed of all elements in $X_{S,E,T}$ with compact support included in $\{ x \mid |x| \le t+\rho \}$ in the variable x for any fixed $t \ge 0$.

Lemma 3.2 *For any $v \in \tilde{X}_{S,E,T}$ we have*

$$\|(v, Dv, D^2v)(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le CE(1+t)^{-1/2}, \quad \forall t \in [0,T] \tag{3.5}$$

and

$$\|(v, Dv, D^2v)(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,q} \le CE(1+t)^{-1/2+1/q}, \quad \forall t \in [0,T] \tag{3.6}$$

where $q \ge 3$ and C is a positive constant.

Proof Noting that $S \ge 5$, by (2.15) (in which we take $n=2$, $N = \left[\frac{S}{2}\right]+1$, so $N + \left[\frac{2}{p}\right] + 1 \le S$, where $p=3$ for v or $p=2$ for Dv and D^2v) and the definition of $X_{S,E,T}$, we immediately get (3.5). By (2.16), similarly we obtain (3.6).

The main result in this section is

Theorem 3.1 *Suppose that $n=2$ and $\alpha=2$. Then under assumptions* (1.13)–(1.19), *for any given integer $S \ge 5$, there exist positive constants ε_0 and C_0 with $C_0\varepsilon_0 \le E_0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, there exists a positive number $T = T(\varepsilon)$ such that Cauchy problem* (1.11)-(1.12) *admits on $[0, T(\varepsilon)]$ a unique classical solution $u \in \tilde{X}_{S,C_0\varepsilon,T(\varepsilon)}$, where $T(\varepsilon)$ can be chosen as follows:*

$$T(\varepsilon) = b\varepsilon^{-6} - 1 \tag{3.7}$$

where b is a positive constant.

Moreover, with eventual modification on a set with zero measure in the variable t, we have

$$u \in C([0,T(\varepsilon)];\ H^{S+1}(R^2)) \tag{3.8}$$

$$u_t \in C([0,T(\varepsilon)];\ H^{S}(R^2)) \tag{3.9}$$

$$u_{tt} \in C([0,T(\varepsilon)];\ H^{S-1}(R^2)) \tag{3.10}$$

In order to prove Theorem 3.1, we define a map

$$M : v \to u = Mv \tag{3.11}$$

by solving the following Cauchy problem for linear wave equations for any $v \in \tilde{X}_{S,E,T}$:

$$\begin{aligned}\Box u &= \hat{F}(v, Dv, D_x Du)\\ &\triangleq \sum_{i,j=1}^{2} b_{ij}(v,Dv)u_{x_i x_j} + 2\sum_{j=1}^{2} a_{0j}(v,Dv)u_{tx_j} + F_0(v,Dv)\end{aligned} \tag{3.12}$$

$$t=0: \quad u = \varepsilon\phi(x), \quad u_t = \varepsilon\psi(x) \tag{3.13}$$

Thus it is only necessary to prove that there exists $C_0 > 0$ such that the map M possesses a unique fixed point in $\tilde{X}_{S,C_0\varepsilon,T(\varepsilon)}$, provided that ε is suitably small and $T(\varepsilon)$ is given by (3.7).

It is not difficult to get the following two lemmas.

Lemma 3.3 *For any $v \in \tilde{X}_{S,E,T}$ we have, with eventual modification on a set with zero measure in t,*

$$u = Mv \in C([0,T]; H^{S+1}(R^2)) \tag{3.14}$$

$$u_t \in C([0,T]; H^S(R^2)) \tag{3.15}$$

$$u_{tt} \in L^\infty(0,T; H^{S-1}(R^2)) \tag{3.16}$$

Moreover, for any fixed $t \geq 0$, $u = u(t,x)$ has compact support included in $\{ x \mid |x| \leq t + \rho\}$ in the variable x.

Lemma 3.4 *For $u = u(t,x) = Mv$, $\partial_t^l u(0,x)$ $(l = 0, 1, \cdots, S+2)$ are independent of $v \in \tilde{X}_{S,E,T}$ and*

$$\partial_t^l u(0,x) = u_l^{(0)}(x) \quad (l = 0, 1, \cdots, S+1) \tag{3.17}$$

Furthermore,

$$\|u(0,\cdot)\|_{\Gamma,S+2,p} \leq C\varepsilon \tag{3.18}$$

where $1 \leq p \leq +\infty$ and C is a positive constant.

Lemma 3.5 *Under the assumptions of Theorem 3.1, for any $v \in \tilde{X}_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \leq \tilde{C}_1\{\varepsilon + (R + \sqrt{R})(E + D_{S,T}(u))\} \tag{3.19}$$

where $\tilde{C}_1$ is a positive constant depending on ρ,

$$R = R(E,T) = E^2(1+T)^{1/3} \tag{3.20}$$

Proof We first estimate $\|u(t,\cdot)\|_{\Gamma,S,3}$.

By (2.8), for any multi-index k with $|k| \leq S$, we have

$$\Box\Gamma^k u = \sum_{|l|\leq|k|} A_{kl}\Gamma^l F(v, Dv, D_x Du) \tag{3.21}$$

Let

$$\Gamma^k u = w_k^0 + w_k \tag{3.22}$$

where w_k satisfies

$$\Box w_k = \sum_{|l|\leq|k|} A_{kl}\Gamma^l \hat{F}(v, Dv, D_x Du) \triangleq \hat{F}_k \tag{3.23}$$

with the zero initial data, while w_k^0 satisfies

$$\Box w_k^0 = 0 \tag{3.24}$$

with the same initial data as $\Gamma^k u$, which has the order $O(\varepsilon)$.

By (2.23), it is easy to see that

$$\|w_k^0(t,\cdot)\|_{L^3(R^2)} \le C\varepsilon(1+t)^{-1/6} \tag{3.25}$$

henceforth C denotes a positive constant.

Noting (3.5), by Lemma 2.3 we have

$$|\hat{F}_k| \le C \sum_{\substack{|l_0|+|l_1|+|l_2|\le|k| \\ |I_0|,|I_1|,|I_2|\le 1}} (|\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}D^{I_1}v)(\Gamma^{l_2}D^{I_2}v)| + |(\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}D^{I_1}v)(\Gamma^{l_2}D_xDu)|) \tag{3.26}$$

where $\hat{F}_k$ is defined by the right-hand side of (3.23). By the positivity of the fundamental solution E, we get

$$\|w_k(t,\cdot)\|_{L^3(R^2)} \le C \sum_{\substack{|l_0|+|l_1|+|l_2|\le|k| \\ |I_0|,|I_1|,|I_2|\le 1}} \Big(\|E*|(\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}D^{I_1}v)(\Gamma^{l_2}D^{I_2}v)|(t,\cdot)\|_{L^3(R^2)} + \|E*|(\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}D^{I_1}v)(\Gamma^{l_2}D_xDu)|(t,\cdot)\|_{L^3(R^2)}\Big) \tag{3.27}$$

To estimate $A \triangleq \|E*|(\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}D^{I_1}v)(\Gamma^{l_2}D^{I_2}v)|(t,\cdot)\|_{L^3(R^2)}$, we use Lemma 2.8. Without loss of generality, we may suppose that $|l_0| \le |l_1| \le |l_2|$, then, by Lemma 2.8 (in which we take $\gamma = \frac{1}{4}$) we get

$$A \le C(1+t)^{-1/12}\Big(\sum_{|I|+|J|\le 1}\int_0^t (1+\tau)^{-1/4} \cdot \|(\Gamma^{l_0}D^{I_0}v)(\Gamma^I\Gamma^{l_0}D^{I_0}v)(\Gamma^J\Gamma^{l_1}D^{I_1}v)(\tau,\cdot)\|_{L^1(R^2)}d\tau\Big)^{1/2} \cdot\Big(\int_0^t (1+\tau)^{-1/4}\|(\Gamma^{l_1}D^{I_1}v)(\Gamma^{l_2}D^{I_2}v)^2(\tau,\cdot)\|_{L^1(R^2)}d\tau\Big)^{1/2} \tag{3.28}$$

By Hölder's inequality and (3.6) (in which we take $q=3$), we have

$$\begin{aligned}\|(\Gamma^{l_0}D^{I_0}v)\,(\Gamma^I\Gamma^{l_0}D^{I_0}v)(\Gamma^J\Gamma^{l_1}D^{I_1}v)(\tau,\cdot)\|_{L^1(R^2)} &\le C\|v(\tau,\cdot)\|^2_{\Gamma,[\frac{S}{2}]+1,3}\|D^{I_1}v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,3} \\ &\le CE^3(1+\tau)^{-1/2}\end{aligned} \tag{3.29}$$

On the other hand, by a similar reason and noting the definition of $X_{S,E,T}$, when $|I_2|=0$, we have

$$\begin{aligned}\|(\Gamma^{l_1}D^{I_1}v)\,(\Gamma^{l_2}D^{I_2}v)^2(\tau,\cdot)\|_{L^1(R^2)} &\le C\|D^{I_1}v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,3}\|v(\tau,\cdot)\|^2_{\Gamma,S,3} \\ &\le CE^3(1+\tau)^{-1/2}\end{aligned} \tag{3.30}$$

while, when $|I_2| = 1$, by (3.5) we have

$$\begin{aligned}\|(\Gamma^{l_1} D^{I_1} v)(\Gamma^{l_2} D^{I_2} v)^2(\tau,\cdot)\|_{L^1(R^2)} &\leq C\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|Dv(\tau,\cdot)\|^2_{\Gamma,S,2} \\ &\leq CE^3(1+\tau)^{-1/2} \qquad (3.31)\end{aligned}$$

Thus, we obtain

$$A \leq CE^3(1+t)^{1/6} \leq C(1+t)^{-1/6}R(E,T)E \qquad (3.32)$$

Similarly, we have

$$\begin{aligned}\|E * |(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_1} D^{I_1} v)(\Gamma^{l_2} D_x Du)|(t,\cdot)\|_{L^3(R^2)} \\ \leq C(1+t)^{-1/6}R(E,T)D_{S,T}(u) \qquad (3.33)\end{aligned}$$

It follows from (3.22), (3.25), (3.27) and (3.32)–(3.33) that

$$\sup_{0\leq t\leq T}(1+t)^{1/6}\|u(t,\cdot)\|_{\Gamma,S,3} \leq C\{\varepsilon + R(E,T)(E + D_{S,T}(u))\} \qquad (3.34)$$

We now estimate $\|D^i u(t,\cdot)\|_{\Gamma,S,2}$ $(i = 1, 2)$.

For any multi-index k ($|k| \leq S$), by respectively applying Γ^k and $\Gamma^k D$ to both sides of (3.12), we can get the following energy integral formula

$$\begin{aligned}&\|D\Gamma^k u(t,\cdot)\|^2_{L^2(R^2)} + \|(\Gamma^k Du(t,\cdot))_t\|^2_{L^2(R^2)} \\ &\quad + \sum_{i,j=1}^{2}\int_{R^2} a_{ij}(v,Dv)(t,\cdot)(\Gamma^k Du(t,\cdot))_{x_i}(\Gamma^k Du(t,\cdot))_{x_j}dx \\ &= \|D\Gamma^k u(0,\cdot)\|^2_{L^2(R^2)} + \|(\Gamma^k Du(0,\cdot))_t\|^2_{L^2(R^2)} \\ &\quad + \sum_{i,j=1}^{2}\int_{R^2} a_{ij}(v,Dv)(0,\cdot)(\Gamma^k Du(0,\cdot))_{x_i}(\Gamma^k Du(0,\cdot))_{x_j}dx \\ &\quad + \sum_{i,j=1}^{2}\int_0^t\int_{R^2}\frac{\partial a_{ij}(v,Dv)(\tau,\cdot)}{\partial\tau}(\Gamma^k Du(\tau,\cdot))_{x_i}(\Gamma^k Du(\tau,\cdot))_{x_j}dxd\tau \\ &\quad -2\sum_{i,j=1}^{2}\int_0^t\int_{R^2}\frac{\partial a_{ij}(v,Dv)(\tau,\cdot)}{\partial x_i}(\Gamma^k Du(\tau,\cdot))_{x_j}(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau \\ &\quad -2\sum_{j=1}^{2}\int_0^t\int_{R^2}\frac{\partial a_{0j}(v,Dv)(\tau,\cdot)}{\partial x_j}(\Gamma^k Du(\tau,\cdot))_\tau(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau \\ &\quad +2\int_0^t\int_{R^2} G_k(\tau,\cdot)(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau \\ &\quad +2\int_0^t\int_{R^2} g_k(\tau,\cdot)(\Gamma^k Du(\tau,\cdot))_\tau dxd\tau\end{aligned}$$

$$= \|D\Gamma^k u(0,\cdot)\|^2_{L^2(\mathbf{R}^2)} + \|(\Gamma^k Du(0,\cdot))_t\|^2_{L^2(\mathbf{R}^2)}$$

$$+ \sum_{i,j=1}^{2} \int_{\mathbf{R}^2} a_{ij}(v, Dv)(0,\cdot)(\Gamma^k Du(0,\cdot))_{x_i}(\Gamma^k Du(0,\cdot))_{x_j} dx$$

$$+I + II + III + IV + V \tag{3.35}$$

in which

$$G_k = \sum_{i,j=1}^{2} \{(\Gamma^k D(b_{ij}(v, Dv)u_{x_i x_j}) - b_{ij}(v, Dv)\Gamma^k Du_{x_i x_j})$$

$$+b_{ij}(v, Dv)(\Gamma^k Du_{x_i x_j} - (\Gamma^k Du)_{x_i x_j})\}$$

$$+2\sum_{j=1}^{2}\{(\Gamma^k D(a_{0j}(v, Dv)u_{tx_j}) - a_{0j}(v, Dv)\Gamma^k Du_{tx_j})$$

$$+a_{0j}(v, Dv)(\Gamma^k Du_{tx_j} - (\Gamma^k Du)_{tx_j})\} \tag{3.36}$$

$$g_k = \Gamma^k DF_0(v, Dv) + \sum_{|i|\le|k|} C_{ki}\Gamma^i F(v, Dv, D_x Du) \tag{3.37}$$

where C_{ki} are constants.

Using (1.16) and (3.5), it is easy to see that

$$|I|, |II|, |III| \le CE^2 \int_0^t (1+\tau)^{-1} d\tau \cdot D^2_{S,T}(u)$$

$$\le CR(E, T)D^2_{S,T}(u) \tag{3.38}$$

Now we estimate the L^2 norm of $G_k(\tau,\cdot)$. By Lemma 2.3 and noting (2.9), we have

$$|G_k| \le C \sum_{\substack{|l_0|+|l_1|+|l_2|\le|k| \\ |I_0|,|I_1|\le 2}} |(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_1} D^{I_1} v)(\Gamma^{l_2} D_x Du)| \tag{3.39}$$

To estimate $B \triangleq \|(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_1} D^{I_1} v)(\Gamma^{l_2} D_x Du)(\tau,\cdot)\|_{L^2(\mathbf{R}^2)}$, we may assume that $|l_0| \le |l_1|$. When $|l_1| \le |l_2|$, by (3.5) and (3.3) we have

$$B \le \|D^{I_0} v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \|D^{I_1} v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \|D_x Du(\tau,\cdot)\|_{\Gamma,S,2}$$

$$\le C(1+\tau)^{-1} E^2 D_{S,T}(u) \tag{3.40-1}$$

when $|l_1| \ge |l_2|$ and $|I_1| \ge 1$, noting (2.15), similarly we have

$$B \le \|D^{I_0} v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty} \|D^I v(\tau,\cdot)\|_{\Gamma,S,2} \|D_x Du\|_{\Gamma,[\frac{S}{2}],\infty}$$

$$\le C(1+\tau)^{-1} E^2 D_{S,T}(u) \tag{3.40-2}$$

while when $|l_1| \geq |l_2|$ and $|I_1| = 0$, by Hölder's inequality and (3.6) (in which we take $q=6$), similarly we still get

$$B \leq \|D^{I_0}v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],6}\|v(\tau,\cdot)\|_{\Gamma,S,3}\|D_xDu\|_{\Gamma,[\frac{S}{2}],\infty}$$

$$\leq C(1+\tau)^{-1}E^2 D_{S,T}(u) \tag{3.40-3}$$

Thus we obtain

$$\|G_k(\tau,\cdot)\|_{L^2(R^2)} \leq C(1+\tau)^{-1}E^2 D_{S,T}(u), \quad \forall \tau \in [0,T] \tag{3.41}$$

then

$$|IV| \leq CE^2\int_0^t (1+\tau)^{-1}d\tau \cdot D^2_{S,T}(u)$$

$$\leq CR(E,T)D^2_{S,T}(u) \tag{3.42}$$

Similarly, we have

$$|V| \leq CR(E,T)(E+D_{S,T}(u))D_{S,T}(u) \tag{3.43}$$

By (3.38) and (3.42)-(3.43), and noticing (2.9), (1.18) and (3.18), it follows from (3.35) that

$$\sum_{i=1}^{2}\sup_{0\leq t\leq T}\|D^iu(t,\cdot)\|_{\Gamma,S,2} \leq C\{\varepsilon + \sqrt{R(E,T)}\,(E+D_{S,T}(u))\} \tag{3.44}$$

The combination of (3.34) and (3.44) yields (3.19).

Similar to Lemma 3.5, we can get (cf. [1]-[2])

Lemma 3.6 *Let $\bar{v},\bar{\bar{v}} \in \tilde{X}_{S,E,T}$. If $\bar{u} = M\bar{v}$ and $\bar{\bar{u}} = M\bar{\bar{v}}$ also satisfy $\bar{u},\bar{\bar{u}} \in \tilde{X}_{S,E,T}$, then*

$$D_{S-1,T}(\bar{u}-\bar{\bar{u}}) \leq \tilde{C}_2(R+\sqrt{R})(D_{S-1,T}(\bar{u}-\bar{\bar{u}}) + D_{S-1,T}(\bar{v}-\bar{\bar{v}})) \tag{3.45}$$

where $\tilde{C}_2$ is a positive constant depending on ρ and $R = R(E,T)$ is still defined by (3.20).

By means of Lemma 3.5 and Lemma 3.6, just as in [9] we can easily use the contraction mapping principle to get Theorem 3.1.

4. Life-span of Classical Solutions in the Special Case $\partial_u^\beta F(0,0,0) = 0$ $(\beta = 3,4)$

In this section we consider Cauchy problem (1.11), (1.12) under hypothesis (1.21). We only point out the essential points in what follows.

Instead of (3.3), we take

$$D_{S,T}(v) = \sum_{i=1}^{2}\sup_{0\leq t\leq T}\|D^iv(t,\cdot)\|_{\Gamma,S,2} + \sup_{0\leq t\leq T}(1+t)^{-1/2}\|v(t,\cdot)\|_{\Gamma,S,2}$$

$$+\sup_{0\leq t\leq T}(1+t)^{1/2}\|v(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \tag{4.1}$$

Then we have

Lemma 4.1 *For any* $v \in \tilde{X}_{S,E,T}$,

$$\|(v, Dv, D^2v)(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \le CE(1+t)^{-1/2}, \quad \forall t \in [0,T] \tag{4.2}$$

where C is a positive constant.

The main result in this section is

Theorem 4.1 *Suppose that (1.21) holds. Then under the assumptions of Theorem 3.1, we have the same conclusion as in Theorem 3.1 with*

$$T(\varepsilon) = \exp\{a\varepsilon^{-2}\} - 1 \tag{4.3}$$

where a is a positive constant.

Lemma 4.2 *Under the assumptions of Theorem 4.1, for any $v \in \tilde{X}_{S,E,T}$, $u = Mv$ satisfies*

$$D_{S,T}(u) \le \tilde{C}_1\{\varepsilon + (R+\sqrt{R})(E + D_{S,T}(u))\} \tag{4.4}$$

where $\tilde{C}_1$ is a positive constant depending on ρ and

$$R = R(E,T) = E^2 \ln(1+T) \tag{4.5}$$

Proof We first estimate $\|u(t,\cdot)\|_{\Gamma,S,2}$.

Noting (1.21), we easily see that $\hat{F}(v, Dv, D_xDu)$ can be rewritten as follows:

$$\hat{F}(v, Dv, D_xDu) = \sum_{i=1}^{2} \partial_i \hat{G}_i(v, Du) + \sum_{i,j=0}^{2} \hat{A}_{ij}(v) v_{x_i} u_{x_j} + \sum_{\substack{i,j,m=0\\ j+m\ge 1}}^{2} \hat{B}_{ijm}(v, Dv) v_{x_i} u_{x_j x_m} + \sum_{i,j=0}^{2} \hat{C}_{ij}(v, Dv) v_{x_i} v_{x_j} + F_0(v,0) \tag{4.6}$$

where in a neighbourhood of the origin we have

$$\hat{G}_i(\bar{\lambda}) = O(|\bar{\lambda}|^3), \quad i = 1, 2, \ \bar{\lambda} = (v, Du) \tag{4.7}$$

and $\hat{G}_i$ is affine in Du,

$$\hat{A}_{ij}(v) = O(|v|), \quad i, j = 0, 1, 2 \tag{4.8}$$

$$\hat{B}_{ijm}(\tilde{\lambda}),\ \hat{C}_{ij}(\tilde{\lambda}) = O(|\tilde{\lambda}|),\ i, j, m = 0, 1, 2,\ \tilde{\lambda} = (v, Dv) \tag{4.9}$$

and

$$F_0(v,0) = O(|v|^5) \tag{4.10}$$

Thus, by (2.8)–(2.9), for any multi-index k with $|k| \le S$, we can suppose that

$$\Gamma^k u = w_{1k} + w_{2k} + w_{3k} \tag{4.11}$$

where w_{1k}, w_{2k} and w_{3k} respectively satisfy

$$\Box w_{1k} = \sum_{i=1}^{2} \partial_i \Big(\sum_{|l| \le k} \tilde{A}_{il} \Gamma^l G_i(v, Du) \Big) \tag{4.12}$$

$$\begin{aligned} \Box w_{2k} = &\sum_{|l| \le |k|} A_{kl} \Gamma^l \Big(\sum_{i,j=0}^{2} \hat{A}_{ij}(v) v_{x_i} u_{x_j} \\ &+ \sum_{\substack{i,j,m=0 \\ j+m \ge 1}}^{2} \hat{B}_{ijm}(v, Dv) v_{x_i} u_{x_j x_m} + \sum_{i,j=0}^{2} \hat{C}_{ij}(v, Dv) v_{x_i} v_{x_j} \Big) \\ \triangleq &\; H_{2k} \end{aligned} \tag{4.13}$$

and

$$\Box w_{3k} = \sum_{|l| \le |k|} A_{kl} \Gamma^l F_0(v, 0) \tag{4.14}$$

with the zero initial data for w_{2k} and w_{3k} and the same initial data as $\Gamma^k u$ for w_{1k}, where $\tilde{A}_{il}$ and A_{kl} are constants.

By Lemma 2.10 and Lemma 2.5, it is easy to get that

$$\|w_{1k}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \le C \Big\{ \varepsilon \sqrt{\ln(2+t)} + \int_0^t \|G_i(v, Du)(\tau, \cdot)\|_{\Gamma, S, 2} d\tau \Big\} \tag{4.15}$$

Noting that, by Hölder's inequality and (4.7) and using (4.1)–(4.2) and (2.19) (in which we take $p = 2$), we have

$$\begin{aligned} &\|G_i(v, Du)\,(\tau, \cdot)\|_{\Gamma, S, 2} \\ &\quad \le C\{ \|v(\tau, \cdot)\|^2_{\Gamma, [\frac{S}{2}], \infty} (\|v(\tau, \cdot)\|_{\Gamma, S, 2} + \|Du(\tau, \cdot)\|_{\Gamma, S, 2}) \\ &\qquad + \|v(\tau, \cdot)\|_{\Gamma, [\frac{S}{2}], \infty} \|Du(\tau, \cdot)\|_{\Gamma, [\frac{S}{2}], \infty} \|v(\tau, \cdot)\|_{\Gamma, S, 2} \} \\ &\quad \le C(1+\tau)^{-1/2} E^2 (E + D_{S,T}(u)) \end{aligned} \tag{4.16}$$

it follows from (4.15) that

$$\|w_{1k}(t, \cdot)\|_{L^2(\mathbb{R}^2)} \le C(1+t)^{1/2} \{ \varepsilon + E^2 (E + D_{S,T}(u)) \} \tag{4.17}$$

In order to estimate $\|w_{2k}(t, \cdot)\|_{L^2(\mathbb{R}^2)}$, we first point out that by Lemma 2.3 and noting (4.9) we have

$$\begin{aligned} &\Big| \Gamma^l \Big(\sum_{i,j=0}^{2} \hat{C}_{ij}(v, Dv) v_{x_i} v_{x_j} \Big) \Big| \\ &\qquad \le C \sum_{\substack{|l_0| + |l_1| + |l_2| \le |k| \\ |l_0| \le 1}} |(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_1} Dv)(\Gamma^{l_2} Dv)| \end{aligned} \tag{4.18}$$

We may assume that $|l_1| \le |l_2|$, then by Lemma 2.9 (in which we take $\gamma = \frac{1}{4}$) and noting the definition of $X_{S,E,T}$, we get

$$\begin{aligned}
&\|E * |(\Gamma^{l_0} D^{I_0} v)\,(\Gamma^{l_1} Dv)(\Gamma^{l_2} Dv)|(t,\cdot)\|_{L^2(\boldsymbol{R}^2)} \\
&\le C(1+t)^{1/4}\Big(\sum_{|I|\le 1}\int_0^t (1+\tau)^{-3/4}\|\Gamma^I\Gamma^{l_1} Dv(\tau,\cdot)\|^2_{L^2(\boldsymbol{R}^2)} d\tau\Big)^{1/2} \\
&\quad\cdot\Big(\int_0^t (1+\tau)^{1/4}\|(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_2} Dv)(\tau,\cdot)\|^2_{L^2(\boldsymbol{R}^2)} d\tau\Big)^{1/2} \\
&\le C(1+t)^{1/4}\Big(\int_0^t (1+\tau)^{-3/4}\|Dv(\tau,\cdot)\|^2_{\Gamma,S,2} d\tau\Big)^{1/2} \\
&\quad\cdot\Big(\int_0^t (1+\tau)^{1/4}\|(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_2} Dv)(\tau,\cdot)\|^2_{L^2(\boldsymbol{R}^2)} d\tau\Big)^{1/2} \\
&\le CE(1+t)^{3/8}\Big(\int_0^t (1+\tau)^{1/4}\|(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_2} Dv)(\tau,\cdot)\|^2_{L^2(\boldsymbol{R}^2)} d\tau\Big)^{1/2}
\end{aligned} \tag{4.19}$$

On the other hand, when $|l_0| \le |l_2|$, by Lemma 4.1 we have

$$\begin{aligned}
\|(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_2} Dv)(\tau,\cdot)\|_{L^2(\boldsymbol{R}^2)} &\le C\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|Dv(\tau,\cdot)\|_{\Gamma,S,2} \\
&\le CE^2(1+\tau)^{-1/2}
\end{aligned}$$

when $|l_0| > |l_2|$ and $|I_0| = 1$, in a similar way we get the same estimate; while when $|l_0| > |l_2|$ and $|I_0| = 0$, by Lemma 2.12 and noting (2.9) we have

$$\begin{aligned}
&\|(\Gamma^{l_0} D^{I_0} v)\,(\Gamma^{l_2} Dv)(\tau,\cdot)\|_{L^2(\boldsymbol{R}^2)} \\
&\le C\|D_x\Gamma^{l_0} v(\tau,\cdot)\|_{L^2(\boldsymbol{R}^2)}\cdot\sum_{|I|\le 1}\|\Gamma^I\Gamma^{l_2} v(\tau,\cdot)\|_{L^\infty(\boldsymbol{R}^2)} \\
&\le C\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \\
&\le CE^2(1+\tau)^{-1/2}
\end{aligned}$$

Thus, by the positivity of the fundamental solution E, it comes from (4.18)–(4.19) that

$$\Big\|E * \Big|\Gamma^l\Big(\sum_{i,j=0}^{2}\tilde{C}_{ij}(v,Dv)v_{x_i}v_{x_j}\Big)\Big|(t,\cdot)\Big\|_{L^2(\boldsymbol{R}^2)} \le C(1+t)^{1/2}E^3 \tag{4.20}$$

Similarly, we have

$$\begin{aligned}
&\Big\|E * \Big|\Gamma^l\Big(\sum_{\substack{i,j,m=0\\ j+m\ge 1}}^{2}\hat{B}_{i,j,m}(v,Dv)v_{x_i}u_{x_jx_m} + \sum_{i,j=0}^{2}\tilde{C}_{ij}(v,Dv)v_{x_i}v_{x_j}\Big)\Big|(t,\cdot)\Big\|_{L^2(\boldsymbol{R}^2)} \\
&\le C(1+t)^{1/2}E^2(E + D_{S,T}(u))
\end{aligned} \tag{4.21}$$

Hence

$$\|w_{2k}(t,\cdot)\|_{L^2(\boldsymbol{R}^2)} \le C(1+t)^{1/2}E^2(E+D_{S,T}(u)) \tag{4.22}$$

We now estimate $\|w_{3k}(t,\cdot)\|_{L^2(\boldsymbol{R}^2)}$. By Lemma 2.3 and noting (4.10), we have

$$|\Gamma^l F_0(v,0)| \le C \sum_{\substack{|l_0|+\cdots+|l_4|\le|k| \\ |l_0|\le\cdots\le|l_4|}} |(\Gamma^{l_0}v)\cdots(\Gamma^{l_4}v)| \tag{4.23}$$

By Lemma 2.9 (in which we take $\gamma=\frac{1}{4}$) and the positivity of the fundamental solution E, we get

$$\|w_{3k}(t,\cdot)\|_{L^2(\boldsymbol{R}^2)} \le C(1+t)^{1/4} \sum_{\substack{|l_0|+\cdots+|l_4|\le|k| \\ |l_0|\le\cdots\le|l_4|}} \Big(\sum_{|I|\le 1}\int_0^t (1+\tau)^{-3/4}\|\Gamma^I(\Gamma^{l_0}v\cdot\Gamma^{l_1}v)(\tau,\cdot)\|^2_{L^2(\boldsymbol{R}^2)}d\tau\Big)^{1/2} \cdot\Big(\int_0^t (1+\tau)^{1/4}\|(\Gamma^{l_2}v)\cdots(\Gamma^{l_4}v)(\tau,\cdot)\|^2_{L^2(\boldsymbol{R}^2)}d\tau\Big)^{1/2} \tag{4.24}$$

Noting that by Hölder's inequality and (4.1)–(4.2) we have

$$\|\Gamma^I(\Gamma^{l_0}v\cdot\Gamma^{l_1}v)(\tau,\cdot)\|_{L^2(\boldsymbol{R}^2)} \le \|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,S,2} \le CE^2$$

and

$$\|(\Gamma^{l_2}v)\cdots(\Gamma^{l_4}v)(\tau,\cdot)\|_{L^2(\boldsymbol{R}^2)} \le \|v(\tau,\cdot)\|^2_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,S,2} \le C(1+t)^{-1/2}E^3$$

it follows from (4.24) that

$$\|w_{3k}(t,\cdot)\|_{L^2(\boldsymbol{R}^2)} \le C(1+t)^{1/2}E^5 \tag{4.25}$$

Thus, we get from (4.11), (4.17), (4.22) and (4.25) that

$$\sup_{0\le t\le T}(1+t)^{-1/2}\|u(t,\cdot)\|_{\Gamma,S,2} \le C\{\varepsilon+R(E,T)(E+D_{S,T}(u))\} \tag{4.26}$$

We next estimate $\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}$.

For any multi-index k with $|k|\le\left[\frac{S}{2}\right]+1$ we still have (4.11)–(4.14).

By Lemma 2.11 and Lemma 2.5, it is easy to see that

$$(1+t)^{1/2}\|w_{1k}(t,\cdot)\|_{L^\infty(\boldsymbol{R}^2)} \le C\Big\{\varepsilon+\sum_{\substack{|l|\le|k| \\ i=1,2}}\Big(\int_0^t(1+\tau)^{1/2}\|\Gamma^l G_i(v,Du)(\tau,\cdot)\|_{L^\infty(\boldsymbol{R}^2)}d\tau + \int_0^t(1+\tau)^{-3/2}\|\Gamma^l G_i(v,Du)(\tau,\cdot)\|_{\Gamma,3,1}d\tau\Big)\Big\} \tag{4.27}$$

By Hölder's inequality and noting (4.7), we have

$$\begin{aligned}&\|\Gamma^l G_i(v,Du)\,(\tau,\cdot)\|_{L^\infty(R^2)}\\&\quad\le C\|v(\tau,\cdot)\|^2_{\Gamma,[\frac{S}{2}]+1,\infty}\left(\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}+\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\right)\\&\quad\le C(1+\tau)^{-3/2}E^2(E+D_{S,T}(u))\end{aligned}$$

and

$$\begin{aligned}&\|\Gamma^l G_i(v,Du)\,(\tau,\cdot)\|_{\Gamma,3,1}\\&\quad\le C\{\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|_{\Gamma,S,2}(\|v(\tau,\cdot)\|_{\Gamma,S,2}+\|Du(\tau,\cdot)\|_{\Gamma,S,2})\\&\qquad+\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^2_{\Gamma,S,2}\}\\&\quad\le C(1+\tau)^{1/2}E^2(E+D_{S,T}(u))\end{aligned}$$

Thus, it follows from (4.27) that

$$(1+t)^{1/2}\|w_{1k}(t,\cdot)\|_{L^\infty(R^2)}\le C\{\varepsilon+R(E,T)(E+D_{S,T}(u))\}\tag{4.28}$$

By Lemma 2.6 we have

$$(1+t)^{1/2}\|w_{2k}(t,\cdot)\|_{L^\infty(R^2)}\le C\sum_{|I|\le 1}\int_0^t(1+\tau)^{-1/2}\|\Gamma^I H_{2k}(\tau,\cdot)\|_{L^1(R^2)}d\tau\tag{4.29}$$

where H_{2k} denotes the right-hand side of (4.13). Similar to (4.18) we have

$$\begin{aligned}|\Gamma^I H_{2k}|\le C&\sum_{\substack{|l_0|+|l_1|+|l_2|\le|k|+1\\|I_0|\le 1}}(|(\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}Dv)(\Gamma^{l_2}Dv)|\\&+|(\Gamma^{l_0}D^{I_0}v)\,(\Gamma^{l_1}Dv)\,(\Gamma^{l_2}D_xDu)|)\end{aligned}\tag{4.30}$$

To estimate $B\triangleq\|(\Gamma^{l_0}D^{I_0}v)(\Gamma^{l_1}Dv)(\Gamma^{l_2}Dv)(\tau,\cdot)\|_{L^1(R^2)}$, we may assume $|l_1|\le|l_2|$. When $|l_0|\le|l_2|$, it is easily seen that

$$\begin{aligned}B&\le C\|D^{I_0}v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|Dv(\tau,\cdot)\|^2_{\Gamma,S,2}\\&\le C(1+\tau)^{-1/2}E^3\end{aligned}$$

when $|l_0|>|l_2|$ and $|I_0|=1$, we have the same estimate; while when $|l_0|>|l_2|$ and $|I_0|=0$, by Lemmas 2.12 and (2.9) it is easy to see that

$$\begin{aligned}B&\le C\|(\Gamma^{l_0}v)(\Gamma^{l_1}Dv)(\tau,\cdot)\|_{L^2(R^2)}\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\\&\le C\|D_x\Gamma^{l_0}v(\tau,\cdot)\|_{L^2(R^2)}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\\&\le C\|Dv(\tau,\cdot)\|^2_{\Gamma,S,2}\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\\&\le C(1+\tau)^{-1/2}E^3\end{aligned}$$

Similarly, we have

$$\|(\Gamma^{l_0} D^{I_0} v)(\Gamma^{l_1} Dv)(\Gamma^{l_2} D_x Du)(\tau,\cdot)\|_{L^1(R^2)} \leq C(1+\tau)^{-\frac{1}{2}} E^2 D_{S,T}(u)$$

Hence

$$\|\Gamma^I H_{2k}(\tau,\cdot)\|_{L^1(R^2)} \leq C(1+\tau)^{-1/2} E^2 D_{S,,T}(u) \tag{4.31}$$

then

$$(1+t)^{1/2}\|w_{2k}(t,\cdot)\|_{L^\infty(R^2)} \leq CR(E,T)(E+D_{S,T}(u)) \tag{4.32}$$

Noting that, by Lemma 2.3 and Eq. (4.10), for any multi-index l with $|l| \leq S$, we have

$$\|\Gamma^l F(v,0)(\tau,\cdot)\|_{L^1(R^2)} \leq C\|v(\tau,\cdot)\|^3_{\Gamma,[\frac{S}{2}],\infty}\|v(\tau,\cdot)\|^2_{\Gamma,S,2}$$

$$\leq C(1+\tau)^{-1/2}E^5 \tag{4.33}$$

by Lemma 2.6 we get

$$(1+t)^{1/2}\|w_{3k}(t,\cdot)\|_{L^\infty(R^2)} \leq CR(E,T)E \tag{4.34}$$

Thus, we get from (4.28), (4.32) and (4.34) that

$$\sup_{0\leq t\leq T}(1+t)^{1/2}\|u(t,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty} \leq C\{\varepsilon + R(E,T)(E+D_{S,T}(u))\} \tag{4.35}$$

Finally we estimate $\|D^i u(t,\cdot)\|_{\Gamma,S,2}$ $(i=1,2)$.

Using Lemma 4.1, similar to (3.38) we still have

$$|I|,|II|,|III| \leq CR(E,T)D^2_{S,T}(u) \tag{4.36}$$

We also have (3.39), (3.40-1) and (3.40-2). Instead of (3.40-3), we use Lemmas 2.12 and (2.9) to still get

$$B \leq C\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|(\Gamma^{l_1}v)(\Gamma^{l_2}D_x Du)(\tau,\cdot)\|_{L^2(R^2)}$$

$$\leq C\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|D_x\Gamma^{l_1}v(\tau,\cdot)\|_{L^2(R^2)}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}$$

$$\leq C\|v(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}\|Dv(\tau,\cdot)\|_{\Gamma,S,2}\|Du(\tau,\cdot)\|_{\Gamma,[\frac{S}{2}]+1,\infty}$$

$$\leq C(1+\tau)^{-1}E^2 D_{S,T}(u) \tag{4.37}$$

Thus we have

$$|IV| \leq CR(E,T)D^2_{S,T}(u) \tag{4.38}$$

Noting (1.21), similarly we have

$$|V| \leq CR(E,T)(E+D_{S,T}(u))D_{S,T}(u) \tag{4.39}$$

Thus, by (4.36) and (4.38)–(4.39), it follows from (3.35) that

$$\sum_{i=1}^{2}\sup_{0\le t\le T}\|D^i u(t,\cdot)\|_{\Gamma,S,2}\le C\{\varepsilon+\sqrt{R(E,T)}(E+D_{S,T}(u))\} \tag{4.40}$$

The combination of (4.26), (4.35) and (4.40) yields (4.4).

Similarly we can prove (cf. [1], [6])

Lemma 4.3 *Let $\bar{v}$, $\bar{\bar{v}}\in\tilde{X}_{S,E,T}$. If $\bar{u}=M\bar{v}$ and $\bar{\bar{u}}=M\bar{\bar{v}}$ also satisfy $\bar{u}$, $\bar{\bar{u}}\in\tilde{X}_{S,E,T}$, then*

$$D_{S-1,T}(\bar{u}-\bar{\bar{u}})\le\tilde{C}_2(R+\sqrt{R})(D_{S-1,T}(\bar{u}-\bar{\bar{u}})+D_{S-1,T}(\bar{v}-\bar{\bar{v}})) \tag{4.41}$$

where $\tilde{C}_2$ is a positive constant depending on ρ and $R=R(E,T)$ is still defined by (4.5).

Theorem 4.1 is a direct consequence of Lemmas 4.2 and 4.3 (see [9]).

References

[1] Li Ta-tsien & Zhou Yi, Life-span of classical solutions to nonlinear wave equations in two space dimensions, to appear in J. Math. Pure et Appl.

[2] Li Ta-tsien & Zhou Yi, Nonlinear stability for two space dimensional wave equations with higher order perturbations, to appear in Nonlinear World.

[3] Lindblad H., Blow up for solutions of $\Box u=|u|^p$ with small initial data, Thesis, Lund, 1989.

[4] Zhou Yi, Blow-up of solutions to the Cauchy problem for nonlinear wave equations with small initial data, Preprint, Fudan University, 1990.

[5] Zhou Y., Life span of classical solutions to $\Box u=|u|^p$ in two space dimensions, Preprint, Fudan University, 1990.

[6] Li Ta-tsien & Zhou Yi, Life-span of classical solutions to fully nonlinear wave equations II, to appear in *Journal of Nonlinear Analysis: Theory, Methods and Applications.*

[7] Klainerman S., Uniform decay estimates and the Lorentz invariance of the classical wave equations, *Comm. Pure Appl. Math.*, **38** (1985), 321–332.

[8] Klainerman S., Global existence for nonlinear wave equations, *Comm. Pure Appl. Math.*, **33** (1980), 43–101.

[9] Li Ta-tsien & Yu Xin, Life-span of classical solutions to fully nonlinear wave equations, *Commu. in Partial Differential Equations*, **16** (1991), 909–940.

[10] Klainerman S., Remarks on the global Sobolev inequalities in the Minkovski space R^{n+1}, *Comm. Pure Appl. Math.*, **40** (1987), 111–116.