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CONDENSATION OF LEAST-ENERGY SOLUTIONS OF A SEMILINEAR NEUMANN PROBLEM*

Pan Xingbin
(Center for Mathematical Sciences, Zhejiang University, Hangzhou 310027, China)
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Abstract This paper is devoted to the study of the least-energy solutions of a singularly perturbed Neumann problem involving critical Sobolev exponents. The condensation rate is given when $n > 4$ and an asymptotic behavior result is obtained.
Key Words Neumann problem; least-energy solutions.
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1. Introduction

This paper is devoted to the study of the condensation behavior of the least-energy solutions, as $d \to 0$, of the following singularly perturbed semilinear Neumann problem

$$\begin{cases} d\Delta u - u + u^\tau = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial v} = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $\Delta = \sum_{j=1}^{n} \dfrac{\partial^2}{\partial x_j^2}$ is the Laplace operator, Ω is a bounded smooth domain in R^n, $n \geq 3$, v is the unit outer normal to $\partial\Omega$, $\tau = \dfrac{n+2}{n-2}$ and $d > 0$ is a constant. By a *least-energy solution* of (1.1) we mean a (classical) solution of (1.1) which minimizes the "energy" functional

$$J_d(u) = \int_\Omega \left\{ \frac{1}{2}(d|\nabla u|^2 + u^2) - \frac{1}{\tau+1} u_+^{\tau+1} \right\} dx$$

where $u_+ = \max(u, 0)$, among all the solutions of (1.1). Such problems have been studied by many authors, see, e.g., [1], [2] and references therein.

It was proved in [3] that the least-energy solution u_d of (1.1) must exhibit "singular point-condensation" character on the boundary $\partial\Omega$ as $d \to 0$. That is, $u_d \to 0$ in Ω

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as $d \to 0$, the (global) maximum of u_d in $\overline{\Omega}$ is assumed exactly at one point P_d which must lie on the boundary $\partial\Omega$ and $\|u_d\|_{L^\infty(\Omega)} \to \infty$ as $d \to 0$.

The purpose of this paper is to establish the condensation rate and the location of the condensation points of u_d as $d \to 0$, and give a detailed description of the convergence under various scalings, in the case when $n > 4$. Throughout this paper, u_d will always denote a least-energy solution of (1.1), α_d and P_d will always denote the maximum and the maximum point of u_d in $\overline{\Omega}$, respectively, i.e. $u_d(P_d) = \|u_d\|_{L^\infty(\Omega)} = \alpha_d$. Let $\beta_d = \alpha_d^{-\frac{2}{n-2}}$.

Before stating our main results, we recall Theorem 3.1, in [3] as follows. Let

$$U(x) = \left[1 + \frac{|x|^2}{n(n-2)}\right]^{-\frac{n-2}{2}}, \quad x \in R^n \tag{1.2}$$

which is a solution of

$$\Delta U + U^\tau = 0 \tag{1.3}$$

in R^n satisfying $U(0) = 1$. Let

$$S = n(n-2)\pi \left[\Gamma\left(\frac{n}{2}\right) \Big/ \Gamma(n)\right]^{2/n} \tag{1.4}$$

which is the best Sobolev constant in R^n in the following sense:

$$S = \inf\left\{\int_\Omega |\nabla u|^2 dx : u \in H_0^1(\Omega) \text{ and } \int_\Omega |U|^{\tau+1} dx = 1\right\} \tag{1.5}$$

Denote $B_\delta(P) = \{x \in R^n : |x - P| < \delta\}$.

Theorem A [3] *Let u_d be a least-energy solution of (1.1). Then for d sufficiently small the maximum of u_d in $\overline{\Omega}$ is attained exactly at one point P_d which must lie on the boundary $\partial\Omega$, and we have*

(i) $\|u_d\|_{L^\infty(\Omega)} \to \infty$ *as* $d \to 0$;

(ii) $u_d \to 0$ *everywhere in* Ω *as* $d \to 0$;

(iii) $d^{-\frac{n}{2}} \int_\Omega u_d^{\tau+1} dx \to \frac{1}{2} S^{n/2}$ *as* $d \to 0$.

Furthermore, for any $\varepsilon > 0$ there exist two positive constants $d_0 = d_0(\Omega, \varepsilon)$ and $R = R(\Omega, \varepsilon)$ such that for $0 < d < d_0$ the following estimates hold:

(iv) $\left|\frac{u_d(x)}{\|u_d\|_{L^\infty(\Omega)}} - U\left[\frac{\Psi_d(x)}{\beta_d\sqrt{d}}\right]\right| < \varepsilon$ *for all* $x \in \Omega \cap B_{\beta_d\sqrt{d}R}(P_d)$;

(v) $u_d(x) < C\varepsilon \exp\left(-\gamma_0 \zeta(x)/\sqrt{d}\right)$ *for all* $x \in Q \backslash B_{\sqrt{d}R}(P_d)$.

where U is given by (1.2), Ψ_d is a diffeomorphism straightening a boundary portion of $\partial\Omega$ around P_d (as described in Section 2), $\zeta(x) = \min\left\{\eta_0, \operatorname{dist}\left(x, \partial\Omega \cap B_{\sqrt{d}R}(P_d)\right)\right\}$, and C, γ_0, η_0 are positive constants only depending on Ω.

Remark 1.1 From the proof of Lemma 3.35 in [3] we actually see that, for *any* $\delta > 0$ and any $\varepsilon > 0$ there is a $d_0 > 0$ such that for $0 < d < d_0$ the estimate (v) holds in $\Omega \backslash B_{\sqrt{d}\delta}(P_d)$.

Let $K(x)$ be the fundamental solution of $-\Delta+1$ in R^n and

$$G(x)=\frac{1}{n}\omega_n[n(n-2)]^{n/2}K(x),\quad x\in R^n\backslash\{0\} \tag{1.6}$$

where ω_n is the area of the unit sphere in R^n. Now we state the main results of this paper as follows.

Theorem 1.1 *Assume $n>4$ and Ω is strictly convex. Assume u_d is the least-energy solution of (1.1). Then the following estimates hold:*

(i) $$\frac{u_d(x)}{\|u_d\|_{L^\infty(\Omega)}}\le CU\left[\frac{x-P_d}{\beta_d\sqrt{d}}\right],\ x\in\Omega,$$

where U is given in (1.2) and $C>0$ only depends on Ω;

(ii) *for any $\delta>0$ and $\varepsilon>0$ there is $d_0>0$ such that for $0<d<d_0$,*

$$\left|\|u_d\|_{L^\infty(\Omega)}u_d(x)-G\left[\frac{1}{\sqrt{d}}\Psi_d(x)\right]\right|<\varepsilon \text{ for all } x\in Q\backslash B_{\sqrt{d}\delta}(P_d)$$

where $G(x)$ is given by (1.6) and Ψ_d is a diffeomorphism straightening a boundary portion around P_d (as described in Section 2);

(iii) $$\lim_{d\to0}\sqrt{d}\|u_d\|^{2/(n-2)}_{L^\infty(\Omega)}=\frac{2}{H(P_0)}\frac{(n-1)(n-3)}{(n-2)(n-4)}\left(\frac{n\pi}{n-2}\right)^{1/2}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}$$

where $H(P_0)$ is the mean curvature of $\partial\Omega$ at $P_0\in\partial\Omega$ (related to the inner normal), P_0 is the limit point of $\{P_d\}$. Moreover when $n>6$ we have $H(P_0)=\max_{P\in\partial\Omega}H(P)$.

Remarks In [2, p.843], W.-M. Ni and I. Takagi conjectured that the least-energy solution u_d attains the maximum near a point $P\in\partial\Omega$ where the mean curvature $H(P)$ of the boundary $\partial\Omega$ at P is the largest. In other words, condensation must happen at the maximum point of the mean curvature. In the subcritical case, i.e. when replacing τ in (1.1) by p with $1<p<\frac{n+2}{n-2}$, and $n\ge3$, the conjecture has been proved by W.-M. Ni and I. Takagi in [4]. In the critical case, our Theorem 1.1 proves the conjecture when $n>6$. Moreover, part (iii) of Theorem 1.1 also means that the mean curvature of the boundary has an important effect on the condensation rate.

Combining part (iv) of Theorem A with part (ii) of Theorem 1.1, we see that u_d has two boundary layers around the maximum point P_d. The behavior of u_d inside the small boundary layer with scale $\beta_d\sqrt{d}$ can be described *via* the function U, while outside the large one with scale $\sqrt{d}$ it can be described *via* the fundamental solution $K(x)$.

The outline of this paper is as follows. In Section 2 we give a precise estimate for the energy of u_d as $d\to0$ and then prove the following result.

Lemma 1.1 *Assume $n>4$ and $\beta_d=\|u_d\|^{-2/(n-2)}_{L^\infty(\Omega)}$. Then*

$$\limsup_{d\to0}\frac{\sqrt{d}}{\beta_d}<\infty \tag{1.7}$$

By virtue of Lemma 1.1 we prove part (i) of Theorem 1.1 in Section 3. In Section 4 we prove parts (ii) and (iii) to complete Theorem 1.1.

Remark 1.2 From the estimate (2.57) in Section 2 we easily see that, when $n > 6$ part (iii) in Theorem 1.1 is true for general domains, without the assumption of convexity, see Corollary 2.1. We shall also see that when $n > 4$, the convexity condition of Ω can be replaced by the following weaker version:

Let P_0 be the limit point of $\{P_d\}$ as $d \to 0$. We assume that all principal curvatures of $\partial\Omega$ at P_0 (related to the inner normal) are positive.

Remark 1.3 In this paper we frequently use the methods and ideas developed by Professors Wei-Ming Ni and Izumi Takagi in their early works, see [4] and references therein. We also use the techniques and estimates from A. Bahri [5], A. Bahri and J. M. Coron [6] and O. Rey [7].

2. Proof of Lemma 1.1

Throughout this section we set

$$V_d(x) = d^{-\frac{n-2}{4}} u_d(x) \tag{2.1}$$

Then V_d satisfies

$$\begin{cases} \Delta V_d - \dfrac{1}{d} V_d + V_d^{\tau} = 0 & \text{in } \Omega \\ \dfrac{\partial V_d}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

and for any given $\rho > 0, \varepsilon > 0$, from Remark 1.1

$$V_d(x) \le C\varepsilon d^{-\frac{(n-2)}{4}} \exp\left[-\frac{\gamma_0 \zeta(x)}{\sqrt{d}}\right] \tag{2.3}$$

for all $x \in \Omega \backslash B_{\rho\sqrt{d}}(P_d)$ and d sufficiently small.

Next, we recall several facts in [1] and [2] concerning a diffeomorphism which straightens a portion of the boundary $\partial\Omega$ around a given point $P \in \partial\Omega$. Through the translation and rotation of the coordinate system we may assume that P is at the origin and the inner normal to $\partial\Omega$ at P points in the direction of the positive x_n-axis. Then there exists a smooth function $\psi(x')$, $x' = (x_1, \cdots, x_{n-1})$, defined in $|x'| < \delta_0$ such that

(i) $\psi(0) = 0,\ \nabla\psi(0) = 0$;

(ii) $\partial\Omega \cap N = \{(x', x_n) \in N : x_n = \psi(x')\}$ and $\Omega \cap N = \{(x', x_n) \in N : x_n > \psi(x')\}$, where N is a neighborhood of P.

Hence around the point $P, \partial\Omega$ can be represented as

$$x_n = \psi(x') = \frac{1}{2}\sum_{i=1}^{n-1} \alpha_i x_i^2 + O(|x'|^3) \tag{2.4}$$

where $\alpha_1, \cdots, \alpha_{n-1}$ are the principal curvatures of $\partial\Omega$ at P (related to the inner normal) and

$$H(P) = \frac{1}{n-1}\sum_{i=1}^{n-1}\alpha_i = \frac{1}{n-1}\Delta\psi(0) \tag{2.5}$$

is the mean curvature of $\partial\Omega$ at P.

For $y \in R^n$ with $|y|$ sufficiently small we define a map $x = \Phi(y) = (\Phi_1(y), \cdots, \Phi_n(y))$ by

$$\Phi_j(y) = \begin{cases} y_j - y_n \dfrac{\partial \psi}{\partial x_j}(y'), & j = 1, \cdots, n-1 \\ y_n + \psi(y'), & j = n \end{cases} \tag{2.6}$$

Then the differential $D\Phi$ of Φ is

$$D\Phi(y) = \begin{bmatrix} \delta_{ij} - \dfrac{\partial^2\psi}{\partial x_i \partial x_j}(y')y_n & -\dfrac{\partial\psi}{\partial x_i}(y') \\ \dfrac{\partial\psi}{\partial x_j}(y') & 1 \end{bmatrix}_{1\le i,j\le n-1} \tag{2.7}$$

where δ_{ij} is the Kronecker symbol. Since $\nabla\psi(0) = 0$, $D\Phi(0) =$ the identity map. Thus Φ has an inverse mapping $y = \Phi^{-1}(x)$ for $|x| < \delta$. We denote Φ^{-1} by $\Psi(x) = (\Psi_1(x), \cdots, \Psi_n(x))$. Without loss of generality we may assume that $\Omega \cap N = \Phi\left(B_\delta^+\right)$ and $\partial\Omega \cap N = \Phi\left(B_\delta^0\right)$, where $B_\delta^+ = B_\delta(0) \cap [y_n > 0]$ and $B_\delta^0 = B_\delta(0) \cap [y_n = 0]$. It was established in [2; Lemma A. 1] that near $y = 0$,

$$\det D\Phi(y) = 1 - (n-1)H(P)y_n + O(|y|^2) \tag{2.8}$$

Obviously the functions Ψ, Φ and ψ depend smoothly on the point $P \in \partial\Omega$ and the number δ may be chosen independent of P.

Let P_d be the maximum point of V_d which must lie on $\partial\Omega$ when d is sufficiently small by Theorem A. Let Ψ be the diffeomorphism straightening a boundary portion of $\partial\Omega$ around P_d (we denote it by Ψ instead of Ψ_d to simplify notation). Let $\Phi = \Psi^{-1}$. Define

$$\hat{g}_{ij}(y) = \sum_{k=1}^n \frac{\partial \Phi_k(y)}{\partial y_i}\frac{\partial \Phi_k(y)}{\partial y_j}, \quad y \in \overline{B_{2\delta}^+} \tag{2.9}$$

$$\hat{g}^{ij}(y) = \sum_{k=1}^n \frac{\partial \Psi_i}{\partial x_k}(\Phi(y))\frac{\partial \Psi_j}{\partial x_k}(\Phi(y)), \quad y \in \overline{B_{2\delta}^+} \tag{2.10}$$

Define for $y = (y', y_n)$

$$g_{ij}(y) = \begin{cases} \hat{g}_{ij}(y) & \text{if } y \in \overline{B_{2\delta}^+} \\ (-1)^{\delta_{in}+\delta_{jn}}\hat{g}_{ij}(y', -y_n) & \text{if } (y', -y_n) \in \overline{B_{2\delta}^+} \end{cases} \tag{2.11}$$

$$g^{ij}(y) = \begin{cases} \hat{g}^{ij}(y) & \text{if } y \in \overline{B_{2\delta}^+} \\ (-1)^{\delta_{in}+\delta_{jn}}\hat{g}^{ij}(y', -y_n) & \text{if } (y', -y_n) \in \overline{B_{2\delta}^+} \end{cases} \tag{2.12}$$

We easily check that

$$\sum_{k=1}^{n} g_{ik}(y)g^{kj}(y) = \delta_{ij}$$

Let

$$\hat{V}_d(y) = V_d(\Phi(y)), \quad y \in \overline{B_{2\delta}^{+}} \tag{2.13}$$

$$\tilde{V}_d(y) = \begin{cases} \hat{V}_d(y) & \text{if } y \in \overline{B_{2\delta}^{+}} \\ \hat{V}_d(y', -y_n) & \text{if } (y', -y_n) \in \overline{B_{2\delta}^{+}} \end{cases} \tag{2.14}$$

Then $\tilde{V}_d$ satisfies in $B_{2\delta}\backslash[y_n = 0]$

$$\Delta_g \tilde{V}_d - \frac{1}{d}\tilde{V}_d + \tilde{V}_d^{\tau} = 0 \tag{2.15}$$

where

$$\Delta_g f = \frac{1}{\sqrt{\det(g_{ij})}} \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(\sqrt{\det(g_{ij})}g^{ij}\frac{\partial f}{\partial y_j}\right)$$

By the proof of [1; Lemma 4.3] we know that $\tilde{V}_d$ is a weak solution of (2.15) in $B_{2\delta}$. Define the measure dg on $\overline{B}_\delta$ by $dg = \sqrt{\det(g_{ij})dy}$ and denote

$$\langle \nabla u, \nabla v \rangle_g := \int_{B_\delta} \sum_{i,j=1}^{n} g^{ij}\frac{\partial u}{\partial y_i}\frac{\partial v}{\partial y_j} dg$$

$$\| \nabla u \|_g = \sqrt{\langle \nabla u, \nabla u \rangle}$$

Then we have

$$\| \nabla \tilde{V}_d \|_g^2 = 2\int_{\Phi(B_\delta)\cap\Omega} |\nabla V_d|^2 dx$$

$$\|\tilde{V}_d\|_{L_g^2}^2 = \int_{B_\delta} \tilde{V}_d^2 dg = 2\int_{\Phi(B_\delta)\cap\Omega} V_d^2 dx$$

$$\|\tilde{V}_d\|_{L_g^{\tau+1}}^{\tau+1} = \int_{B_\delta} \tilde{V}_d^{\tau+1} dg = 2\int_{\Phi(B_\delta)\cap\Omega} V_d^{\tau+1} dx$$

From (2.3) we have for all small d

$$\tilde{V}_d(y) \le cd^{-\frac{n-2}{4}} \exp\left[-\frac{2\gamma|y|}{\sqrt{d}}\right] \tag{2.16}$$

for $|y| \ge \rho\sqrt{d}$ with $\gamma > 0$ independent of d. Especially

$$\tilde{V}_d(y) \le cd^{-\frac{n-2}{4}} \exp\left[-\frac{2\gamma\delta}{\sqrt{d}}\right] = O(e^{-\gamma/\sqrt{d}}) \text{ on } \partial B_\delta \tag{2.17}$$

when d is sufficiently small.

Let $H^*(B_\delta) = \{V \in W^{1,2}(B_\delta) : V(y', -y_n) = V(y', y_n)\}$ and $H_0^*(B_\delta) = H^*(B_\delta) \cap W_0^{1,2}(B_\delta)$. Define the projection $P : W^{1,2}(B_\delta) \to W_0^{1,2}(B_\delta)$ by letting $u = PV$ be the solution of

$$\begin{cases} \Delta_g u = \Delta_g V & \text{in } B_\delta \\ u = 0 & \text{on } \partial B_\delta \end{cases}$$

From (2.11) (2.12) we see that $\det(g_{ij}(y', -y_n)) = \det(g_{ij}(y', y_n))$ and $(\Delta_g)u(y', -y_n) = (\Delta_g u)(y', -y_n)$. Hence $P(H^*(B_\delta)) \subset H_0^*(B_\delta)$. Let

$$U(y) = \left[1 + \frac{|y|^2}{n(n-2)}\right]^{-\frac{n-2}{n}}, \quad U_{x,\varepsilon}(y) = \varepsilon^{-\frac{n-2}{2}} U\left(\frac{y-x}{\varepsilon}\right)$$

for $\varepsilon > 0$ and $x \in B_\delta^0 \equiv B_\delta \cap [y_n = 0]$. Define

$$M = \left\{cPU_{x,\varepsilon} : c \in R^1, x \in B_\delta^0, \varepsilon > 0\right\}$$

which is an $(n+1)$-dimensional submanifold of $H_0^*(B_\delta)$. Observe that for any $V \in H^*(B_\delta)$, $x = (x', 0) \in B_\delta^0$,

$$\left\langle \nabla V, \nabla \frac{\partial}{\partial x_n} PU_{x,\varepsilon} \right\rangle_g = 0 \tag{2.18}$$

since

$$\int_{B_\delta \cap [y_n < 0]} \sum_{i,j=1}^n g^{ij} \frac{\partial V}{\partial y_i} \frac{\partial}{\partial y_j} \left(\frac{\partial}{\partial x_n} PU_{x,\varepsilon}\right) dg = -\int_{B_\delta^+} \sum_{i,j=1}^n g^{ij} \frac{\partial V}{\partial y_i} \frac{\partial}{\partial y_j} \left(\frac{\partial}{\partial x_n} PU_{x,\varepsilon}\right) dg$$

by (2.11) (2.12). Now for $x \in B_\delta^0$, $\varepsilon > 0$, we define

$$\begin{aligned} E_{x,\varepsilon}^* = \Big\{ V \in H_0^*(B_\delta) : \langle \nabla V, \nabla PU_{x,\varepsilon} \rangle_g &= \left\langle \nabla V, \nabla \frac{\partial}{\partial \varepsilon} PU_{x,\varepsilon} \right\rangle_g \\ &= \left\langle \nabla V, \nabla \frac{\partial}{\partial x_i} PU_{x,\varepsilon} \right\rangle_g = 0, \quad 1 \le i \le n-1 \Big\} \end{aligned} \tag{2.19}$$

For $x = (x', 0)$ with $|x'|$ sufficiently small, denote

$$\varphi_{x,\varepsilon}(y) = U_{x,\varepsilon}(y) - PU_{x,\varepsilon}(y) \tag{2.20}$$

Then $\Delta_g \varphi_{x,\varepsilon} = 0$ and $\varphi_{x,\varepsilon} = U_{x,\varepsilon}$ on ∂B_δ. By the maximum principle we have as $\varepsilon \to 0$,

$$0 < \varphi_{x,\varepsilon}(y) = O\left(\varepsilon^{\frac{n-2}{2}}\right) \quad \text{in } B_\delta \tag{2.21}$$

$$\| \nabla \varphi_{x,\varepsilon} \|_g, \|\varphi_{x,\varepsilon}\|_{L_g^{\tau+1}}, \left\| \frac{\partial}{\partial x_i} \varphi_{x,\varepsilon} \right\|_{L_g^{\tau+1}} = O\left(\varepsilon^{\frac{n-2}{2}}\right) \tag{2.22}$$

see, e.g. [7; Proposition 1]. Denote

$$h_d(y) = \tilde{V}_d(y) - P\tilde{V}_d(y) \tag{2.23}$$

Then we have from (2.17)

$$0 < h_d(y) = O\left(e^{-\tau/\sqrt{d}}\right) \quad \text{in } B_\delta \tag{2.24}$$

By Green formula

$$\|\nabla \tilde{V}_d\|_g = \|\nabla (P\tilde{V}_d)\|_g + O\left(e^{-\tau/\sqrt{d}}\right) \tag{2.25}$$

Let

$$\mathrm{dist}(P\tilde{V}_d, M) = \inf_{\varphi\in M} \|\nabla (P\tilde{V}_d) - \nabla\varphi\|_g$$

Lemma 2.1 *When d is sufficiently small.* $\mathrm{dist}(P\tilde{V}, M)$ *is assumed by* $C_d PU_{x_d,\varepsilon_d} \in M$. *Let* $f_d = P\tilde{V}_d - C_d PU_{x_d,\varepsilon_d}$. *Then* $f_d \in E^*_{x_d,\varepsilon_d}$. *Moreover, as* $d \to 0$,

(i) $x_d = (x'_d, 0)$, $|x'_d| = O\left(\beta_d\sqrt{d}\right)$,

(ii) $\lim_{d\to 0} \frac{\varepsilon_d}{\beta_d\sqrt{d}} = 1$.

Proof We follow the line in [6] and [7]. Let

$$\tilde{W}_d(y) = \left(\beta_d\sqrt{d}\right)^{(n-2)/2} \tilde{V}_d\left(\beta_d\sqrt{d}y\right)$$

From Theorem A, $\tilde{W}_d(y) \to U(y)$ in $C^2_{\mathrm{loc}}(R^n)$ as $d \to 0$ and

$$\|\nabla \tilde{W}_d\|^2_{L^2(B_\delta/\beta_d\sqrt{d})} \to S^n = \|\nabla U\|^2_{L^2(R^n)}$$

Hence $\|\nabla \tilde{W}_d - \nabla U\|_{L^2(B_\delta/\beta_d\sqrt{d})} \to 0$. By the invariance of the norm $\|\nabla V\|_{L^2}$ in the rescaling, $\|\nabla \tilde{V}_d - \nabla U_{0,\beta_d\sqrt{d}}\|_{L^2(B_\delta)} \to 0$. Noting $g_{ij}(y) = \delta_{ij} + O(|y|)$, we have

$$\|\nabla \tilde{V}_d - \nabla U_{0,\beta_d\sqrt{d}}\|_g \to 0 \quad \text{as } d \to 0 \tag{2.26}$$

From (2.22) (2.25) we have

$$\mathrm{dist}(P\tilde{V}_d, M) \le \|\nabla P\tilde{V}_d - \nabla PU_{0,\beta_d\sqrt{d}}\|_g \to 0 \tag{2.27}$$

For d sufficiently small, let $\{C_j PU_{x_j,\varepsilon_j}\}$ be the minimizing sequence of $\mathrm{dist}(P\tilde{V}_d, M)$. From (2.16), (2.27) we easily see that $\{C_j\}$ is bounded, $\{x_j\} \subset B^0_{\delta/2}$ and $\{\varepsilon_j\}$ is bounded between two positive constants $0 < a(d) < \varepsilon_j < A(d)$. Therefore $\mathrm{dist}(P\tilde{V}_d, M)$ is achieved by $C_d PU_{x_d,\varepsilon_d}$ with $C_d \in R^1$, $x_d = (x'_d, 0) \in B_\delta$, $\varepsilon_d > 0$. Hence we can write

$$P\tilde{V}_d = C_d PU_{x_d,\varepsilon_d} + f_d \tag{2.28}$$

with $\|\nabla f_d\|_g \to 0$ as $d \to 0$, which implies

$$\|\nabla P\tilde{V}_d\|^2_g = C_d^2 \|PU_{x_d,\varepsilon_d}\|^2_g + o(1) \tag{2.29}$$

(2.25) (2.26) (2.29) imply

$$\|\nabla U_{0,\beta_d\sqrt{d}}\|_g = C_d^2 \|PU_{x_d,\varepsilon_d}\|^2_g + o(1) \tag{2.30}$$

From (2.16) (2.28) we see that $\varepsilon_d \le C$, hence $\|PU_{x_d,\varepsilon_d}\|_g$ is bounded away from zero. Note that $\|\nabla U_{0,\beta_d\sqrt{d}}\|^2_g = S^{n/2} + o(1)$. (2.30) implies C_d is bounded and

$$C_d \left\langle \nabla U_{0,\beta_d\sqrt{d}}, \nabla PU_{x_d,\varepsilon_d}\right\rangle_g = S^{n/2} + o(1) \tag{2.31}$$

Using the Green formula

$$\begin{aligned}\left\langle \nabla U_{0,\beta_d\sqrt{d}}, \nabla PU_{x_d,\varepsilon_d}\right\rangle_g &= -\int_{B_\delta} (\Delta_g U_{0,\beta_d\sqrt{d}}) PU_{x_d,\varepsilon_d} dg \\ &= -\int_{B_\delta} (\Delta U_{0,\beta_d\sqrt{d}} + O(|y|) |\nabla U_{0,\beta_d\sqrt{d}}|) PU_{c_d,\varepsilon_d} dg \\ &= -\int_{B_\delta} (\Delta U_{0,\beta_d\sqrt{d}}) PU_{x_d,\varepsilon_d} dy + o(1) \\ &= \int_{B_\delta} U^{\tau}_{0,\beta_d\sqrt{d}} PU_{x_d,\varepsilon_d} dy + o(1) \\ &\le \int_{B_\delta} U^{\tau}_{0,\beta_d\sqrt{d}} U_{x_d,\varepsilon_d} dy + o(1) \quad (\text{since } PU_{x_d,\varepsilon_d} < U_{x_d,\varepsilon_d}) \\ &= O\left[\frac{\varepsilon_d}{\beta_d\sqrt{d}} + \frac{\beta_d\sqrt{d}}{\varepsilon_d} + \frac{1}{\varepsilon_d\beta_d\sqrt{d}}|x_d|^2\right]^{-\frac{n-2}{2}} + o(1)\end{aligned}$$

by the estimate in [5]. Together with (2.31), the above estimate implies $|x_d'|^2 = |x_d|^2 = O(\beta_d\sqrt{d}\varepsilon_d)$ and

$$C_1\beta_d\sqrt{d} \le \varepsilon_d \le C_2\beta_d\sqrt{d}$$

for some positive constants C_1, C_2. From (2.22) (2.30), (2.31) we have $C_d \to 1$ as $d \to 0$, and

$$\int_{B_\delta} |\nabla U_{0,\beta_d\sqrt{d}} - \nabla U_{x_d,\varepsilon_d}|^2 dy \to 1$$

Hence $\dfrac{\varepsilon_d}{\beta_d\sqrt{d}} \to 1$.

Since $C_d PU_{x_d,\varepsilon_d}$ is the minimizer of $\operatorname{dist}(P\tilde{V}_d, M)$, we know that $f_d = P\tilde{V}_d - C_d PU_{x_d,\varepsilon_d}$ must lie in $E^*_{x_d,\varepsilon_d}$. Now Lemma 2.1 is complete.

In the following we estimate f_d. From (2.11) (2.12) and (2.8), computation shows that, for $V \in H^*(B_\delta)$,

$$\begin{aligned}\|\nabla V\|_g^2 = &\int_{B_\delta} |\nabla V(y)|^2\{1-(n-1)H(P_d)|y_n| + O(|y|^2)\}dy \\ &+ 2\sum_{j=1}^{n-1}\alpha_j(P_d)\int_{B_\delta}|y_n|\left(\frac{\partial V}{\partial y_j}\right)^2 dy \end{aligned} \tag{2.32}$$

and for $p \ge 1$,

$$\int_{B_\delta}|V(y)|^p dg = \int_{B_\delta}|V(y)|^p\{1-(n-1)H(P_d)|y_n| + O(|y|^2)\}dy \tag{2.33}$$

where $H(P_d)$ and $\alpha_j(P_d)$, $1 \le j \le n-1$, are the mean curvature and the principal curvatures of $\partial\Omega$ at P_d. In particular if $V(y) = V(|y|)$, then

$$\|\nabla V\|_g^2 = \omega_n\int_0^\eta r^{n-1}|V'(r)|^2 dr$$

$$-\frac{2(n-1)^2\pi^{\frac{n-1}{2}}H(P_d)}{(n+1)\Gamma\left(\frac{n+1}{2}\right)}\int_0^\delta r^n(1+O(r))V'(r)^2dr \tag{2.34}$$

$$\int_{B_\delta}|V(y)|^p dg = \omega_n\int_0^\eta r^{n-1}|V(r)|^p dr - \frac{4\pi^{\frac{n-1}{2}}H(P_d)}{\Gamma\left(\frac{n-1}{2}\right)}\int_0^\delta r^n(1+O(r))|V(r)|^p dr \tag{2.35}$$

From (2.34), (2.35) we have, for small d, as $\varepsilon\to 0$,

$$\|\nabla U_{0,\varepsilon}\|_g^2 = S^{n/2} - a_n\xi_1(\varepsilon)H(P_d) + \begin{cases} O(\varepsilon^{n-2}) & \text{if } n>4 \\ O\left(\varepsilon^2\log\frac{1}{\varepsilon}\right) & \text{if } n=4 \\ O(\varepsilon) & \text{if } n=3 \end{cases} \tag{2.36}$$

where

$$\xi_1(\varepsilon) = \begin{cases} \varepsilon & \text{if } n\geq 4 \\ \varepsilon\log\frac{1}{\varepsilon} & \text{if } n=3 \end{cases}$$

and

$$a_n = \begin{cases} \pi^{\frac{n-1}{2}}\frac{(n-1)^2(n-2)^2}{n-3}[n(n-2)]^{\frac{n-1}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma(n)} & \text{if } n\geq 4 \\ 6\pi & \text{if } n=3 \end{cases}$$

$$\int_{B_\delta} U_{0,\varepsilon}^{\tau+1}(y)dg = S^{n/2} - b_nH(P_d)\varepsilon + O(\varepsilon^2) \tag{2.37}$$

where

$$b_n = 2\pi^{\frac{n-1}{2}}[n(n-2)]^{\frac{n+1}{2}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}$$

$$\int_{B_\delta} U_{0,\varepsilon}^2(y)dg = C_n\xi_2(\varepsilon) + o(\xi_2(\varepsilon))$$

where

$$\xi_2(\varepsilon) = \begin{cases} \varepsilon^2 & \text{if } n>4 \\ \varepsilon^2\log\frac{1}{\varepsilon} & \text{if } n=4 \\ O(\delta)\varepsilon & \text{if } n=3 \end{cases}$$

and

$$C_n = \begin{cases} \frac{4(n-1)}{n-4}S^{n/2} & \text{if } n>4 \\ 128\pi^2 & \text{if } n=4 \\ 12\pi & \text{if } n=3 \end{cases}$$

$$\|U_{0,\varepsilon}\|_{L_g^{2n/(n+2)}} \equiv \left(\int_{B_\delta} U_{0,\varepsilon}^{2n/(n+2)}(y)dg\right)^{\frac{n+2}{2n}} = O(\xi_3(\varepsilon)) \quad (2.38)$$

where

$$\xi_3(\varepsilon) = \begin{cases} \varepsilon^2 & \text{if } n > 6 \\ \varepsilon^2\left(\log\frac{1}{\varepsilon}\right)^{2/3} & \text{if } n = 6 \\ \varepsilon^{\frac{n-2}{2}} & \text{if } 3 \le n \le 5 \end{cases}$$

$$\int_{B_\delta} |y|^2 |\nabla U_{0,\varepsilon}(y)|^2 dg = O(\xi_2(\varepsilon)) \quad (2.39)$$

Lemma 2.2 *Let f_d be given in Lemma 2.1. Then as $d \to 0$,*

$$\|\nabla f_d\|_g^2 + \frac{1}{d}\int_{B_\delta} f_d^2 dg = O\left(\xi_2^{1/2}(\varepsilon_d) + \frac{1}{d}\xi_3(\varepsilon_d) + e^{-\gamma\sqrt{d}}\right)^2 \quad (2.40)$$

Proof Since $f_d \in E^*_{x_d,\varepsilon_d}$ we have $\langle \nabla f_d, \nabla PU_{x_d,\varepsilon_d}\rangle_g = 0$. Multiplying (2.15) by f_d we get

$$\|\nabla f_d\|_g^2 + \frac{1}{d}\int_{B_\delta} f_d^2 dg + \frac{1}{d}\int_{B_\delta}(C_d PU_{x_d,\varepsilon_d} + h_d) f_d dg = \int_{B_\delta}(C_d PU_{x_d,\varepsilon_d} + h_d + f_d)^\tau f_d dg$$

Let $\sigma = \min\{3, \tau+1\}$. Then from (2.24),

$$\begin{aligned}\int_{B_\delta}(C_d PU_{x_d,\varepsilon_d} + h_d + f_d)^\tau f_d dg =& C_d^\tau \int_{B_\delta}(PU_{x_d,\varepsilon_d})^\tau f_d dg + C_d^{\tau-1}\tau\int_{B_\delta}(PU_{x_d,\varepsilon_d})^{\tau-1} f_d^2 dg \\ &+ O\left(\|\nabla f_d\|_g^\sigma\right) + O\left(e^{-\gamma/\sqrt{d}}\right)\end{aligned}$$

Since $f_d \in E^*_{x_d,\varepsilon_d} \subset H_0^*(B_\delta)$ and $|x_d| = O(\varepsilon_d)$,

$$\begin{aligned} 0 = \langle \nabla f_d, \nabla PU_{x_d,\varepsilon_d}\rangle_g &= -\int_{B_\delta} f_d \Delta_g PU_{x_d,\varepsilon_d} dg \\ &= -\int_{B_\delta} f_d \Delta_g U_{x_d,\varepsilon_d} dg = \langle \nabla f_d, \nabla U_{x_d,\varepsilon_d}\rangle_g \\ &= \int_{B_\delta} \nabla f_d \cdot \nabla U_{x_d,\varepsilon_d} dy + \int_{B_\delta} O(|y|)|\nabla f_d||\nabla U_{x_d,\varepsilon_d}| dy \\ &= -\int_{B_\delta} f_d \Delta U_{x_d,\varepsilon_d} dy + \int_{B_\delta} O(|x_d| + |y - x_d|)|\nabla f_d||\nabla U_{x_d,\varepsilon_d}| dy \\ &= \int_{B_\delta} f_d U^\tau_{x_d,\varepsilon_d} dy + \int_{B_\delta} O(\varepsilon_d + |y - x_d|)|\nabla f_d||\nabla U_{x_d,\varepsilon_d}| dy \end{aligned}$$

Hence by (2.39)

$$\begin{aligned}\int_{B_\delta}(PU^\tau_{x_d,\varepsilon_d}) f_d dg &\le C\int_{B_\delta} U^\tau_{x_d,\varepsilon_d}|f_d| dy \\ &\le \Big[O(\varepsilon_d)\|\nabla U_{x_d,\varepsilon_d}\|_{L^2(B_\delta)}\end{aligned}$$

$$+\left(\int_{B_\delta} O(|y-x_d|^2)\,|\nabla U_{x_d,\varepsilon_d}|^2\,dy\right)^{1/2}\Bigg]\|\nabla f_d\|_g$$
$$=\left(O(\varepsilon_d)+O\left(\xi_2^{1/2}(\varepsilon_d)\right)\right)\|\nabla f_d\|_g$$
$$=O\left(\xi_2^{1/2}(\varepsilon_d)\right)\|\nabla f_d\|_g \tag{2.41}$$

and by (2.38)

$$\int_{B_\delta} PU_{x_d,\varepsilon_d}|f_d|dg \le \|U_{x_d,\varepsilon_d}\|_{L_g^{2n/(n+2)}}\|f_d\|_{L_g^{\tau+1}} = O(\xi_3(\varepsilon_d))\|\nabla f_d\|_g \tag{2.42}$$

Using (2.24) we obtain

$$\|\nabla f_d\|_g^2+\frac{1}{d}\int_{B_\delta} f_d^2dg - C_d^{\tau-1}\tau\int_{B_\delta}(PU_{x_d,\varepsilon_d})^{\tau-1}f_d^2dg$$
$$=O\left(\xi_2^{1/2}\right)(\varepsilon_d)+\frac{1}{d}\xi_3(\varepsilon_d)+e^{-\gamma/\sqrt{d}})\|\nabla f_d\|_g+O\left(\|\nabla f_d\|_g^\sigma\right) \tag{2.43}$$

By the next lemma, when d is sufficiently small, the left-hand term in (2.43) is larger than $\overline{\rho}\left(\|\nabla f_d\|_g^2+\frac{1}{d}\int_{B_\delta} f_d^2dg\right)+O\left(\varepsilon_d^{n-2}+e^{-\gamma/\sqrt{d}}\right)$, where $\overline{\rho}>0$ is independent of d. Therefore we have

$$\|\nabla f_d\|_g^2+\frac{1}{d}\int_{B_\delta} f_d^2dg = O\left(\xi_2^{1/2}(\varepsilon_d)+\frac{1}{d}\xi_3(\varepsilon_d)+e^{-\gamma/\sqrt{d}}\right)\|\nabla f_d\|_g$$
$$+O\left(\|f_d\|_g^\sigma\right)+O\left(\varepsilon_d^{n-2}+e^{-\gamma/\sqrt{d}}\right)$$

which gives the estimate (2.40).

Lemma 2.3 *Let f_d be given in* Lemma 2.1. *Then there exists a positive constant $\rho_1>0$ such that for d sufficiently small,*

$$\|\nabla f_d\|_g^2+\frac{1}{d}\int_{B_\delta} f_d^2dg \ge (\tau+\rho_1)\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}f_d^2dg+O\left(\varepsilon_d^{n-2}+e^{-\gamma/\sqrt{d}}\right) \tag{2.44}$$

Proof First we note that, for any $V\in H^*(B_\delta)$, $x=(x',0)\in B_\delta^0$, (2.18) holds. Let

$$V(x_d,\varepsilon_d)=\text{span}\left\{PU_{x_d,\varepsilon_d},\frac{\partial}{\partial\varepsilon}PU_{x_d,\varepsilon_d},\frac{\partial}{\partial x_i}PU_{x_d,\varepsilon_d},1\le i\le n\right\}$$

Define the inner product in $E^*_{x_d,\varepsilon_d}$, by

$$\langle u,v\rangle=\int_{B_\delta}\nabla u\cdot\nabla v dy$$

and let $\varphi_1^d,\cdots,\varphi_{n+2}^d$ be the orthonormal bases of $V(x_d,\varepsilon_d)$. Let

$$f_d=\sum_{i=1}^{n+2}\alpha_i^d\varphi_i^d+\overline{f}_d$$

where $\left\langle \overline{f}_d, \varphi_i^d \right\rangle = 0$, $1 \le i \le n+2$. By [7; Appendix D]

$$\int_{B_\delta} |\nabla \overline{f}_d|^2 dy \ge (\tau + \rho') \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1} \overline{f}_d^2 dy$$

for d sufficiently small, where $\rho' > 0$ is independent of d. Hence

$$\int_{B_\delta} |\nabla f_d|^2 dy = \sum_{i=1}^{n+2} \left(\alpha_i^d\right)^2 + \int_{B_\delta} |\nabla \overline{f}_d|^2 dy \ge (\tau + \rho') \int_{B_\delta} U_{x_d;\varepsilon_d}^{\tau-1} \overline{f}_d^2 dy$$

Since $f_d = P\tilde{V}_d - C_d P U_{x_d,\varepsilon_d} = \tilde{V}_d - h_d - C_d U_{x_d,\varepsilon_d} + C_d \varphi_{x_d,\varepsilon_d}$, from (2.16), (2.22), (2.24) and Lemma 2.1, we choose $\eta > 0$ sufficiently small and estimate

$$\begin{aligned} \| \nabla f_d \|_g^2 &\ge \int_{B_\delta} \| \nabla f_d|^2 dy - C_1 \int_{B_\delta} |y| | \nabla f_d|^2 dy \\ &\ge \int_{B_\delta} |\nabla f_d|^2 dy - C_1 \eta \int_{|y| \le \eta} |\nabla f_d|^2 dy - C_1 \delta \int_{|y| \ge \eta} |\nabla f_d|^2 dy \\ &\ge (1 - C_1 \eta) \int_{B_\delta} |\nabla f_d|^2 dy + O\left(\varepsilon_d^{n-2} + e^{-\gamma/\sqrt{d}}\right) \end{aligned}$$

where C_1 does not depend on d. Choose η sufficiently small we obtain

$$\| \nabla f_d \|_g^2 \ge \left(\tau + \frac{1}{2}\rho'\right) \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1} \overline{f}_d^2 dy + O\left(\varepsilon_d^{n-2} + e^{-\gamma/\sqrt{d}}\right) \tag{2.45}$$

Next denote $\tilde{f}_d = f_d / \| \nabla f_d \|_{L^2(B_\delta)}$, $\hat{f}_d = \overline{f}_d / \| \nabla f_d \|_{L^2(B_\delta)}$. Define

$$\begin{aligned} \psi_i^d &= \frac{\partial}{\partial x_i} P U_{x_d,\varepsilon_d} \Big/ \left\| \nabla \frac{\partial}{\partial x_i} P U_{x_d,\varepsilon_d} \right\|_{L^2(B_\delta)}, \quad 1 \le i \le n \\ \psi_{n+1}^d &= \frac{\partial}{\partial \varepsilon} P U_{x_d,\varepsilon_d} \Big/ \left\| \nabla \frac{\partial}{\partial \varepsilon} P U_{x_d,\varepsilon_d} \right\|_{L^2(B_\delta)} \\ \psi_{n+2}^d &= P U_{x_d,\varepsilon_d} / \| \nabla P U_{x_d,\varepsilon_d} \|_{L^2(B_\delta)} \end{aligned}$$

Noting $PU_{x_d,\varepsilon_d} = U_{x_d,\varepsilon_d} - \varphi_{x_d,\varepsilon_d}$, from (2.22) we can obtain the estimates for ψ_j^d, $1 \le j \le n+2$, as in [7, Appendix B] and

$$\int_{B_\delta} \nabla \psi_i^d \nabla \psi_j^d dy = \delta_{ij} + O\left(\varepsilon_d^{\frac{n-2}{2}}\right), \quad 1 \le i, j \le n+2$$

Then we can write

$$\tilde{f}_d = \sum_{j=1}^{n+2} b_j^d \psi_j^d = \hat{f}_d, \quad b_j^d \in R^1 \tag{2.46}$$

For $1 \le i \le n+2$, from (2.46) we have

$$\int_{B_\delta} \nabla \tilde{f}_d \cdot \nabla \psi_i^d dy = \sum_{j=1}^{n+2} b_j^d \int_{B_\delta} \nabla \psi_j^d \nabla \psi_i^d dy$$

Since $0=\left\langle \nabla \tilde{f}_d, \nabla\psi_i\right\rangle_g = \int_{B_\delta} \nabla\tilde{f}_d\cdot\nabla\psi_i^d dy + \int_{B_\delta} O(|y|)|\nabla \tilde{f}_d||\nabla\psi_i^d|dy$ and $|x_d| = |x_d'| = O(\varepsilon_d)$,

$$\int_{B_\delta} \nabla\tilde{f}_d\cdot\nabla\psi_i^d dy = \int_{B_\delta} O(|y|)|\nabla \tilde{f}_d||\nabla\psi_i^d|dy$$
$$\le C\|\nabla \tilde{f}_d\|_{L^2(B_\delta)}\left(\int_{B_\delta}|y|^2|\nabla\psi_i^d|^2dy\right)^{1/2} = O(\varepsilon_d)$$

Hence we have $b_1^d = O(\varepsilon_d)$, $1\le i\le n+2$.

$$\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy = \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\left(\tilde{f}_d - \sum_{j=1}^{n+2} b_j^d\psi_j^d\right)^2 dy$$
$$= \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy - 2\sum_{j=1}^{n+2} b_j^d\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d\psi_j^d dy + \sum_{i,j=1}^{n+2} b_ib_j\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\psi_i^d\psi_j^d dy$$
$$\ge \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy + \sum_{j=1}^{n+2} O(\varepsilon_d)\left(\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy\right)^{\frac12}\left(\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\left(\psi_j^d\right)^2dy\right)^{\frac12}$$
$$+\sum_{i,j=1}^{n+2} O\left(\varepsilon_d^2\right)\left(\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\left(\psi_i^d\right)^2dy\right)^{1/2}\left(\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\left(\psi_j^d\right)^2dy\right)^{1/2}$$
$$= \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy + \sum_{j=1}^{n+2} O(\varepsilon_d)\left(\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau+1}dy\right)^{2/n}\|\nabla \tilde{f}_d\|_{L^2(B_\delta)}\cdot\|\nabla\psi_j^d\|_{L^2(B_\delta)}$$
$$+\sum_{i,j=1}^{n+2} O(\varepsilon_d^2)\left(\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau+1}dy\right)^{2/n}\|\nabla\psi_i^d\|_{L^2(B_\delta)}\|\nabla\psi_j^d\|_{L^2(B_\delta)}$$
$$= \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy + O(\varepsilon_d)$$
$$= \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy + \int_{B_\delta} O(|y|)U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy + O(\varepsilon_d)$$
$$= \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dy + O\left(\int_{B_\delta}|y|^{n/2}U_{x_d,\varepsilon_d}^{\tau+1}dy\right)^{2/n}\|\nabla \tilde{f}_d\|_{L^2(B_\delta)} + O(\varepsilon_d)$$
$$= \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\tilde{f}_d^2dg + O(\varepsilon_d)$$

since $|x_d| = O(\varepsilon_d)$. Therefore

$$\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}\overline{f}_d^2dy \ge \int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1}f_d^2dy + O(\varepsilon_d)\|\nabla f_d\|_{L^2(B_\delta)}^2 \tag{2.47}$$

From (2.45) (2.47) we obtain (2.44).

Now we need another characterization of the least-energy solution u_d. Define for $V\in W^{1,2}(\Omega)\backslash\{0\}$.

$$I_d(V) = \frac{\|\nabla V\|_{L^2(\Omega)}^2 + \frac{1}{d}\|V\|_{L^2(\Omega)}^2}{\|V\|_{L^{\tau+1}(\Omega)}^2}$$

Then u_d is the least-energy solution of (1.1) if and only if $V_d = d^{-\frac{n-2}{4}} u_d$ is the minimizer of I_d in $W^{1,2}(\Omega)\backslash\{0\}$, module a constant multiple. Define for $\tilde{V} \in H^*(B_\delta)$

$$\tilde{I}_d(\tilde{V}) = \frac{\|\nabla \tilde{V}\|_g^2 + \frac{1}{d}\|V\|_{L_g^2}^2}{\|\tilde{V}\|_{L_g^{\tau+1}}^2}$$

Then we have from (2.3)

$$I_d(V_d) = \left(\frac{1}{2}\right)^{2/n} \tilde{I}_d(\tilde{V}_d) + O\left(e^{-\gamma/\sqrt{d}}\right)$$

Lemma 2.4 *There exists $\overline{\rho} > 0$ independent of d such that as $d \to 0$,*

$$\begin{aligned}\tilde{I}_d(\tilde{V}_d) \geq & \tilde{I}_d(U_{x_d,\varepsilon_d}) + \overline{\rho}\left(\|\nabla f_d\|_g^2 + \frac{1}{d}\int_{B_\delta} f_d^2 dg\right) \\ & + O\left(\xi_2^{1/2}(\varepsilon_d) + \frac{1}{d}\xi_3(\varepsilon_d)\right)\|\nabla f_d\|_g + O\left(\varepsilon_d^{\frac{n-2}{2}} + e^{-\gamma/\sqrt{d}}\right)\end{aligned} \tag{2.48}$$

Proof Write $\psi_d(y) = f_d + h_d - c_d\varphi_{x_d,\varepsilon_d}$. Then

$$\tilde{V}_d = C_d U_{x_d,\varepsilon_d} + \psi_d$$

From Lemma 2.2, (2.22), (2.25) we have

$$\|\nabla \psi_d\|_g^2 + \frac{1}{d}\int_{B_\delta} \psi_d^2 dg = O\left(\xi_2^{1/2}(\varepsilon_d) + \frac{1}{d}\xi_3(\varepsilon_d) + e^{-\gamma/\sqrt{d}}\right)^2$$

$$\begin{aligned}\|\nabla \tilde{V}_d\|_g^2 + \frac{1}{d}\int_{B_\delta} \tilde{V}_d^2 dg =& C_d^2\|\nabla U_{x_d,\varepsilon_d}\|_g^2 + \frac{1}{d}C_d^2\int_{B_\delta} U_{x_d,\varepsilon_d}^2 dg \\ & + \|\nabla f_d\|_g^2 + \frac{1}{d}\int_{B_\delta} f_d^2 dg + \frac{2C_d}{d}\int_{B_\delta} U_{x_d,\varepsilon_d} f_d dg \\ & + O\left(\varepsilon_d^{\frac{n-2}{2}}\right) + O\left(e^{-\gamma/\sqrt{d}}\right) \\ =& C_d^2\left[\|\nabla U_{x_d,\varepsilon_d}\|_g^2 + \frac{1}{d}\int_{B_\delta} U_{x_d,\varepsilon_d}^2 dg\right] + \|\nabla f_d\|_g^2 + \frac{1}{d}\int_{B_\delta} f_d^2 dg \\ & + O\left(\frac{1}{d}\xi_3(\varepsilon_d)\right)\|\nabla f_d\|_g + O\left(\varepsilon_d^{\frac{n-2}{2}} + e^{-\gamma/\sqrt{d}}\right)\end{aligned}$$

$$\begin{aligned}\int_{B_\delta} \tilde{V}_d^{\tau+1} dg =& \int_{B_\delta} (C_d U_{x_d,\varepsilon_d} + \psi_d)^{\tau+1} dg \\ =& C_d^{\tau+1}\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau+1} dg + (\tau+1)C_d^\tau \int_{B_\delta} U_{x_d,\varepsilon_d}^\tau f_d dg \\ & + \frac{1}{2}\tau(\tau+1)C_d^{\tau-1}\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau-1} f_d^2 dg + O(\|\nabla f_d\|_g^\sigma) \\ & + O\left(\varepsilon_d^{\frac{n-2}{2}} + e^{-\gamma/\sqrt{d}}\right)\end{aligned}$$

Hence

$$
\begin{aligned}
\frac{1}{\|\tilde V_d\|^2_{L_g^{\tau+1}}} =&\frac{1}{C_d^2\|U_{x_d,\varepsilon_d}\|^2_{L_g^{\tau+1}}}\Big\{1+\frac{\tau(\tau+1)}{2C_d^2\|U_{x_d,\varepsilon_d}\|^{\tau+1}_{L_g^{\tau+1}}}\int_{B_\delta}U^{\tau-1}_{x_d,\varepsilon_d}f_d^2dg\\
&+O\left(\xi_2^{1/2}(\varepsilon_d)\right)\|\nabla f_d\|_g+O\left(\|\nabla f_d\|_g^\sigma\right)+O\left(\varepsilon_d^{\frac{n-2}{2}}+e^{-\gamma/\sqrt{d}}\right)\Big\}^{-2/(\tau+1)}\\
=&\frac{1}{C_d^2\|U_{x_d,\varepsilon_d}\|^2_{L_g^{\tau+1}}}\Big\{1-\frac{\tau}{C_d^2\|U_{x_d,\varepsilon_d}\|^{\tau+1}_{L_g^{\tau+1}}}\int_{B_\delta}U^{\tau-1}_{x_d,\varepsilon_d}f_d^2dg\\
&+O\left(\xi_2^{1/2}(\varepsilon_d)\right)\|\nabla f_d\|_g+O\left(\|\nabla f_d\|_g^\sigma\right)+O\left(\varepsilon_d^{\frac{n-2}{2}}+e^{-\gamma/\sqrt{d}}\right)\Big\}
\end{aligned}
$$

Then

$$
\begin{aligned}
\tilde I_d(\tilde V_d)=&\tilde I_d(U_{x_d,\varepsilon_d})+\frac{1}{C_d^2\|U_{x_d,\varepsilon_d}\|^2_{L_g^{\tau+1}}}\Big\{\|\nabla f_d\|_g^2+\frac1d\int_{B_\delta}f_d^2dg\\
&-\frac{\tau\tilde I_d(U_{x_d,\varepsilon_d})}{\|U_{x_d,\varepsilon_d}\|^{\tau-1}_{L_g^{\tau+1}}}\int_{B_\delta}U^{\tau-1}_{x_d,\varepsilon_d}f_d^2dg\Big\}+O\Big(\xi_2^{1/2}(\varepsilon_d)\\
&+\frac1d\xi_3(\varepsilon_d)\Big)\|\nabla f_d\|_g^\sigma+O\left(\|\nabla f_d\|_g^\sigma\right)+O\left(\varepsilon_d^{\frac{n-2}{2}}+e^{-\gamma/\sqrt{d}}\right)
\end{aligned}
$$

Since $\tilde I_d(U_{x_d,\varepsilon_d})\to S$ and $\|U_{x_d,\varepsilon_d}\|^{\tau-1}_{L_g^{\tau+1}}\to S$ as $d\to0$ by (2.36) (2.37), using Lemma 2.3 we obtain

$$
\begin{aligned}
\tilde I_d(\tilde V_d)\ge&\tilde I_d(U_{x_d,\varepsilon_d})+\bar\rho\left(\|\nabla f_d\|_g^2+\frac1d\int_{B_\delta}f_d^2dg\right)\\
&+O\left(\xi_2^{1/2}(\varepsilon_d)+\frac1d\xi_3(\varepsilon_d)\right)\|\nabla f_d\|_g+O\left(\varepsilon^{\frac{n-2}{2}}+e^{-\gamma/\sqrt{d}}\right)
\end{aligned}
$$

Now we estimate $\tilde I_d(U_{x_d,\varepsilon_d})$. Since $x_d=(x'_d,0)$ and $|x'_d|=O(\varepsilon_d)$, from (2.32), (2.36) we have

$$
\begin{aligned}
\|\nabla U_{x_d,\varepsilon_d}\|_g^2=&\int_{B_\delta(x_d)}|\nabla U_{0,\varepsilon_d}|^2\{1-(n-1)H(P_d)|y_n|+O(|x_d+y|^2)\}dy\\
&+2\sum_{j=1}^{n-1}\alpha_j(P_d)\int_{B_\delta(x_d)}|y_n|\left(\frac{\partial U_{0,\varepsilon_d}}{\partial y_j}\right)^2dy\\
=&S^{n/2}-a_nH(P_d)\xi_1(\varepsilon_d)+O(\xi_2(\varepsilon_d))
\end{aligned}
$$

From (2.33), (2.38)

$$
\begin{aligned}
\int_{B_\delta}U^2_{x_d,\varepsilon_d}dg=&\int_{B_\delta(x_d)}|U_{0,\varepsilon_d}|^2\{1-(n-1)H(P_d)|y_n|+O(|x_d+y|^2)\}dy\\
=&C_n\xi_2(\varepsilon_d)+o(\xi_2(\varepsilon_d))
\end{aligned}
$$

From (2.33), (2.37),

$$\int_{B_\delta} U_{x_d,\varepsilon_d}^{\tau+1} dg = \int_{B_\delta(x_d)} U_{0,\varepsilon_d}^{\tau+1}\{1-(n-1)H(P_d)|y_n| + O(|x_d+y|^2)\}dy$$

$$= S^{n/2} - b_n H(P_d)\varepsilon_d + O\left(\varepsilon_d^2\right)$$

$$\frac{1}{\|U_{x_d,\varepsilon_d}\|_{L_g^{\tau+1}}^2} = S^{-\frac{n-2}{2}}\left\{1 + \frac{(n-2)b_n}{nS^{n/2}} H(P_d)\varepsilon_d + O\left(\varepsilon_d^2\right)\right\}$$

Hence

$$\tilde I_d(U_{x_d,\varepsilon_d}) = S - p_n H(P_d)\xi_1(\varepsilon_d) + \frac{1}{d} q_n \xi_2(\varepsilon_d) + O(\xi_2(\varepsilon_d)) + o\left(\frac{1}{d}\xi_2(\varepsilon_d)\right) \tag{2.49}$$

where

$$p_n = \begin{cases} \dfrac{4}{n(n-3)} \pi^{\frac{n-1}{2}} [n(n-2)]^{\frac{n+3}{2}} \dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n+1)} S^{-\frac{n-2}{2}} & \text{if } n \geq 4 \\ a_n S^{-\frac{n-2}{2}} & \text{if } n = 3 \end{cases}$$

and $q_n = C_n S^{-\frac{n-2}{2}}$.

From (2.40) (2.48) (2.49) we have

$$\tilde I_d(\tilde V_d) \geq S - p_n H(P_d)\xi_1(\varepsilon_d) + \frac{1}{d} q_n \xi_2(\varepsilon_d) + o\left(\frac{1}{d}\xi_2(\varepsilon_d)\right)$$
$$+ O\left(\xi_2^{1/2}(\varepsilon_d) + \frac{1}{d}\xi_3(\varepsilon_d)\right)^2 + O\left(\varepsilon_d^{\frac{n-2}{2}} + e^{-\gamma/\sqrt{d}}\right) \tag{2.50}$$

Proof of Lemma 1.1 When $n > 4$, suppose (1.8) is not true. Then $\beta_d = o(\sqrt{d})$ as $d \to 0$. Since $\frac{\varepsilon_d}{\beta_d\sqrt{d}} \to 1$, we have

$$\left(\xi_2^{1/2}(\varepsilon_d) + \frac{1}{d}\xi_3(\varepsilon_d)\right)^2 = \left(\varepsilon_d + \frac{1}{d}\xi_3(\varepsilon_d)\right)^2$$
$$= O\left(\varepsilon_d^2 + \frac{1}{d^2}\xi_3^2(\varepsilon_d)\right)$$
$$= \begin{cases} O\left(\beta_d^2 d + \beta_d^4\right) & \text{if } n > 6 \\ O\left(\beta_d^2 d + \beta_d^4\left[\log\dfrac{1}{\beta_d\sqrt{d}}\right]\right)^{4/3} & \text{if } n = 6 \\ O\left[\beta_d^2 d + \dfrac{\beta_d^3}{\sqrt{d}}\right] & \text{if } n = 5 \end{cases}$$
$$= o\left(\beta_d^2\right)$$

Hence

$$2^{2/n} I_d(V_d) = \tilde I_d(\tilde V_d) + O\left(e^{-\gamma/\sqrt{d}}\right) \geq S - p_n H(P_d)\beta_d\sqrt{d} + q_n\beta_d^2$$

$$+ o\left(\beta_d\sqrt{d} + \beta_d^2\right) + O\left(e^{-\gamma/\sqrt{d}}\right) \tag{2.51}$$

Next choose $P_0 \in \partial\Omega$ so that $H(P_0) > 0$, and choose $\eta > 0$ sufficiently small. Let Ψ_0 be the diffeomorphism straightening a portion of boundary $\partial\Omega$ around P_0 and define

$$\overline{V}_d(x) = U_{0,\eta d}(\Psi_0(x))\psi(x)$$

where $\psi(x)$ is a cut-off function. Since V_d is the minimizer of I_d we have when $n > 4$

$$\begin{aligned} 2^{2/n} I_d(V_d) &\le 2^{2/n} I_d(\overline{V}_d) = \tilde{I}_d(U_{0,\eta d}) + O(d^2) \\ &= S - p_n H(P_0)\eta d + q_n \eta^2 d + o(d) \\ &= S - \eta(P_n H(P_0) - q_n \eta)d + o(d) \end{aligned} \tag{2.52}$$

Fix $\eta > 0$ so that $p_n H(P_0) \ge 2q_n\eta$. From (2.51) (2.52) we have

$$[p_n H(P_d) + o(1)]\beta_d\sqrt{d} - q_n\beta_d^2 \ge q_n\eta^2 d + o(d) + O\left(e^{-\gamma/\sqrt{d}}\right) \ge \frac{1}{2}q_n\eta^2 d$$

when d is sufficiently small. This is a contradiction because we assume $\beta_d = o(\sqrt{d})$. Now proof of Lemma 1.1 is complete.

Lemma 2.5 *Assume $n > 6$ and P_0 is a limit point of $\{P_d\}$ as $d \to 0$. Then*

$$H(P_0) = \max_{P\in\partial\Omega} H(P)$$

Proof Since $\dfrac{\varepsilon_d}{\beta_d\sqrt{d}} \to 1$ as $d \to 0$, when $n > 6$ we have

$$\left(\xi_2^{1/2}(\varepsilon_d) + \frac{1}{d}\xi_3(\varepsilon_d)\right)^2 = O\left(\beta_d^2 d + \beta_d^4\right)$$

From (2.50) we have

$$\tilde{I}_d(\tilde{V}_d) \ge S - p_n H(P_d)\beta_d\sqrt{d} + q_n\beta_d^2 + o\left(\beta_d\sqrt{d} + \beta_d^2\right) \tag{2.53}$$

since $e^{-\gamma/\sqrt{d}} = o\left(\beta_d^2\right)$ from Lemma 1.1. Assume $H(P_0) < \max\limits_{P\in\partial\Omega} H(P)$. Let $\overline{P} \in \partial\Omega$ with $H(\overline{P}) = \max\limits_{P\in\partial\Omega} H(P)$. Let $\overline{\Psi}$ be the diffeomorphism straightening a portion of boundary $\partial\Omega$ around $\overline{P}$ and define

$$\overline{V}_d(x) = U_{0,\beta_d\sqrt{d}}(\overline{\Psi}(x))\psi(x)$$

where $\psi(x)$ is a cut-off function. Then we have

$$\begin{aligned} 2^{2/n} I_d(V_d) &\le 2^{2/n} I_d(\overline{V}_d) = \tilde{I}_d(U_{0,\beta_d\sqrt{d}}) + O\left(\beta_d^2 d\right) \\ &= S - p_n H(\overline{P})\beta_d\sqrt{d} + q_n\beta_d^2 + O\left(\beta_d^2 d\right) \end{aligned}$$

Hence

$$\tilde{I}_d(\tilde{V}_d) \le S - p_n H(\overline{P})\beta_d\sqrt{d} + q_n\beta_d^2 + O\left(\beta_d^2 d\right) + O\left(e^{-\gamma/\sqrt{d}}\right) \tag{2.54}$$

From (2.53) (2.54) we have

$$p_n(H(\overline{P}) - H(P_d))\beta_d\sqrt{d} \le o\left(\beta_d^2 + \beta_d\sqrt{d}\right)$$

Hence $\sqrt{d} = o(\beta_d)$. Using this conclusion to (2.53) we have

$$\tilde{I}_d(\tilde{V}_d) \ge S + q_n\beta_d^2 + o\left(\beta_d^2\right) > S$$

when d is sufficiently small. Obviously $\tilde{I}_d(\tilde{V}_d) < S$, we reach a contradiction.

Corollary 2.1 *When $n > 6$, part (iii) of Theorem 1.1 holds for any smooth bounded domain Ω in R^n.*

Proof Fix $d > 0$ and introduce a function f_d defined by

$$f_d(\beta) = p_n H(P_d)\sqrt{d}\beta - q_n\beta^2$$

Set $\beta_d^* = (2q_n)^{-1}p_n H(P_d)\sqrt{d}$. Then $\max\limits_{\beta>0} f_d(\beta) = f_d(\beta_d^*) = (4q_n)^{-1}(P_n H(P_d))^2 d$. From (2.57) we have

$$\tilde{I}_d(\tilde{V}_d) \ge S - f_d(\beta_d^*) + o(d), \quad \text{as } d \to 0 \tag{2.55}$$

On the other hand, let Ψ_d be the diffeomorphism straightening a portion of boundary $\partial\Omega$ around P_d and define $\overline{V}_d(x) = U_{0,\beta_d^*\sqrt{d}}(\Psi_d(x))\psi(x)$, where $\psi(x)$ is a cut-off function. Then we have

$$2^{2/n}I_d(V_d) \le 2^{2/n}I_d(\overline{V}_d) = I_d(U_{0,\beta_d^*\sqrt{d}}) + o(d) = S - f_d(\beta_d^*) + o(d)$$

Therefore

$$\tilde{I}_d(\tilde{V}_d) \le S - f_d(\beta_d^*) + o(d), \quad \text{as } d \to 0 \tag{2.56}$$

From (2.55), (2.56) we see that

$$f_d(\beta_d) = f_d(\beta_d^*) + o(d)$$

and hence $\beta_d/\beta_d^* \to 1$ as $d \to 0$. Thus

$$\frac{\sqrt{d}}{\beta_d} \to \frac{2q_n}{p_n H(P_0)} \quad \text{since } H(P_d) \to H(P_0)$$

Remark 2.1 When $n > 6$, from Corollary 2.1 we compute

$$I_d(V_d) = \left(\frac{1}{2}\right)^{2/n} S\left[1 - \delta_n H(P_0)^2 d\right] + o(d), \quad \text{as } d \to 0 \tag{2.57}$$

where

$$\delta_n = \frac{(n-2)^3(n-4)}{\pi n(n-1)(n-3)^2}\left[\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right]^2 \tag{2.58}$$

Note that

$$I_d(V_d) = \inf\left\{\frac{\|\nabla V\|^2_{L^2(\Omega)} + \frac{1}{d}\|V\|^2_{L^2(\Omega)}}{\|V\|^2_{L^{\tau+1}(\Omega)}} : V \in W^{1.2}(\Omega)\backslash\{0\}\right\}$$

$$= \inf\left\{\frac{\|\nabla \varphi\|^2_{L^2(\Omega/\sqrt{d})} + \|\varphi\|^2_{L^2(\Omega/\sqrt{d})}}{\|\varphi\|^2_{L^{\tau+1}(\Omega/\sqrt{d})}} : \varphi \in W^{1.2}(\Omega/\sqrt{d})\backslash\{0\}\right\}$$

(2.57) gives the asymptotic estimate of the best Sobolev constant for the imbedding $W^{1.2}\left[\frac{\Omega}{\sqrt{d}}\right] \to L^{\tau+1}\left[\frac{\Omega}{\sqrt{d}}\right]$, as the domain $\frac{\Omega}{\sqrt{d}}$ expands to infinity.

3. Uniform Estimates

This section is devoted to the proof of Theorem 1.1 (i), under the assumption that Ω is strictly convex. Recall that $P_d \in \partial\Omega$ when d is sufficiently small. Without loss of generality we may assume $P_d \to P_0$ as $d \to 0$, $P_0 \in \partial\Omega$. From the convexity condition, the $n-1$ principal curvatures $\alpha_j(P_0)$, $1 \le j \le n-1$, are all positive. Hence we may assume $\alpha_j(P_d) \ge \alpha_0 > 0$ for all small d and $j = 1, \cdots, n-1$.

When $\Omega \ne B_1(0)$, by rescaling the coordinate system we may assume $\alpha_0 > 1$. Then through translation and rotation of the coordinate system we assume that $P_d = (0, \cdots, 0, -1)$, and the inner normal to $\partial\Omega$ at P_d points in the direction of the positive x_n-axis. Now we denote the new domain by Ω_d. Choose $\delta > 0$ such that around the point $P_d = (0, \cdots, 0, -1)$, $\partial\Omega_d$ can be represented by

$$x_n = \psi_d(x') = -1 + \frac{1}{2}\sum_{j=1}^{n-1}\alpha_j(P_d)x_j^2 + O(|x'|^3), \quad |x'| < 2\delta \tag{3.1}$$

Now we have

$$\psi_d(x') \ge -1 + \frac{\alpha_0}{4}|x'|^2, \quad |x'| < 2\delta \tag{3.2}$$

and $\Omega_d \cap B_{2\delta}(P_d) \subset B_1(0)$. Observe that ψ_d varies smoothly in d, and δ can be chosen independent of d.

Next we define a map $F : B_1 \to R^n_+$ by

$$F(x', x_n) = \left(\frac{4x'}{(1+x_n)^2 + |x'|^2}, \frac{2(1-|x|^2)}{(1+x_n)^2 + |x'|^2}\right) \tag{3.3}$$

$F(B_1) = R^n_+$, $F(P_d) = \infty$, $F(\partial B_1) = \partial B^n_+ \cup \{\infty\}$. The inverse map $F^{-1} : R^n_+ \to B_1$ is given by

$$F^{-1}(y', y_n) = \left(\frac{4y'}{(2+y_n)^2 + |y'|^2}, \frac{4-|y|^2}{(2+y_n)^2 + |y'|^2}\right) \tag{3.4}$$

Denote $\Omega_d^* = F(\Omega_d \cap B_{2\delta}(P_d))$, $\Gamma_d = F(\partial\Omega_d \cap B_{2\delta}(P_d)\backslash\{P_d\})$. Let V_d be defined in (2.1) and define for $y \in \Omega_d^* \cup \Gamma_d$

$$\overline{V}_d(y) = 4^{\frac{n-2}{2}}\left[(2+y_n)^2 + |y'|^2\right]^{-\frac{n-2}{2}} V_d(F^{-1}y) \tag{3.5}$$

Computation shows that

$$\triangledown \overline{V}_d - 16d^{-1}\left[(2+y_n)^2 + |y'|^2\right]^{-2} \overline{V}_d + \overline{V}_d^{\tau} = 0 \quad \text{in } \Omega_d^* \tag{3.6}$$

$$\frac{\partial \tilde{V}_d}{\partial v^*} + \frac{n-2}{(2+y_n)^2 + |y'|^2} b_d(y)\overline{V}_d = 0 \quad \text{on } \Gamma_d \tag{3.7}$$

where

$$b_d(y) = (2+y_n)v_n^* + \sum_{j=1}^{n-1} y_j v_j^* \tag{3.8}$$

and $v^*(y) = (v_1^*, \cdots, v_n^*)$ is the unit outer normal to Γ_d. In fact for $y = F(x)$,

$$(1+x_n)^2 + |x'|^2 = 16\left[(2+y_n)^2 + |y'|^2\right]^{-1} \tag{3.9}$$

Let $u(y) = 4^{-\frac{n-2}{2}}\left[(2+y_n)^2 + |y'|^2\right]^{\frac{n-2}{2}}$. Then we can write $V_d(x) = u(y)\overline{V}_d(y)$ with $y = F(x)$.

$$\frac{\partial V_d}{\partial x_i}(x) = \sum_{k=1}^{n}\left[u(y)\frac{\partial \overline{V}_d}{\partial y_k}\frac{\partial F_k}{\partial x_i} + \overline{V}_d \frac{\partial u(y)}{\partial y_k}\frac{\partial F_k}{\partial x_i}\right] \tag{3.10}$$

$$\begin{aligned}\Delta V_d(x) =& u(y)\sum_{k,l=1}^{n} \triangledown F_k(x) \triangledown F_l(x) \frac{\partial^2 \overline{V}_d}{\partial y_k \partial y_l} \\ &+ u(y)\sum_{k=1}^{n}\Delta F_k(x)\frac{\partial \overline{V}_d}{\partial y_k} + 2\sum_{k,l=1}^{n} \triangledown F_k(x)\cdot\triangledown F_l(x)\frac{\partial u}{\partial y_k}\frac{\partial \overline{V}_d}{\partial y_l} \\ &+ \overline{V}_d(y)\left[\sum_{k=1}^{n}\Delta F_k(x)\frac{\partial u}{\partial y_k} + \sum_{k,l=1}^{n}\triangledown F_k(x)\cdot\triangledown F_l(x)\frac{\partial^2 u}{\partial y_k \partial y_l}\right]\end{aligned} \tag{3.11}$$

From (3.3), (3.9)

$$\triangledown F_k(x)\cdot\triangledown F_l(x) = \frac{16\delta_{kl}}{\left[(1+x_n)^2 + |x'|^2\right]^2}, \quad 1 \le k, l \le n \tag{3.12}$$

$$\Delta F_k(x) = -\frac{8(n-2)x_k}{\left[(1+x_n)^2 + |x'|^2\right]^2}, \quad 1 \le k \le n-1 \tag{3.13}$$

$$\Delta F_n(x) = -\frac{8(n-2)(1+x_n)}{\left[(1+x_n)^2 + |x'|^2\right]^2} \tag{3.14}$$

Plugging (3.12), (3.13) and (3.14) into (3.11) we find

$$\Delta V_d(x) = \frac{16}{\left[(1+x_n)^2 + |x'|^2\right]^2} u(y)\sum_{k=1}^{n}\frac{\partial^2 \overline{V}_d}{\partial y_k^2}$$

Now from (2.2) we have

$$\frac{16}{[(1+x_n)^2+|x'|^2]^2}u(y)\Delta\overline{V}_d(y)-\frac{1}{d}u(y)\overline{V}_d+u(y)^\tau\overline{V}_d^\tau=0$$

Since $u(y)^{\tau-1}=\frac{1}{16}\left[(2+y_n)^2+|y'|^2\right]^2=\frac{16}{[(1+x_n)^2+|x'|^2]^2}$, we obtain (3.6).

Let $v(x)$ be the unit outer normal to $\partial\Omega_d\cap B_{2\delta}(P_d)$ and $v^*(y)$ be the unit outer normal to Γ_d with $y=F(x)$. Then

$$v_i(x)=\frac{1}{4}\Big[(1+x_n)^2+|x'|^2\Big]\sum_{k=1}^n\frac{\partial F_k(x)}{\partial x_i}v_k^*(y)$$

$$v(x)\cdot\nabla F_k(x)=\frac{1}{4}\left[(1+x_n)^2+|x'|^2\right]\sum_{l=1}^n\nabla F_k(x)\cdot\nabla F_l(x)v_l^*(y)$$

$$=\frac{4}{[(1+x_n)^2+|x'|^2]}v_k^*(y)$$

Since

$$0=\frac{\partial V_d}{\partial v}=\sum_{k=1}^n\left[u(y)\frac{\partial\overline{V}_d}{\partial y_k}+\overline{V}_d(y)\frac{\partial u(y)}{\partial y_k}\right](v\cdot\nabla F_k)$$

$$=\frac{4}{(1+x_n)^2+|x'|^2}\left[u(y)\frac{\partial\overline{V}_d}{\partial v^*}+\overline{V}_d(y)\frac{\partial u(y)}{\partial v^*}\right]$$

we have

$$\frac{\partial\overline{V}_d}{\partial v^*}+\frac{1}{u(y)}\frac{\partial u(y)}{\partial v^*}\overline{V}_d=0\quad\text{on }\Gamma_d$$

Since

$$\frac{1}{u(y)}\frac{\partial u(y)}{\partial v^*}=\frac{n-2}{(2+y_n)^2+|y'|^2}\left[(2+y_n)v_n^*+\sum_{k=1}^{n-1}y_kv_k^*\right]$$

we obtain (3.7).

Next we estimate $b_d(y)$. Computation shows

$$v_i^*(y)=\frac{1}{4}\left[(2+y_n)^2+|y'|^2\right]\sum_{k=1}^n v_k(x)\frac{\partial F_k^{-1}}{\partial y_i}(y),\quad 1\le i\le n$$

where $F^{-1}=\left(F_1^{-1},\cdots,F_n^{-1}\right)$ and $y=F(x)$. Hence

$$b_d(y)=(2+y_n)v_n^*+\sum_{i=1}^{n-1}y_iv_i^*$$

$$=\frac{1}{4}\left[(2+y_n)^2+|y'|^2\right]\sum_{k=1}^n v_k(x)\left[(2+y_n)\frac{\partial F_k^{-1}(y)}{\partial y_n}+\sum_{i=1}^{n-1}y_i\frac{\partial F_k^{-1}}{\partial y_i}(y)\right]$$

$$=-(2+y_n)v_n(x)-\sum_{i=1}^{n-1}y_kv_i(x)$$

$$= -\frac{4}{(1+x_n)^2} + |x'|^2 \left[(1+x_n)v_n(x) + \sum_{k=1}^{n-1} x_k v_k(x) \right]$$

Since at $P_d = (0,\cdots,0,-1), v(P_d) = (0,\cdots,0,-1)$, we can choose $\delta > 0$ sufficiently small (and independent of d) so that by (3.1)

$$(1+x_n)v_n(x) + \sum_{k=1}^{n-1} x_k v_k(x) = O\left(|x'|^2 + (1+x_n)^2\right)$$

Hence

$$|b_d(y)| \le C_* \quad \text{uniformly on } \Gamma_d \tag{3.15}$$

By computation

$$\det DF(x) = -\left[\frac{4}{(1+x_n)^2 + |x'|^2}\right]^n = -\frac{1}{4^n}\left[(2+y_n)^2 + |y'|^2\right]^n \tag{3.16}$$

Since F reverses the orientation, from (3.5) (3.16)

$$\int_{\Omega_d \cap B_{2\delta}(P_d)} V_d^{\tau+1}(x)dx = \int_{\Omega_d^*} \overline{V}_d^{\tau+1}(y)dy \tag{3.17}$$

From [3; (3.34)], for any $\varepsilon > 0$ there exist $d_0 > 0$ and $R > 0$ such that when $0 < d < d_0$

$$\int_{\Omega_d \setminus B_{\beta_d \sqrt{d} R}(P_d)} V_d^{\tau+1}(x)dx < \varepsilon \tag{3.18}$$

Hence

$$\int_{\Omega_d^* \cap B_{\frac{4}{\beta_d \sqrt{d} R}}(Q_0)} \overline{V}_d^{\tau+1}(y)dy < \varepsilon \tag{3.19}$$

where $Q_0 = (0,\cdots,0,-2)$.

Now we denote $W_d(z) = \frac{1}{\alpha_d} u_d(P_d + \beta_d \sqrt{d} z)$. We shall prove

$$W_d(z) \le CU(z), \quad z \in \frac{\Omega_d - P_d}{\beta_d \sqrt{d}} \tag{3.20}$$

where U is given in (1.2). Recalling (2.3), Lemma 1.1, and noting

$$\begin{aligned} W_d(z) &= \left(\beta_d \sqrt{d}\right)^{\frac{n-2}{2}} V_d\left(P_d + \beta_d \sqrt{d}\, z\right) \\ &= \left(\beta_d \sqrt{d}\right)^{\frac{n-2}{2}} V_d\left(\beta_d \sqrt{d} z', \beta_d \sqrt{d} z_n - 1\right) \\ &= \left(\beta_d \sqrt{d}\right)^{\frac{n-2}{2}} 4^{\frac{n-2}{2}} \left|\beta_d \sqrt{d} z\right|^{-(n-2)} \overline{V}_d\left(F\left(P_d + \beta_d \sqrt{d} z\right)\right) \\ &= 4^{\frac{n-2}{2}} \left(\beta_d \sqrt{d}\right)^{-\frac{n-2}{2}} |z|^{-(n-2)} \overline{V}_d(y) \end{aligned}$$

where $y = F(P_d + \beta_d \sqrt{d} z)$, we only need to prove that, there exist $R > 0$ and $C > 0$, both independent of d, such that for all $|z| \ge R$ and $d > 0$ sufficiently small,

$$\overline{V}_d\left(F\left(P_d + \beta_d \sqrt{d} z\right)\right) \le C\left(\beta_d \sqrt{d}\right)^{\frac{n-2}{2}} \tag{3.21}$$

that is,

$$\overline{V}_d(y) \le C\varepsilon_d^{\frac{n-2}{2}} \quad \text{for all } y \in \Omega_d^* \cap B_{\frac{4}{\varepsilon_d R}}(Q_0) \tag{3.22}$$

where $\varepsilon_d = \beta_d\sqrt{d}$.

In the following we set

$$\overline{W}_d(z) = \varepsilon_d^{-\frac{n-2}{2}}\overline{V}_d\left(\frac{z}{\varepsilon_d}\right) \tag{3.23}$$

and $\tilde{\Omega}_d = \varepsilon_d\Omega_d^*$, $\tilde{\Gamma}_d = \varepsilon_d\Gamma_d$. We only need to prove that, there exists $\mu > 0$ such that

$$\overline{W}_d(z) \le C \quad \text{for all } z \in \tilde{\Omega}_d \cap B_\mu(0) \tag{3.24}$$

Obviously $\overline{W}_d$ satisfies

$$\Delta\overline{W}_d - \frac{16\beta_d^2}{[(2\varepsilon_d + z_n)^2 + |z'|^2]^2}\overline{W}_d + \overline{W}_d^{\tau} = 0 \quad \text{in } \tilde{\Omega}_d \tag{3.25}$$

$$\frac{\partial\overline{W}_d}{\partial v} + \frac{(n-2)\varepsilon_d}{(2\varepsilon_d + z_n)^2 + |z'|^2}b_d\left(\frac{z}{\varepsilon_d}\right)\overline{W}_d = 0 \quad \text{on } \tilde{\Gamma}_d \tag{3.26}$$

where v is the unit outer normal to $\tilde{\Gamma}_d$ and by (3.15)

$$\left|b_d\left(\frac{z}{\varepsilon_d}\right)\right| \le C_* \quad \text{on } \tilde{\Gamma}_d \tag{3.27}$$

Denote $\tilde{\Gamma}_d^A = \left\{z \in \tilde{\Gamma}_d : (2\varepsilon_d + z_n)^2 + |z'|^2 \le A\varepsilon_d\right\}$. When $z \notin \tilde{\Gamma}_d^A$, $\frac{(n-2)\varepsilon_d}{(2\varepsilon_d + z_n)^2 + |z'|^2} \cdot b_d\left(\frac{z}{\varepsilon_d}\right) \le \frac{(n-2)C_*}{A}$. When $z \in \tilde{\Gamma}_d^A$ and $y = \frac{z}{\varepsilon_d}$, $(2 + y_n)^2 + |y'|^2 \le \frac{A}{\varepsilon_d}$ and from (3.9), $|x - P_d^2| = (1 + x_n)^2 + |x'|^2 \ge \frac{16\varepsilon_d}{A} = \frac{16\beta_d\sqrt{d}}{A} \ge \frac{16d}{AC_0}$ by Lemma 1.1, where $C_0 > 0$ does not depend on d, and $x = F^{-1}y$. From Remark 1.1 we can find $d_0(A) > 0$ such that for $0 < d < d_0(A)$, $u_d(x) \le e^{-\gamma_0|x-P_d|/\sqrt{d}}$. Hence

$$\begin{aligned}
\overline{V}_d(y) &= 4^{\frac{n-2}{2}}\left[(2 + y_n)^2 + |y'|^2\right]^{-\frac{n-2}{2}}V_d(x) \\
&\le 4^{\frac{n-2}{2}}d^{-\frac{n-2}{2}}\left[(2 + y_n)^2 + |y'|^2\right]^{-\frac{n-2}{2}}e^{-4\gamma_0/\sqrt{d}}\left[(2 + y_n)^2 + |y'|^2\right]^{-1/2} \\
&\le d^{-\frac{n-2}{2}}e^{-2\gamma_0/\sqrt{d}}
\end{aligned}$$

since $y_n > 0$. By Lemma 1.1, when d is sufficiently small,

$$\begin{aligned}
\overline{W}_d(z) &= \varepsilon_d^{-\frac{n-2}{2}}\overline{V}_d\left(\frac{z}{\varepsilon_d}\right) \le \left(\beta_d\sqrt{d}\right)^{-\frac{n-2}{2}}d^{-\frac{n-2}{4}}e^{-2\gamma_0/\sqrt{d}} \\
&= (d\beta_d)^{-(n-2)/2}e^{-2\gamma_0/\sqrt{d}} \le C_0^{-(n-2)/2}d^{-(n-2)/2}e^{-2\gamma_0/\sqrt{d}} \le e^{-(3/2)\gamma_0/\sqrt{d}}
\end{aligned}$$

Hence on $\tilde{\Gamma}_d^A$ we have, since $z_n > 0$,

$$\frac{(n-2)\varepsilon_d}{(2\varepsilon_d + z_n)^2 + |z'|^2}\left|b_d\left(\frac{z}{\varepsilon_d}\right)\right|\overline{W}_d(z) \le \frac{(n-2)C_*}{4\varepsilon_d}e^{-(3/2)\gamma_0/\sqrt{d}}$$

$$\leq \frac{(n-2)C_*}{4C_0} d^{-1} e^{-(3/2)\gamma_0/\sqrt{d}} \leq e^{-\gamma_0/\sqrt{d}} \tag{3.28}$$

Now we prove (3.24) by using the arguments in [3; Lemma 2.13]. Before starting the proof we make some remarks.

First, we note that, there exists $\mu_0 > 0$ independent of d so that $\partial\tilde{\Omega}_d \cap B_{\mu_0}(0) \subset \tilde{\Gamma}_d$.

Second, we observe that, $\{\tilde{\Omega}_d \cap B_{\mu_0}(0), 0 < d < d_0\}$ satisfies the uniform cone condition. In fact, from (3.1), (3.4), (3.9), Γ_d can be represented by

$$\frac{4-|y|^2}{(2+y_n)^2+|y'|^2} = -1 + \frac{1}{2}\sum_{j=1}^{n-1} \frac{16\alpha_j y_j^2}{[(2+y_n)^2+|y'|^2]^2} + g(y)$$

where $g(y) = O\left(\frac{1}{[(2+y_n)^2+|y'|^2]^{3/2}}\right)$. Hence

$$(2+y_n)^3 + |y'|^2(2+y_n) = 2\sum_{j=1}^{n-1} \alpha_j y_j^2 + \overline{g}(y)$$

where $\overline{g}(y) = O\left([(2+y_n)^2+|y'|^2]^{1/2}\right)$. Letting $y = \frac{z}{\varepsilon_d}$ in the above equation we obtain the representation $z_n = z_n(z')$ of $\tilde{\Gamma}_d \cap B_{\mu_0}(0)$:

$$(2\varepsilon_d + z_n)^3 + |z'|^2(2\varepsilon_d + z_n) = 2\varepsilon_d \sum_{j=1}^{n-1} \alpha_j z_j^2 + f_{\varepsilon_d}(z) \tag{3.29}$$

where $f_{\varepsilon_d}(z) = \varepsilon_d^3 \overline{g}\left(\frac{z}{\varepsilon_d}\right) = O\left(\varepsilon_d^2 [(2\varepsilon_d + z_n)^2 + |z'|^2]^{1/2}\right)$. From (3.29),

$$2\varepsilon_d + z_n \leq 2\varepsilon_d \sum_{j=1}^{n-1} \alpha_j \frac{z_j^2}{|z'|^2} + O\left(\varepsilon_d^2\right)$$

for $z \in \tilde{\Gamma}_d \cap B_{\mu_0}(0)$. Hence

$$0 \leq z_n \leq C\varepsilon_d \quad \text{on} \quad \tilde{\Gamma}_d \cap B_{\mu_0}(0) \tag{3.30}$$

For $1 \leq i \leq n-1$, $z \in \tilde{\Gamma}_d \cap B_{\mu_0}(0)$,

$$\frac{\partial z_n}{\partial z_i} = \frac{4\varepsilon_d(\alpha_i - 1)z_i - 2z_i z_n}{3(2\varepsilon_d + z_n)^2 + |z'|^2} + O(\varepsilon_d) \tag{3.31}$$

(3.30) (3.31) imply that

$$\left|\frac{\partial z_n}{\partial z_i}\right| \leq C', \ 1 \leq i \leq n-1, \ z \in \tilde{\Gamma}_d \cap B_{\mu_0}(0), \ 0 < d < d_0 \tag{3.32}$$

Thus we obtain the second observation above. (3.30) also implies that $\tilde{\Omega}_d \cap B_{\mu_0}(0)$ converges to $B^+_{\mu_0}$ in C^α, $0 < \alpha < 1$.

Now we claim that we can use Sobolev imbedding theorems in $\tilde{\Omega}_d \cap B_{\mu_0}(0)$, with the Sobolev constants independent of d. Moreover, when μ_0 is sufficiently small,

$$\int_{\tilde{\Gamma}_d \cap B_{\mu_0}(0)} dS \leq 1, \quad \int_{\tilde{\Omega}_d \cap B_{\mu_0}(0)} dz \leq 1 \tag{3.33}$$

In the following we let η denote a smooth cut-off function and $b > 1$ denote an arbitrary number, both to be specified later. We use C to denote a general positive constant which is independent of d, but varies from line to line. Multiplying (3.25) by $\eta^2 \overline{W}_d^b$, where supp $\eta \subset B_{\mu_0}(0)$, we compute, by Green's identity and (3.28)

$$\begin{aligned}\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{\tau+b} dy &\geq -\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^b \Delta \overline{W}_d dy \\ &= \int_{\tilde{\Omega}_d} \nabla(\eta^2 \overline{W}_d^b) \cdot \nabla \overline{W}_d dy - \int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^b \frac{\partial \overline{W}_d}{\partial v} dS \\ &\geq \frac{b}{2} \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b-1} |\nabla \overline{W}_d|^2 dy - \frac{2}{b} \int_{\tilde{\Omega}_d} |\nabla \eta|^2 \overline{W}_d^{b+1} dy \\ &\quad - \frac{(n-2)C_*}{A} \int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^{b+1} dS - e^{-\gamma_0/\sqrt{d}} \int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS\end{aligned}$$

Hence

$$\begin{aligned}\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b-1} |\nabla \overline{W}_d|^2 dy &\leq \frac{2}{b} \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{\tau+b} dy + \frac{4}{b^2} \int_{\tilde{\Omega}_d} \overline{W}_d^{b+1} |\nabla \eta|^2 dy \\ &\quad + \frac{2(n-2)C_*}{Ab} \int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^{b+1} dS + \frac{2}{b} e^{-\gamma_0/\sqrt{d}} \int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS\end{aligned} \tag{3.34}$$

By Sobolev inequality

$$\begin{aligned}\lambda_0 \left\| \eta \overline{W}_d^{\frac{b+1}{2}} \right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)} &\leq \int_{\tilde{\Omega}_d} \left| \nabla \left(\eta \overline{W}_d^{\frac{b+1}{2}} \right) \right|^2 dy + \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b+1} dy \\ &\leq \frac{1}{2}(b+1)^2 \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b-1} |\nabla \overline{W}_d|^2 dy + 2\int_{\tilde{\Omega}_d} |\nabla \eta|^2 \overline{W}_d^{b+1} dy + \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b+1} dy\end{aligned} \tag{3.35}$$

where λ_0 is independent of d, see the remark above. Let

$$k = \frac{n-1}{n-2}$$

Then

$$\begin{aligned}\lambda_1 \left\| \eta \overline{W}_d^{\frac{b+1}{2}} \right\|^2_{L^{2k}(\tilde{\Gamma}_d \cap B_{\mu_0})} &\leq \int_{\tilde{\Omega}_d} \left| \nabla \left(\eta \overline{W}_d^{\frac{b+1}{2}} \right) \right|^2 dy + \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b+1} dy \\ &\leq \frac{1}{2}(b+1)^2 \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b-1} |\nabla \overline{W}_d|^2 dy + 2\int_{\tilde{\Omega}_d} |\nabla \eta|^2 \overline{W}_d^{b+1} dy + \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b+1} dy\end{aligned} \tag{3.36}$$

and

$$\lambda_2 \left\| \eta \overline{W}_d^{\frac{b+1}{2}} \right\|^2_{L^2(\tilde{\Gamma}_d \cap B_{\mu_0})} \leq \int_{\tilde{\Omega}_d} \left| \nabla \left(\eta \overline{W}_d^{\frac{b+1}{2}} \right) \right|^2 dy + \int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b+1} dy$$

$$\leq \frac{1}{2}(b+1)^2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_2^{b-1}|\nabla\overline{W}_d|^2dy+2\int_{\tilde{\Omega}_d}|\nabla\eta|^2\overline{W}_2^{b+1}dy+\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{b+1}dy \quad (3.37)$$

where λ_1, λ_2 are independent of d.

Fix $A=\dfrac{2(n-2)C_*(\tau+1)^2}{\lambda_2\tau}$. Then for $0<d<d(A)$, from (3.28), (3.33) and from (3.34), (3.37) with $b=\tau$,

$$\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau-1}|\nabla\overline{W}_d|^2dy\leq\frac{2}{\tau}\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{2\tau}dy+\frac{4}{\tau^2}\int_{\tilde{\Omega}_d}|\nabla\eta|^2\overline{W}_d^{\tau+1}dy$$
$$+\frac{1}{2}\left[\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau-1}|\nabla\overline{W}_d|^2dy+2\int_{\tilde{\Omega}_d}|\nabla\eta|^2\overline{W}_d^{\tau+1}dy\right.$$
$$\left.+2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau+1}dy\right]+e^{-(\tau+1)\gamma_0/\sqrt{d}}$$

Hence

$$\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau-1}|\nabla\overline{W}_d|^2dy\leq\frac{4}{\tau}\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{2\tau}dy+\left(\frac{8}{\tau^2}+2\right)\int_{\tilde{\Omega}_d}|\nabla\eta|^2\overline{W}_d^{\tau+1}dy$$
$$+2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau+1}dy+2e^{-(\tau+1)\gamma_0/\sqrt{d}}$$

Then from (3.35)

$$\lambda_0\left\|\eta\overline{W}_d^{\frac{\tau+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)}\leq 4(\tau+1)^2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau+1}dy+20\int_{\tilde{\Omega}_d}|\nabla\eta|^2\overline{W}_d^{\tau+1}dy$$
$$+2(\tau+1)^2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau+1}dy+(\tau+1)^2e^{-(\tau+1)\gamma_0/\sqrt{d}} \quad (3.38)$$

From (3.19) we can choose $\eta_0>0$ such that for all $0<d<d_1$,

$$\int_{\tilde{\Omega}_d\cap B_{\eta_0}(0)}\overline{W}_d^{\tau+1}dy\leq\left[\frac{\lambda_0}{8(\tau+1)^2}\right]^{n/2} \quad (3.39)$$

By Hölder inequality

$$4(\tau+1)^2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{2\tau}dy\leq 4(\tau+1)^2\left[\int_{\tilde{\Omega}_d\cap B_{\eta_0}(0)}\overline{W}_d^{\tau+1}dy\right]^{\frac{2}{n}}\left\|\eta\overline{W}_d^{\frac{\tau+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)}$$
$$\leq\frac{\lambda_0}{2}\left\|\eta\overline{W}_d^{\frac{\tau+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)}$$

Then by (2.38)

$$\lambda_0\left\|\eta\overline{W}_d^{\frac{\tau+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)}\leq 4(\tau+1)^2\int_{\tilde{\Omega}_d}\eta^2\overline{W}_d^{\tau+1}dy$$
$$+40\int_{\tilde{\Omega}_d}|\nabla\eta|^2\overline{W}_d^{\tau+1}dy+2(\tau+1)^2e^{-(\tau+1)\gamma_0/\sqrt{d}} \quad (3.40)$$

Fix $0 < r < \frac{1}{4}\mu_0$. Let η be a smooth cut-off function supported in $B_{4r}(0)$, $\eta \equiv 1$ in $B_{2r}(0)$, $0 \le \eta \le 1$ and $|\nabla \eta| \le \frac{1}{r}$. From (3.39) (3.40)

$$\left\|\overline{W}_d^{\frac{\tau+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d \cap B_{2r})} \le C\left(1+\frac{1}{r^2}\right)\int_{\tilde{\Omega}_d \cap B_{\mu_0}(0)} \overline{W}_d^{\tau+1} dy + Ce^{-(\tau+1)\gamma_0/\sqrt{d}} \le C \quad (3.41)$$

where C only depends on n, λ_0 and r. The same reasoning gives

$$\left\|\overline{W}_d^{\frac{\tau+1}{2}}\right\|^2_{L(\Gamma_d \cap B_{2r})} \le C \quad (3.41)'$$

In the following we shall use an iteration process to establish (3.24). Let η be a smooth cut-off function supported in $B_{2r}(0)$ which will be specified later in each step of the iteration process. First we use Hölder inequality to obtain

$$\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{\tau+b} dy \le \left(\int_{\tilde{\Omega}_d \cap B_{2r}} \overline{W}_d^{(\tau-1)q/2} dy\right)^{2/q} \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{2q/(q-2)}(\tilde{\Omega}_d)}$$

where $q = \frac{n^2}{n-2}$. Since $q > n, 2 < \frac{2q}{q-2} < \frac{2n}{n-2} = \tau + 1$. For any $\varepsilon > 0$,

$$\left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|_{L^{2q/(q-2)}(\tilde{\Omega}_d)} \le \varepsilon \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|_{L^{2q/(q-2)}(\tilde{\Omega}_d)} + \varepsilon^{-\sigma}\left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|_{L^2(\tilde{\Omega}_d)}$$

where $\sigma = \frac{n-2}{2}$. By (3.41)

$$\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{\tau+b} dy \le C\varepsilon^2 \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)} + C\varepsilon^{-(n-2)}\left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^2(\tilde{\Omega}_d)} \quad (3.42)$$

From (3.34) (3.35) (3.42)

$$\begin{aligned}\lambda_0 \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)} &\le 2(b+1)C\varepsilon^2 \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)} + 10\int_{\tilde{\Omega}_d} |\nabla \eta|^2 \overline{W}_d^{b+1} dy \\ &+ 3(b+1)C\varepsilon^{-(n-2)}\left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^2(\tilde{\Omega}_d)} + \frac{1}{A}2(n-2)C_*(b+1)\int_{\tilde{\Gamma}_d}^2 \overline{W}_d^{b+1} dS \\ &+ 2(b+1)e^{-\gamma_0/\sqrt{d}}\int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS \end{aligned} \quad (3.43)$$

Choose $\varepsilon = \left[\frac{\lambda_0}{4C(b+1)}\right]^{1/2}$. Then we have

$$\begin{aligned}\left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)} &\le C(b+1)^{n/2}\left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^2(\tilde{\Omega}_d)} + C\int_{\tilde{\Omega}_d} |\nabla \eta|^2 \overline{W}_d^{b+1} dy \\ &+ C(b+1)\int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^{b+1} dS + C(b+1)e^{-\gamma_0/\sqrt{d}}\int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS \end{aligned} \quad (3.44)$$

Also by Hölder inequality and (3.41)

$$\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{\tau+b} dy \leq \left[\int_{\tilde{\Omega}_d \cap B_{2r}} \overline{W}_d^{\tau+1} dy\right]^{2/n} \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)} \leq C \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{\tau+1}(\tilde{\Omega}_d)}$$

Then by (3.34) (3.44)

$$\int_{\tilde{\Omega}_d} \eta^2 \overline{W}_d^{b-1} |\nabla \overline{W}_d|^2 dy \leq 4C^2(b+1)^{\frac{n-2}{2}} \left\|\eta \overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^2(\tilde{\Omega}_d)} + \frac{4C^2}{b} \int_{\tilde{\Omega}_d} |\nabla \eta|^2 \overline{W}_d^{b+1} dy$$
$$+ 4C^2 \int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^{b+1} dS + 4C^2 e^{-\gamma_0/\sqrt{d}} \int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS \quad (3.45)$$

Now for $r \leq r_2 < r_1 \leq 2r$ we choose η so that $\eta \equiv 1$ in $B_{r_2}(0)$, $\eta \equiv 0$ in $R^n \backslash B_{r_1}(0)$, $0 \leq \eta \leq 1$ and $|\nabla \eta| \leq \dfrac{2}{r_1 - r_2}$. From (3.36) (3.45)

$$\lambda_1 \left\|\overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{2k}(\tilde{\Gamma}_d \cap B_{r_2})} \leq \left[\int_{\tilde{\Gamma}_d} \left(\eta \overline{W}_d^{\frac{b+1}{2}}\right)^{2k} dS\right]^{1/k}$$
$$\leq \frac{4C^2(b+1)^{\frac{n+2}{2}}}{(r_1-r_2)^2} \left\|\overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^2(\tilde{\Omega}_d \cap B_{r_1})} + 2C^2(b+1)^2 \int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^{b+1} dS$$
$$+ 2C^2(b+1)^2 e^{-\gamma_0/\sqrt{d}} \int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS$$

Since $\tilde{\Gamma}_d^A \subset \tilde{\Gamma}_d \cap B_{\sqrt{A\varepsilon_d}}(0) \subset \tilde{\Gamma}_d \cap B_r \subset \tilde{\Gamma}_d \cap B_{r_2}$, by Hölder inequality and (3.33)

$$\int_{\tilde{\Gamma}_d^A} \eta^2 \overline{W}_d^b dS \leq \left(\int_{\tilde{\Gamma}_d} \eta^2 \overline{W}_d^{b+1} dS\right)^{\frac{b}{b+1}}$$

Hence for another constant C,

$$\left\|\overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^{2k}(\tilde{\Gamma}_d \cap B_{r_2})} \leq \frac{C(b+1)^{\frac{n+2}{2}}}{(r_1-r_2)^2}) \left\|\overline{W}_d^{\frac{n+1}{2}}\right\|^2_{L^2(\tilde{\Omega}_d \cap B_{r_1})}$$
$$+ C(b+1)^2 \left[\left\|\overline{W}_d^{\frac{b+1}{2}}\right\|^{2b/(b+1)}_{L^2(\tilde{\Gamma}_d \cap B_{r_1})} + \left\|\overline{W}_d^{\frac{b+1}{2}}\right\|^2_{L^2(\tilde{\Gamma}_d \cap B_{r_1})}\right] \quad (3.46)$$

Let $h = b+1$. Then we have

$$\|\overline{W}_d\|^h_{L^{kh}(\tilde{\Gamma}_d \cap B_{r_2})} \leq \frac{Ch^{\frac{n+2}{2}}}{(r_1-r_2)^2} \|\overline{W}_d\|^h_{L^h(\tilde{\Omega}_d \cap B_{r_1})} + Ch^2 \|\overline{W}_d\|^h_{L^h(\tilde{\Gamma}_d \cap B_{r_1})}$$
$$+ Ch^2 \|\overline{W}_d\|^{h-1}_{L^h(\tilde{\Gamma}_d \cap B_{r_1})} \quad (3.47)$$

Observe that, we only need to prove for $0 < d < d_1$

$$\liminf_{h \to \infty} \|\overline{W}_d\|_{L^h(\tilde{\Omega}_d \cap B_r)} < \infty$$

Suppose this conclusion is not true, then for a sequence $d_j \to 0$,

$$\liminf_{h\to\infty} \|\overline{W}_{d_j}\|_{L^h(\tilde{\Omega}_{d_j}\cap B_r)} \to \infty$$

Thus we may assume, for such $d = d_j$, there is $h_d > 0$ so that when $h > h_d$, $\|\overline{W}_d\|_{L^k(\tilde{\Omega}_d\cap B_r)} > 1$. Hence

$$\|\overline{W}_d\|^{h-1}_{L^h(\tilde{\Gamma}_d\cap B_{r_1})} \le \max\{\|\overline{W}_d\|^h_{L^h(\tilde{\Gamma}_d\cap B_{r_1})}, \|\overline{W}_d\|_{L^h(\tilde{\Omega}_d\cap B_r)}\} \tag{3.48}$$

Thus after replacing C by $2C$, the last term in (3.47) can be omitted.

Next for $r \le r_3 < r_2$, choose η so that $\eta \equiv 1$ in $B_{r_3}(0)$, $\eta \equiv 0$ in $R^n\setminus B_{r_2}(0)$, $0 \le \eta \le 1$ and $|\bigtriangledown \eta| \le \dfrac{2}{r_2 - r_3}$. Let $\alpha = \dfrac{n}{n-2}$ and $h = b+1$. From (3.33) (3.44) (3.45) and the remark before (3.48),

$$\begin{aligned}\|\overline{W}_d\|^h_{L^{kh}(\tilde{\Omega}_d\cap B_{r_3})} \le& \|\overline{W}_d\|^h_{L^{\alpha h}(\tilde{\Omega}_d\cap B_{r_1})}\\ \le& \frac{Ch^{\frac{n+2}{2}}}{(r_2-r_3)^2}\|\overline{W}_d\|^h_{L^h(\tilde{\Omega}_d\cap B_{r_1})} + Ch\|\overline{W}_d\|^h_{L^h(\tilde{\Gamma}\cap B_{r_1})}\end{aligned} \tag{3.49}$$

Let $r_2 = \dfrac{r_1 + r_3}{2}$. From (3.47) (3.48) (3.49)

$$\begin{aligned}&\|\overline{W}_d\|^h_{L^{kh}(\tilde{\Omega}_d\cap B_{r_3})} + \|\overline{W}_d\|^h_{L^{kh}(\tilde{\Gamma}_d\cap B_{r_3})}\\ &\le \frac{Ch^{\frac{n+2}{2}}}{(r_1-r_3)^2}\left[\|\overline{W}_d\|^h_{L^h(\tilde{\Omega}_d\cap B_{r_1})} + \|\overline{W}_d\|^h_{L^h(\tilde{\Gamma}_d\cap B_{r_1})}\right]\end{aligned}$$

Hence

$$\begin{aligned}&\left[\|\overline{W}_d\|^{kh}_{L^{kh}(\tilde{\Omega}_d\cap B_{R_2})} + \|\overline{W}_d\|^{kh}_{L^{kh}(\tilde{\Gamma}_d\cap B_{R_2})}\right]^{\frac{1}{kh}} \le \left[\|\overline{W}_d\|^h_{L^{kh}(\tilde{\Omega}_d\cap B_{R_2})} + \|\overline{W}_d\|^h_{L^{kh}(\tilde{\Gamma}_d\cap B_{R_2})}\right]^{\frac{1}{h}}\\ &\le \left[\frac{C}{(R_1-R_2)^2}\right]^{\frac{1}{h}}\left[\|\overline{W}_d\|^h_{L^h(\tilde{\Gamma}_d\cap B_{R_1})} + \|\overline{W}_d\|^h_{L^h(\tilde{\Gamma}_d\cap B_{R_1})}\right]^{\frac{1}{h}}\end{aligned} \tag{3.50}$$

where $R_1 = r_1, R_2 = r_3$. Define

$$N(p,R) = \left[\|\overline{W}_d\|^p_{L^p(\tilde{\Gamma}_d\cap B_R)} + \|\overline{W}_d\|^p_{L^p(\tilde{\Omega}_d\cap B_R)}\right]^{\frac{1}{p}}$$

Let $h = h_m = (\tau+1)k^m, R = R_m = r(1+2^{-m}), m = 0,1,2,\cdots$. By standard iteration process we obtain from (3.50)

$$\limsup_{m\to\infty} N\left((\tau+1)k^{m+1}, R_{m+1}\right) \le CN(\tau+1, 2r)$$

From (3.39), (3.50) we see that $N(\tau+1, 2r) \le C$. Hence

$$\sup_{\tilde{\Omega}_d\cap B_r} \overline{W}_d = \lim_{s\to\infty} \|\overline{W}_d\|^{1/s}_{L^s(\tilde{\Omega}_d\cap B_r)} \le \limsup_{m\to\infty} N\left((\tau+1)k^{m+1}, R_{m+1}\right) \le C$$

Now Theorem 1.1 (i) is complete.

4. Condensation Rate

First we shall prove Theorem 1.1 (ii). Let

$$W_d(z)=\frac{1}{\alpha_d}u_d\left(P_d+\beta_d\sqrt{d}z\right) \tag{4.1}$$

From Theorem 1.1 part (i),

$$W_d(z)\le CU(z),\quad z\in\frac{\Omega-P_d}{\beta_d\sqrt{d}} \tag{4.2}$$

Now we define $V_d(x)$, $\tilde V_d(y)$ by (2.1), (2.13), (2.14) and define

$$\tilde W_d(z)=\left(\beta_d\sqrt{d}\right)^{\frac{n-2}{2}}\tilde V_d\left(\beta_d\sqrt{d}z\right),\quad z\in B_{\delta/\beta_d\sqrt{d}} \tag{4.3}$$

Let $m_d=\inf\limits_{B_\delta}|D\Phi_d(y)|$ and $M_d=\sup\limits_{B_\delta}|D\Phi_d(y)|$. Then $0<m_0\le m_d\le M_d\le M_0$ for all small d and hence

$$m_0\beta_d\sqrt{d}|z|\le\left|\Phi_d\left(\beta_d\sqrt{d}z\right)-P_d\right|\le M_0\beta_d\sqrt{d}|z|$$

for all $z\in B_{\delta/\beta_d\sqrt{d}}$. By (4.2), for $z\in B^+_{\delta/\beta_d\sqrt{d}}$,

$$\begin{aligned}\tilde W_d(z)&=\left(\beta_d\sqrt{d}\right)^{\frac{n-2}{2}}\tilde V_d\left(\beta_d\sqrt{d}z\right)=\left(\beta_d\sqrt{d}\right)^{\frac{n-2}{2}}V_d\left(\Phi_d\left(\beta_d\sqrt{d}z\right)\right)\\&=\left(\beta_d\sqrt{d}\right)^{\frac{n-2}{2}}V_d\left(P_d+\left(\Phi_d\left(\beta_d\sqrt{d}z\right)-P_d\right)\right)=W_d\left[\frac{\Phi_d\left(\beta_d\sqrt{d}z\right)-P_d}{\beta_d\sqrt{d}}\right]\\&\le\max_{m_0|z|\le|\hat z|\le M_0|z|}W_d(\hat z)\le CU(m_0|z|)\le Cm_0^{2-n}U(z)\end{aligned}$$

Hence

$$\tilde W_d(z)\le CU(z),\quad z\in B_{\delta/\beta_d\sqrt{d}} \tag{4.4}$$

Let $\tilde G_d(z)=\alpha_d^2\tilde W_d\left(\dfrac{z}{\beta_d}\right)$. Then for $z\in B_{\delta/\sqrt{d}}$, by (4.4)

$$\tilde G_d(z)\le C\alpha_d^2U\left(\frac{z}{\beta_d}\right)=C\left[\beta_d^2+\frac{|z|^2}{n(n-2)}\right]^{-\frac{n-2}{2}} \tag{4.5}$$

Hence on any compact subset of $R^n\backslash\{0\}$, $\{G_d\}$ is uniformly bounded. Let g_{ij} and g^{ij} be defined in (2.9)–(2.12) and define

$$g_{ij}^d(z)=g_{ij}\left(\sqrt{d}z\right),\quad g_d^{ij}(z)=g^{ij}\left(\sqrt{d}z\right) \tag{4.6}$$

Then set

$$\Delta_d=\frac{1}{\sqrt{\det\left(g_{ij}^d\right)}}\sum_{i,j=1}^n\frac{\partial}{\partial z_i}\left(\sqrt{\det\left(g_{ij}^d\right)}g_d^{ij}\frac{\partial}{\partial z_j}\right) \tag{4.7}$$

From (2.15) (4.3) We have

$$\Delta_d \tilde{G}_d - \tilde{G}_d + \beta_d^2 \tilde{G}_d^\tau = 0 \tag{4.8}$$

in $B_{\delta/\sqrt{d}} \backslash [z_n = 0]$, and $\tilde{G}_d$ is a weak solution of (4.8) in $B_{\delta/\sqrt{d}}$. From (4.5), after passing to a subsequence we have

$$\tilde{G}_d \to G \quad \text{in } C^2_{\text{loc}}(R^n \backslash \{0\})$$

and $0 \le G(z) \le C^*|z|^{2-n}$ for $z \neq 0$, where G satisfies

$$-\Delta G + G = 0 \quad \text{in } R^n \backslash \{0\} \tag{4.9}$$

since $g^d_{ij}(z) \to \delta_{ij}$ in $C^2_{\text{loc}}(R^n)$.

For any $\varphi \in C^\infty_0(R^n)$, by (4,4), (4.8)

$$\begin{aligned}
&\int_{R^n} G(z)[(-\Delta+1)\varphi(z)]dz = \lim_{d\to 0}\int_{B_{\delta/\sqrt{d}}} \tilde{G}_d(z)[(-\Delta+1)\varphi(z)]dz \\
&= \lim_{d\to 0}\int_{B_{\delta/\sqrt{d}}} \varphi(z)[-\Delta_d \tilde{G}_d + \tilde{G}_d]dz = \lim_{d\to 0}\beta_d^2 \int_{B_{\delta/\sqrt{d}}} \varphi(z)\tilde{G}^\tau_d(z)dz \\
&= \lim_{d\to 0}\beta_d^2\alpha_d^{2\tau}\int_{B_{\delta/\sqrt{d}}} \varphi(z)\tilde{W}^\tau_d\left(\frac{z}{\beta_d}\right)dz = \lim_{d\to 0}\beta_d^{n+2}\alpha_d^{2\tau}\int_{B_{\delta/\beta_\delta\sqrt{d}}} \varphi(\beta_d z)\tilde{W}^\tau_d(z)dz \\
&= \varphi(0)\int_{R^n} U^\tau(z)dz = \frac{1}{n}\omega_n[n(n-2)]^{n/2}\varphi(0)
\end{aligned}$$

since $\tilde{W}_d \to U$ in $C^2_{\text{loc}}(R^n)$. Hence

$$(-\Delta+1)G(z) = \frac{1}{n}\omega_n[n(n-2)]^{n/2}\delta(z) \quad \text{in} \quad R^n\backslash\{0\}$$

where δ is the Dirac measure. Hence

$$G(z) = \frac{1}{n}\omega_n[n(n-2)]^{n/2}K(z)$$

where $K(z)$ is the fundamental solution of $-\Delta+1$ in R^n.

Next we prove Theorem 1.1 (iii). Although we only need to prove (iii) when $n = 5, 6$, the following estimates are valid for any $n > 4$. Assume $P_d \in \partial\Omega$, $P_d \to P_0$ as $d \to 0$. When $n > 6$, by Lemma 2.5, $H(P_0) = \max_{P\in\partial\Omega} H(P) > 0$. When $n = 5, 6$, by the assumption that Ω is a strictly convex domain we also have $H(P_0) > 0$.

From (1.1) and Pohozaev identity

$$\int_\Omega u_d^2 dx = \int_{\partial\Omega}(x - P_d, v)\left[\frac{d}{2}|\nabla u_d|^2 - \frac{n-2}{2n}u_d^{\tau+1} + \frac{1}{2}u_d^2\right]dS \tag{4.10}$$

As in Section 2 we assume $P_d = 0$. Let N be a neighborhood of P_0 and $\delta > 0$ be chosen so small that (4.4) holds for all small d. From (2.4)

$$x \cdot v(x)dS = \left[\frac{1}{2}\sum_{i=1}^{n-1}\alpha_i x_i^2 + o(|x'|^2)\right]dx', \quad |x'| < \delta$$

where $\alpha_i = \alpha_i(P_d)$, $i = 1, \cdots, n-1$, are the principle curvatures of $\partial\Omega$ at P_d.

$$H(P_d) = \frac{1}{n-1}\sum_{i=1}^{n-1}\alpha_i$$

Now we shall estimate each term in (4.10). In the following we shall use (4.2), (4.4) frequently without mentioning, and use H. O. T. to denote higher order terms, which may vary from line to line.

$$\begin{aligned}
I_1 \equiv& \frac{d}{2}\int_{\partial\Omega}(x-P_d, v)|\nabla u_d|^2 dS = \frac{d}{2}\int_{|x'|<\delta}\left(\frac{1}{2}\sum_{i=1}^{n-1}\alpha_i x_i^2\right)|\nabla u_d|^2 dx' + H.O.T.\\
=& \frac{1}{4}d^{\frac{n}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{|x'|<\delta}x_i^2|\nabla V_d|^2 dx' + H.O.T.\\
=& \frac{1}{4}d^{\frac{n}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{|y'|<\delta}y_i^2|\nabla \tilde{V}_d(y')D\Psi_d(\Phi(y'))|^2\det(D\Phi_d(y'))dy' + H.O.T.\\
=& \frac{1}{4}d^{\frac{n}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{|y'|<\delta}y_i^2|\nabla \tilde{V}_d(y')|^2 dy + H.O.T.\\
=& \frac{1}{4}\beta_d d^{\frac{n+1}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{|z'|<\delta/\beta_d\sqrt{d}}z_i^2|\nabla \tilde{W}_d(z')|^2 dz' + H.O.T.\\
=& \frac{1}{4}\beta_d d^{\frac{n+1}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{R^{n-1}}z_i^2|\nabla U(z')|^2 dz' + H.O.T.\\
=& \frac{1}{4n^2}\omega_{n-1}H(P_0)\beta_d d^{\frac{n+1}{2}}\int_0^\infty r^{n+2}\left[1+\frac{r^2}{n(n-2)}\right]^{-n}dr + H.O.T.\\
=& \frac{n+1}{8n^2(n-3)}[n(n-2)]^{\frac{n+3}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}\omega_{n-1}H(P_0)\beta_d d^{\frac{n+1}{2}}\\
&+ H.O.T.
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
I_2 \equiv& \int_{\partial\Omega}\frac{n-2}{2n}(x-P_d, \nu)u_d^{\tau+1}(x)dS\\
=& \frac{n-2}{4n}\sum_{i=1}^{n-1}\alpha_i\int_{|x'|<\delta_0}x_i^2 u_d^{\tau+1}(x')dx' + H.O.T.\\
=& \frac{n-2}{4n}d^{n/2}\sum_{i=1}^{n-1}\alpha_i\int_{|y'|<\delta}y_i^2\tilde{V}_d^{\tau+1}(y')\det(D\Phi_d(y'))dy' + H.O.T.\\
=& \frac{n-2}{4n}\beta_d d^{\frac{n+1}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{|z'|<\delta/\beta_d\sqrt{d}}z_i^2\tilde{W}_d^{\tau+1}(z')dz' + H.O.T.\\
=& \frac{n-2}{4n}\beta_d d^{\frac{n+1}{2}}\sum_{i=1}^{n-1}\alpha_i\int_{R^{n-1}}z_i^2 U^{\tau+1}(z')dz' + H.O.T.
\end{aligned}$$

$$=\frac{n-2}{4n}\beta_d d^{\frac{n+2}{2}}\omega_{n-1}H(P_0)\int_0^\infty r^n\left[1+\frac{r^2}{n(n-2)}\right]^{-n}dr+H.O.T.$$

$$=\frac{n-2}{8n}[n(n-2)]^{\frac{n+1}{2}}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}\omega_{n-1}H(P_0)\beta_d d^{\frac{n+1}{2}}+H.O.T. \quad (4.12)$$

From (4.2) we can choose $R>0$ so that

$$u_d(P_d+x)\le C(\beta_d d)^{\frac{n-2}{2}}|x|^{2-n},\ |x|\ge R\sqrt{d}$$

$$I_3\equiv\frac{1}{2}\int_{\partial\Omega}(x-P_d,\nu)u_d^2(x)dS=\frac{1}{4}\int_{|x'|<\delta}\left(\sum_{i=1}^{n-1}\alpha_i x_i^2\right)u_d^2(x')dx'+H.O.T.$$

$$\le C\int_{|x'|<R\sqrt{d}}|x'|^2u_d^2(P_d+x')dx'+C\int_{R\sqrt{d}\le|x'|\le\delta}|x'|^2u_d^2(P_d+x')dx'+H.O.T.$$

$$\int_{|x'|<R\sqrt{d}}|x'|^2u_d^2(P_d+x')dx'\le R\sqrt{d}\int_{|x'|<R\sqrt{d}}|x'|u_d^2(P_d+x')dx'$$

$$=R\beta_d^2d^{\frac{n+1}{2}}\int_{|z'|<R/\beta_d}|z'|W_d^2(z')dz'$$

$$\le CR\beta_d^2d^{\frac{n+1}{2}}\int_{R^{n-1}}r^{n-1}\left(1+\frac{r^2}{n(n-2)}\right)^{-(n-2)}dr\le CR\beta_d^2d^{\frac{n+1}{2}}$$

$$\int_{R\sqrt{d}\le|x'|\le\delta_0}|x'|^2u_d^2(P_d+x')dx'\le C(\beta_d d)^{n-2}\int_{R\sqrt{d}\le|x'|\le\delta}|x'|^{6-2n}dx'$$

$$\le C(\beta_d d)^{n-2}\int_{R\sqrt{d}}^{\delta}r^{4-n}dr=\begin{cases}O\left(\beta_d^{n-2}d^{\frac{n+1}{2}}\right) & \text{if } n>5\\ O\left(\beta_d^3d^3\log\frac{1}{d}\right) & \text{if } n=5\end{cases}$$

Hence

$$I_3\equiv\frac{1}{2}\int_{\partial\Omega}(x-P_d,\nu)u_d^2(x)dS=\begin{cases}O\left(\beta_d^2d^{\frac{n+1}{2}}\right) & \text{if } n>5\\ O\left(\beta_d^2d^3+\beta_d^3d^3\log\frac{1}{d}\right) & \text{if } n=5\end{cases}$$

$$=o\left(\beta_d^2d^{\frac{n}{2}}\right) \quad (4.13)$$

$$I_0\equiv\int_\Omega u_d^2(x)dx=d^{\frac{n-2}{2}}\int_{B_\delta^+}\tilde{V}_d^2(y)\det(D\Phi_d(y))dy+H.O.T.$$

$$=\beta_d^2d^{\frac{n}{2}}\int_{B^+_{\delta/\beta_d\sqrt{d}}}\tilde{W}_d^2(z)dz+H.O.T.=\beta_d^2d^{\frac{n}{2}}\int_{R^n_+}U^2(z)dz+H.O.T.$$

$$=\frac{n-1}{n-4}[n(n-2)]^{\frac{n}{2}}\frac{\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{\Gamma(n)}\omega_n\beta_d^2d^{\frac{n}{2}}+H.O.T. \quad (4.14)$$

Hence $I_3 = o(I_0)$ as $d \to 0$. From (4.11)-(4.14)

$$\frac{n-1}{n-4}[n(n-2)]^{\frac{n}{2}}\frac{\left(\Gamma\left(\frac{n}{2}\right)\right)^2}{\Gamma(n)}\omega_n\beta_d^2 d^{\frac{n}{2}} = \left\{\frac{1}{8n^2}\frac{n+1}{n-3}[n(n-2)]^{\frac{n+3}{2}}\right.$$

$$\left.-\frac{n-2}{8n}[n(n-2)]^{\frac{n+1}{2}}\right\}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma(n)}\omega_{n-1}H(P_0)\beta_d d^{\frac{n+1}{2}} + H.O.T.$$

Hence

$$\lim_{d\to 0}\frac{\beta_d}{\sqrt{d}} = \frac{(n-2)(n-4)}{2n(n-1)(n-3)}\left[\frac{n(n-2)}{\pi}\right]^{\frac{1}{2}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}H(P_0)$$

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