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# A NOTE ON THE VALUATION OF AMERICAN OPTION*

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Dedicated to the 80th birthday of Professor Zhou Yulin
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**Abstract** American options give holder a right to exercise it at any time at will, the holder should to make the exercise policy in such a way that the expected payoff from the option will be maximized. In this note we prove that it is equivalent to a fact which makes the option value and option delta continuous.
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## 1. Introduction

An American put option is a contract which give the holder a right to sell the underlying assert at any time before or at the expiry date. The holder should choose best strategy to exercise the contract. It means that the holder should make his exercise policy in such a way that the expected payoff from the option will be maximized. It has been showed that the maximize principle is equivalent to the fact which makes the option value and option delta continuous in the case of the perpetual American put option in which the option pricing can be found in a close form [1].

In this note we prove that the equivalence is still true for American options with a finite time to expiry. We claim that for American put option the option value is maximized if and only if the owner of option exercises such that $\Delta = \frac{\partial V}{\partial s}$ is continuous.

## 2. Assumptions and Basic Facts

In the risk neutral world, the underlying assert price $S_t$ is assumed to follow the lognormal diffusion process

$$\frac{dS_t}{S_t} = (r-q)dt + \sigma dW_t$$

where $r$ and $q$ are the constant riskless interest rate and dividend yield, respectively, $\sigma$ is the constant volatility and $dW_t$ is the standard Wiener process (See [1] and [2])

$$E(dW_t) = 0,$$

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$$Var(dW_t) = dt.$$

Now let us consider an American option on the underlying asset $S_t$. American option can be exercised at any time up to expiration date. It gives holder a right to do something, but holder does not have to exercise this right. So threr is a cost to entering into an option contract(the holder has to pay premium for the contract initially). This cost is the price of American option.

How to define the price of an American option? It is well known that the value of European option is governed by Black-Scholes equation ([1] and [2]). In view of probability theory, the pricing problem of American optionis can be reduced to a optimal stopping problem and then a free boundary problem of a parabolic partial differential equation [3]. In this paper, we prove that the pricing problem of American optionis is equivalent to free boundary problem by using the PDE approach. We consider the American put option only.

Let $K$ be the striking price of option, $s$ be the price of stock and $V^*(s,t)$ be the value of American put option with payoff $(K-s)^+$ until time expiry $T$. According to arbitrage-free principle, we conclude that the price of American option $V^*(s,t)$ is a nonnegative continuous function. For American option with same strike price, the owner of long-life option has all the exercise opportunities open to the owner of short-life option and more, so the value of option is monotone increase with respect to the life time of the option $T-t$ (Ch.8 in [4]). Hence $V^*(s,t)$ is monotone decreasing function of $t$. Mathematically, we have

$$\begin{aligned} &V^*(s,T) = (K-s)^+ \\ &V^*(0,t) = K \\ &V^*(\infty,t) = 0. \end{aligned}$$

And arbitrage-free principle implies

$$(K-s)^+ \le V^*(s,t) \le K.$$

For American put option, based on $\Delta$-hedging argument and Ito's lemma, we deduce that $V^*(s,t)$ satisfies the following Black-Scholes equation [1]

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0, \tag{1}$$

in the continuation region

$$\mathcal{D}^* = \{(s,t) \mid V^*(s,t) > (K-s)^+\}.$$

Let

$$s^*(t) = \sup\{s \in [0,+\infty) | V^*(s,t) = (K-s)^+\}, \quad t \in [0,T).$$

Since $V^*(s,t)$ is monotone decrease with respect to $t$, we see that $s^*(t)$ is monotone increase. By the maximum principle, we can derive estimates

$$0 < s^*(t) \le \min\{K, \frac{r}{q}K\}.$$

So we can define $s^*(T) = \lim_{t\to T} s^*(t)$. We obtain, by using the maximum principle again

$$\mathcal{D}^* = \{(s,t)|s > s^*(t)\}.$$

Hence for American put option $V^*(s,t)$, there are two regions: continuation region

$$\mathcal{D}^* = \{(s,t)|V^*(s,t) > (K-s)^+\} = \{(s,t)|s > s^*(t)\}$$

and stopping region

$$\{(s,t)|s < s^*(t)\}.$$

And between them there is an optimal exercise boundary $s = s^*(t)$. To find $(V^*(s,t), s^*(t))$ is the main object of our paper.

Now let us define a class of curve

$$\Gamma = \{s = s(t)|s(.) \in C^1[0,T], 0 < s(t) \le \min\{K, \frac{r}{q}K\}, s(.)\text{increase}\}.$$

For any given curve $s = s(t) \in \Gamma$, we solve the following partial differential equation

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0, \text{ in } \mathcal{D} = \{(s,t)|s > s(t)\}$$

with terminal-boundary conditions

$$\begin{aligned} &V(s,T) = (K-s)^+,\\ &V(s(t),t) = K - s(t),\\ &V(\infty,t) = 0. \end{aligned}$$

The existence and uniqueness of solution $V = V(s,t) = V(s,t;s(.)) \in C^{2,1}(\mathcal{D})$ and $V \in C^{1,0}(\overline{\mathcal{D}})$ for above problem can be derived from the theory of liner partial differential equation [5].

It is obvious that $V^*(s,t) = V(s,t;s^*(.))$ is the American put option's value and it is the solution of the following problem

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0, \text{ in } \mathcal{D}^* \tag{2}$$

with terminal-boundary conditions

$$\begin{aligned} &V(s,T) = (K-s)^+,\\ &V(s^*(t),t) = K - s^*(t),\\ &V(\infty,t) = 0. \end{aligned}$$

But $s = s^*(t)$ is still an unknow curve. To determine $(V^*(s,t), s^*(t))$, we have to impose an another condition on the $s^*(t)$. The principle of "to maximize the option's value as possible as we can" implies

$$V^*(s,t) = \max_{s(.)\in\Gamma} V(s,t;s(.)), \tag{3}$$

i.e. the pricing problem of American put option is equivalent to above maximizing problem (3). In this note, we will prove the following main theorm

**Main Theorem** *The pair* $(V^*(s,t), s^*(t))$ *is the solution of maximizing problem* (3) *if and only if*

$$\frac{\partial V^*}{\partial s}(s^*(t)^+, t) = -1. \tag{4}$$

*Hence* $(V^*(s,t), s^*(t))$ *is the solution of maximizing problem* (3) *if and only if it is the solution of the following parabolic free boundary problem*

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0, \ \textit{in } \mathcal{D} \tag{5}$$

*with terminal-boundary conditions*

$$\begin{aligned}
&V(s,T) = (K-s)^+,\\
&V(s(t),t) = K - s(t),\\
&V(\infty,t) = 0,\\
&\frac{\partial V}{\partial s}(s(t),t) = -1.
\end{aligned}$$

## 3. Necessary Condition

Consider the maximizing problem: Find $(V^*(s,t), s^*(t))$ such that

$$V^*(s,t) = V(s,t;s^*(.)) = \max_{s(.)\in\Gamma} V(s,t;s(.)). \tag{6}$$

Suppose that $(V^*(s,t), s^*(t))$ is the solution of this problem, we want to find conditions which $(V^*(s,t), s^*(t))$ must satisfy. Since $V^*(s,t) \geq (K-s)^+$ and $V^*(s^*(t),t) = K - s^*(t)$, we get

$$\frac{\partial V^*}{\partial s}(s^*(t)^+, t) \geq -1. \tag{7}$$

We want to derive the inverse inequality. To this end, we need the following

**Lemma 1** *Let* $(V^*, s^*)$ *be the solution of problem* (6) *and* $V^\epsilon$ *be the solution of the following equation*

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0 \ \textit{in } \mathcal{D}_\epsilon$$

*with terminal-boundary conditions*

$$\begin{aligned}
&V(s,T)=(K-s)^+,\\
&V(s_\epsilon^*(t),t)=K-s_\epsilon^*(t),\\
&V(\infty,t)=0,
\end{aligned}$$

*where* $\mathcal{D}_\epsilon=\{s>s_\epsilon^*(t),0\le t<T\}$ *and* $s_\epsilon^*(t)=e^{-\epsilon}s^*(t)$. *Then we have*

$$\lim_{\epsilon\to 0^+}\frac{\partial V^\epsilon}{\partial s}(s_\epsilon^*(t)^+,t)=\frac{\partial V^*}{\partial s}(s^*(t)^+,t)$$

*for any* $t\in[0,T)$.

**Proof** Let $w(s,t)=V^\epsilon(e^{-\epsilon}s,t)$, then $w$ satisfies

$$\mathcal{L}w=\frac{\partial w}{\partial t}+\frac{\sigma^2}{2}s^2\frac{\partial^2 w}{\partial s^2}+(r-q)s\frac{\partial w}{\partial s}-rw=0,\ \text{in } \mathcal{D}^*$$

with terminal-boundary conditions

$$\begin{aligned}
&w(s,T)=(K-e^{-\epsilon}s)^+,\\
&w(s^*(t),t)=K-e^{-\epsilon}s^*(t),\\
&w(\infty,t)=0.
\end{aligned}$$

By using the boundary estimates for $w-V^*$ we complete the proof [5].

According to this lemma, we can deduce necessary conditon which $(V^*(s,t),s^*(t))$ must satisfy.

**Theorem 1** *The solution of maximizing problem* (3) $(V^*(s,t),s^*(t))$ *satisfies*

$$\frac{\partial V^*}{\partial s}(s^*(t)^+,t)=-1. \tag{8}$$

*Hence it is the solution of the following parabolic free boundary problem*

$$\mathcal{L}V=\frac{\partial V}{\partial t}+\frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2}+(r-q)s\frac{\partial V}{\partial s}-rV=0,\ \text{in } \mathcal{D} \tag{9}$$

*with terminal-boundary conditions*

$$\begin{aligned}
&V(s,T)=(K-s)^+,\\
&V(s^*(t),t)=K-s(t),\\
&V(\infty,t)=0,\\
&\frac{\partial V}{\partial s}(s(t),t)=-1.
\end{aligned}$$

**Proof** Since (7), we only need to prove

$$\frac{\partial V^*}{\partial s}(s^*(t)^+,t)\le -1. \tag{10}$$

In view of Lemma 1, (10) can be deduced from

$$\frac{\partial V^\epsilon}{\partial s}(s_\epsilon^*(t)^+, t) \le -1. \tag{11}$$

On the other hand, (11) can be derived from

$$V^\epsilon(s,t) \le K - s \ \text{ in } \mathcal{O} = \{(s,t)|s_\epsilon^*(t) < s < s^*(t),\ 0 \le t < T\}. \tag{12}$$

For, recalling $V^\epsilon(s_\epsilon^*(t), t) = K - s_\epsilon^*(t)$, (11) follows from (12) directly.

Now we prove by contradiction that (12) holds.

If we assume that (12) is invalid, then $V^\epsilon(s,t) - (K-s)$ achieves its positive maximum in the interior of $\mathcal{O}$ by observing that $V^\epsilon(s,t) - (K-s)$ is non positive on $\partial_p\mathcal{O}$, parabolic boundary of $\mathcal{O}$ (The maximality of $V^*$ implies that $V^\epsilon \le V^* = (K-s)$ on the boundary $s = s^*(t)$). Let $(s_0, t_0) \in \mathcal{O}$ be positive maximum point. We deduce that $\mathcal{L}(V^\epsilon(s,t) - (K-s))(s_0,t_0) < 0$ because $\mathcal{L}$ is the maximum principle operator. But we have

$$\mathcal{L}(V^\epsilon(s,t) - (K-s))(s_0,t_0) = rK - qs_0$$

by calculation. This is in contradiction with $s_0 \le s^*(t_0) \le \frac{r}{q}K$. The proof of Theorem 1 is completed.

## 4. Sufficient Condition

In this section we will prove that the solution of parabolic free boundary problem is American put option's value.

Consider the parabolic free boundary problem: Find $(V(s,t), s(t))$ such that

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0 \tag{13}$$

with terminal-boundary conditions

$$\begin{aligned} &V(s,T) = (K-s)^+,\\ &V(s(t),t) = K - s(t),\\ &V(\infty,t) = 0,\\ &\frac{\partial V}{\partial s}(s(t),t) = -1. \end{aligned}$$

is a unique solution $(V^*(s,t), s^*(t))$ for above problem. Let

$$\mathcal{D}^* = \{(s,t)|s > s^*(t)\}.$$

From the regularity theory of parabolic free boundary problem of linear partial differential equation [6], we conclude that

$$s^*(t) \in C^\infty \text{ and } V^* \in C^\infty(\mathcal{D}^*) \cap C^{2,1}(\overline{\mathcal{D}^*} \setminus \{t = T\}).$$

Furthermore $V^*(s,t)$ is a positive continuous function and monotone decrease with respect to $t$. Meanwhile $s^*(t)$ is monotone increase function and

$$V^*(s,t) \geq (K-s)^+, \quad 0 < s^*(t) \leq \min\{K, \frac{r}{q}K\}.$$

**Theorem 2** *The solution of free boundary problem is the solution of maximizing problem* (3).

**Proof** For any given curve $s = s(t) \in \Gamma$, we solve the following problem

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \frac{\sigma^2}{2}s^2\frac{\partial^2 V}{\partial s^2} + (r-q)s\frac{\partial V}{\partial s} - rV = 0$$

with terminal-boundary conditions

$$\begin{aligned} &V(s,T) = (K-s)^+, \\ &V(s(t),t) = K - s(t), \\ &V(\infty,t) = 0 \end{aligned}$$

to obtain the solution $V(s,t) = V(s,t;s(.))$. We will prove that $V(s,t) = V(s,t;s(.)) \leq V^*(s,t)$ for all $s \geq \max\{s^*(t), s(t)\}, 0 \leq t \leq T$.

Define a new function

$$\hat{V}(s,t) = \begin{cases} V^*(s,t), & \text{if } s > s^*(t), \\ K - s, & \text{if } s(t) < s \leq s^*(t), \end{cases}$$

in $\mathcal{D} = \{(s,t)|s > s(t), 0 \leq t < T\}$. It is easy to see that $\hat{V} \in C^1(\mathcal{D})$. Letting $w = V - \hat{V}$ we have from the properity of $V^*$ that $w \leq 0$ on the parabolic boundary $\partial_p\mathcal{D}$. We claim that $w \leq 0$ on $\overline{\mathcal{D}}$. If on the contrary this is invalid, then we may assume $M = \max_{\overline{\mathcal{D}}} w = w(s_0,t_0) > 0$. We consider three cases:

(1) $s_0 > s^*(t_0)$. In this case, we have $\mathcal{L}w(s_0,t_0) = 0$, which is impossible because $(s_0,t_0)$ is the interior positive maximum point of $w$ and $\mathcal{L}$ is the maximum principle operator.

(2) $s_0 < s^*(t_0)$. Locally, $w = V-(K-s)$ achieves its positive maximun M at $(s_0,t_0)$, then we deduce that

$$\begin{aligned} &\frac{\partial V}{\partial s}(s_0,t_0) = -1, \quad \frac{\partial^2 V}{\partial s^2}(s_0,t_0) \leq 0; \\ &\frac{\partial V}{\partial t}(s_0,t_0) \leq 0, \quad V(s_0,t_0) = K - s_0 + M. \end{aligned}$$

From these estimates, we have

$$\mathcal{L}V(s_0,t_0) \leq qs_0 - rK - rM < qs_0 - rK.$$

This inequality is in contradiction with $s_0 \leq \frac{r}{q}K$.

(3) $s_0 = s^*(t_0)$. In this case, we have $M = \max_{\overline{\mathcal{D}}}(V - \hat{V}) = (V - \hat{V})(s_0, t_0) > 0$. Again we get

$$\frac{\partial V}{\partial s}(s_0, t_0) = -1, \quad V(s_0, t_0) = K - s_0 + M.$$

We wish to show

$$\frac{\partial V}{\partial t}(s_0, t_0) \leq 0, \quad \frac{\partial^2 V}{\partial s^2}(s_0, t_0) \leq 0.$$

We have $(V - \hat{V})(s_0, t) \leq (V - \hat{V})(s_0, t_0)$ if $0 \leq t - t_0 \ll 1$. Hence $V(s_0, t) \leq (V - \hat{V})(s_0, t_0) + \hat{V}(s_0, t)$. But $\hat{V}(s_0, t) = K - s_0$ for $0 \leq t - t_0 \ll 1$. so we get $V(s_0, t) \leq V(s_0, t_0)$ if $0 \leq t - t_0 \ll 1$. This implies $\frac{\partial V}{\partial t}(s_0, t_0) \leq 0$.

Again we have $(V - \hat{V})(s, t_0) \leq (V - \hat{V})(s_0, t_0)$ for all $s > s(t_0)$. By the definition of $\hat{V}$, we obtain

$$\begin{aligned} V(s, t_0) &\leq (V - \hat{V})(s_0, t_0) + \hat{V}(s, t_0) \\ &= \begin{cases} V(s_0, t_0) + V^*(s, t_0) - (K - s_0), & \text{if } s > s_0, \\ V(s_0, t_0) + s_0 - s, & \text{if } s \leq s_0, \end{cases} \end{aligned}$$

Hence

$$V(s, t_0) \leq V(s_0, t_0) + s_0 - s, \text{ if } s \leq s_0.$$

We conclude that

$$\frac{\partial^2 V}{\partial s^2}(s_0, t_0) \leq 0$$

by observing that $\frac{\partial V}{\partial s}(s_0, t_0) = -1$. Following the path used to prove case 2, we complete the proof of Theorem 2.

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