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GLOBAL ATTRACTORS OF REACTION-DIFFUSION SYSTEMS AND THEIR HOMOGENIZATION*

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Abstract In this paper, we study the existence of the global attractor $\mathcal{A}^\varepsilon$ of reaction-diffusion equation

$$\partial_t u^\varepsilon(x,t) = A_\varepsilon u^\varepsilon(x,t) - f(x, \varepsilon^{-1}x, u^\varepsilon(x,t)),$$

and the homogenized attractor $\mathcal{A}^0$ of the corresponding homogenized equation, then give explicit estimates for the distance between the attractor $\mathcal{A}^\varepsilon$ and the homogenized attractor $\mathcal{A}^0$.
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1. Introduction and Main Results

We consider the reaction-diffusion system

$$\begin{cases} \partial_t u^\varepsilon(x,t) = A_\varepsilon u^\varepsilon(x,t) - f(x, \varepsilon^{-1}x, u^\varepsilon(x,t)), & (x,t) \in \Omega \times \mathbf{R}^+, \\ u^\varepsilon(x,t)|_{\partial\Omega} = 0, \quad u^\varepsilon(x,t)|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^3$ and $0 < \varepsilon \le \varepsilon_0 < 1$. Here $u^\varepsilon = u^\varepsilon(x,t) = (u_\varepsilon^1, \cdots, u_\varepsilon^k)$ is an unknown vector-valued function. The second order elliptic differential operators A_ε have the form as follows:

$$A_\varepsilon u := \mathrm{diag}(A_\varepsilon^1 u^1, \cdots, A_\varepsilon^k u^k), \tag{1.2}$$

with

$$A_\varepsilon^l u^l = \sum_{i,j=1}^{3} \partial_{x_i}(a_{ij}^l(\varepsilon^{-1}x)\partial_{x_j} u^l(x)), \tag{1.3}$$

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where the functions $a_{ij}^l(y)$, $l=1,\cdots,k$, $y\in\mathbf{R}^3$, are assumed to be symmetric, smooth and $\mathbf{Y}$-periodic with respect to $y\in\mathbf{R}^3$, where $\mathbf{Y}\subset\mathbf{R}^3$ is a fixed cube. The uniform ellipticity condition

$$\sum_{i,j=1}^{3} a_{ij}^l(y)\zeta_i\zeta_j \geq \nu|\zeta|^2, \quad \forall y,\zeta\in\mathbf{R}^3, \tag{1.4}$$

is also assumed (with an appropriate $\nu>0$) to be valid for operators A_ε^l. We impose that $f(x,y,u)$ is almost-periodic ([1]) with respect to $y\in\mathbf{R}^3$ and satisfies the conditions as follows:

$$f\in C^1(\mathbf{R}^k,\mathbf{R}^k), \quad \partial_z f(x,y,z)\zeta\zeta\geq -C_2\zeta\zeta, \quad \forall\zeta\in\mathbf{R}^k, \tag{1.5}$$

$$|f(x,y,u)|\leq C(1+|u|^p), \quad \forall(x,y)\in\Omega\times\mathbf{R}^3, \tag{1.6}$$

$$\sum_{l=1}^{k} f^l u^l|u^l|^{p_l}\geq C\sum_{l=1}^{k}|u^l|^{p_l+2}-C_1, \quad \forall u\in\mathbf{R}^k, \tag{1.7}$$

where $p\geq 1, p_i\geq 2(p-1)$, $i=1,\cdots,k$. It is assumed also that the initial data $u_0\in(L^2(\Omega))^k$.

Efendiev and Zelik (see [2]) studied the problem (1.1) when $f(x,y,u)$ is independent of y. Fiedler and Vishik (see [3]) studied the case when the $A_\varepsilon u$ in (1.1) is replaced by $a\Delta u$. In fact, one can obtain the existence of solutions and attractors for (1.1) with $f(x,y,u)$ depending on y by the standard method as those in [4]. However, when estimate the distance between the attractors for (1.1) and the attractors of the homogenized equation, the arguments in [2] or [3] don't work. We have to overcome these difficulties by combining the ideas in [3], [2] and analyzing carefully the properties of periodic and almost-periodic functions.

In order to simplify our expression, we denote $H=(L^2(\Omega))^k$, $V=(W_0^{1,2}(\Omega))^k$, $F=(L^\infty(\Omega))^k$, $\|\cdot\|_{(W^{l,p}(\Omega))^k}=\|\cdot\|_{l,p}$.

Theorem 1.1 *If the assumptions* (1.2) $-$ (1.7) *hold, and the initial data* $u_0\in H$, *then for any* $T>0$, $\varepsilon>0$, *the problem* (1.1) *possesses a unique solution* $u^\varepsilon(x,t)\in L^\infty([0,T];H)\cap L^2([0,T];V)$, $u^\varepsilon\in C(R^+;H)$. *The mapping* S_t^ε: $u_0\longrightarrow u^\varepsilon(x,t)$ *defines a continuous semigroup* S_t^ε: $H\longrightarrow H$. *If, furthermore,* $u_0\in V$, *then* $u^\varepsilon(x,t)\in L^\infty([0,T];V)\cap L^2([0,T];W^{2,2}(\Omega))$, $u^\varepsilon\in C(R^+;V)$.

Theorem 1.2 *If the assumptions* (1.2) $-$ (1.7) *hold, and* $u_0\in H$, *then for every* $\varepsilon>0$, *the semigroup* S_t^ε *generated by the equation* (1.1) *possesses a global compact attractor* $\mathcal{A}^\varepsilon$ *in* H.

Theorem 1.1 can be proved by the Faedo-Galerkin method with the help of R.Temam [4], and the details of the proof are omitted. Similar arguments as in [4] for the problem (1.1) yield the a prior estimates needed about $u^\varepsilon(x,t)$ in H and V, and we omit the details. Then Theorem 1.2, whose proof is also omitted, can be easily proved by the standard arguments [4, Theorem 1.1.1].

By the standard homogenization theory, one can obtain the homogenized problem (2.11), for which one can prove the similar results to Theorems 1.1 and 1.2. In order to estimate the L^2-distance between the attractors for (1.1) and the attractors of the homogenized equation (2.11), we can obtain the a prior estimates required, whose proofs are also omitted, by the similar arguments as those in [2] under the better initial data condition (see Section 2). Under some additional assumptions (mainly the so-called Diophantine conditions (2.21)), we have

Theorem 1.3 *Let the assumptions of Theorem 1.2, (2.1), (2.2) and the assumptions of Proposition 2.2 (see Section 2) hold. Let* $u_0 \in F \cap V$ *and let* $u^\varepsilon(x,t)$ *be the solution, defined in Theorem 1.1, of the problem (1.1),* $u^0(x,t) \in L^\infty([0,T];H) \cap L^2([0,T];V)$ *be the solution of the problem (2.11), then* $\forall\ t > 0$*, we have*

$$\|u^\varepsilon(x,t) - u^0(x,t)\|_H \le C\varepsilon^{\frac{2}{3}} e^{\beta t},$$

where the constant $C > 0$ *depends only on* $\|u_0\|_{F\cap V}$ *and* $\beta > 0$ *is a constant independent of* u^ε *and* u^0.

Theorem 1.4 *Let the assumptions of Theorem 1.3 and (2.39) hold. Let* $\mathcal{A}^\varepsilon$ *be the global attractor of the equation (1.1) and* $\mathcal{A}^0$ *be the global attractor of the homogenized equation (2.11), and define the fractional convergence rate* $k = \dfrac{2\rho}{3\rho + 3\beta}$*, then there exists a constant* $C > 0$ *such that*

$$d(\mathcal{A}^\varepsilon, \mathcal{A}^0) := \mathrm{dist}_H(\mathcal{A}^\varepsilon, \mathcal{A}^0) \le C\varepsilon^k, \quad 0 < \varepsilon \le \varepsilon_0.$$

2. The Homogenization and the Estimates of Errors

First, we study the homogenization of the problem (1.1). In addition to the assumptions (1.2)–(1.7), we assume the initial data $u_0 \in F \cap V$ and the $f(x,y,z)$ satisfies the conditions as follows:

$$f^l(x,y,z) = \sum_{j=1}^{q} b_l^j(x,y) f_{jl}(z), \quad |b_l^j(x,y)| \le C, \tag{2.1}$$

where $f^l(x,y,z)$, $l = 1,...,k$, are the components of $f(x,y,z)$. Let

$$\sum_{l=1}^{k} |\partial_z f^l(x,y,z)| \le C_1(|z|^4 + 1). \tag{2.2}$$

Recall that $w \in AP(\mathbf{R}^3)$ (the set of almost-periodic functions) possesses the mean value which can be calculated by :

$$\langle w\rangle = \langle w\rangle_x := \lim_{T\to\infty} \frac{1}{2^3 T^3} \int_{[-T,T]^3} w(x)\mathrm{d}x, \tag{2.3}$$

and the Fourier expansion as follows (see [5])

$$w(x) = \sum_{\hat{w}(\xi)\neq 0} \hat{w}(\xi) e^{i(x,\xi)}, \tag{2.4}$$

where the amplitudes $\hat{w}(\xi) \in \mathrm{C}$, $\xi \in \mathbf{R}^3$, defined by $\hat{w}(\xi) = \langle w(x)e^{-i(x,\xi)}\rangle$. We denote by $\mathrm{Trig}(\mathbf{R}^3)$ the space of all finite trigonometric polynomials of the form (2.4)

$$\mathrm{Trig}(\mathbf{R}^3) := \left\{ w(x) = \sum_{k=1}^{K} w_k e^{i(x,\xi_k)} : K \in \mathbf{N}, \xi_k \in \mathbf{R}^3, w_k \in \mathrm{C} \right\}. \tag{2.5}$$

We state a classical result in the homogenization theory:

Proposition 2.1([6, 7]) *Let $g \in W^{-1,2}(\Omega)$ and $v^\varepsilon \in V$ be the solution of the equation $A_\varepsilon v^\varepsilon = g$, where the operator A_ε is defined by* (1.3). *Then,*

$$\begin{cases} v^\varepsilon \rightharpoonup v^0 \quad \text{weakly in } V, \\ A_\varepsilon v^\varepsilon \rightharpoonup A_0 v^0 \quad \text{weakly in } H, \end{cases} \tag{2.6}$$

where $v^0 \in V$ is a unique solution of the homogenized problem $A_0 v^0 = g$. The operator A_0 is defined by the form as follows:

$$A_0^l v^{0l} = \sum_{i,j=1}^{3} \partial_{x_i}(a_{ij}^{0l} \partial_{x_j} v^{0l}), \quad A_0 v := \mathrm{diag}(A_0^1 v^1, \cdots, A_0^k v^k), \tag{2.7}$$

and the so-called homogenized coefficients $a_{ij}^{0l} = \langle a_{ij}^l(y)\rangle + \sum_{m=1}^3 \langle a_{im}^l(y)\partial_{y_m} N_m^l(y)\rangle$ are constants, where the $\mathbf{Y}$-periodic correctors $N_m^l(y)$, $m = 1, 2, 3$, $l = 1, \cdots, k$, are the solutions of the auxiliary periodic problem as follows:

$$\sum_{i,j=1}^{3} \partial_{y_i}(a_{ij}^l(y)\partial_{y_j} N_m^l(y)) = -\sum_{i=1}^{3} \partial_{y_i}(a_{im}^l(y)), \quad y \in \mathbf{R}^3. \tag{2.8}$$

And the homogenized matrix A_0 satisfies the coerciveness condition (1.4).

The following lemma, whose proof is easy and so omitted, will be used in the sequel.

Lemma 2.1 *Let Assumptions* (1.6), (2.1) *hold and $f(x, y, u^\varepsilon)$ be almost-periodic in y, assume $u^\varepsilon \to u^0$ in H ($\varepsilon \to 0$), and denote $f_0(x, u^0) := \langle f(x, y, u^0)\rangle_y$, then we have the result as follows:*

$$f(x, \varepsilon^{-1}x, u^\varepsilon) \rightharpoonup f_0(x, u^0) \quad \text{weakly in } H. \tag{2.9}$$

$$f^l(x, u^0) = \sum_{j=1}^{q} b_l^{0j}(x) f_{jl}(u^0). \tag{2.10}$$

Now by the standard homogenization theory we obtain the homogenized problem

$$\begin{cases} \partial_t u^0 = A_0 u^0 - f_0(x, u^0), \quad (x,t) \in \Omega \times \mathbf{R}^+, \\ u^0|_{\partial\Omega} = 0, \quad u^0|_{t=0} = u_0. \end{cases} \tag{2.11}$$

Note that this equation satisfies all assumptions of the equation (1.1), consequently, it admits a unique solution $u^0(x,t) \in L^\infty([0,T]; H) \bigcap L^2([0,T]; V)$ and (2.11) possesses a global attractor $\mathcal{A}^0$ in H.

We now specify additional conditions which enable us to estimate the distance between the solutions $u^\varepsilon(x,t)$ and $u^0(x,t)$ in the norm of H. In order to give the distance estimate of $u^\varepsilon(x,t)$ and $u^0(x,t)$ in H, we need three propositions (see [2, 3]).

First, we introduce some results about divergence representations. Let $h(x,y) = h(x_1,\cdots,x_3,y_1,\cdots,y_3)$ be a sufficiently smooth function which is almost-periodic in $y=(y_1,\cdots,y_3)$, ie :

(i) there exists a function $H(x,w_1,\cdots,w_3) = H(x_1,\cdots,x_3,w_{11},\cdots,w_{1k_1},\cdots,w_{31},\cdots,w_{3k_3})$ which is 2π -periodic with respect to each w_{ij}. Here $w_i = (w_{i1},\cdots,w_{ik_i}) \in \mathbf{R}^{k_i}$. $(i=1,\cdots,3)$

(ii) there exists rationally independent frequency $\alpha_{11},\cdots,\alpha_{1k_1},\cdots,\alpha_{3k_3}$ such that

$$h(x,y) = H(x_1,\cdots,x_3,\alpha_1 y,\cdots,\alpha_3 y), \tag{2.12}$$

where $\alpha_l = (\alpha_{l1},\cdots,\alpha_{lk_l})$. Let $\tilde{H}(x,w) = H(x,w) - H_0(x)$, where

$$H_0(x) = |T^k|^{-1}\int_{T^k} H(x,w_1,\cdots,w_3)\mathrm{d}w_1\cdots\mathrm{d}w_3, \tag{2.13}$$

where $T^k = T^{k_1}\times\cdots\times T^{k_3}$, and $T^{k_i} = \mathbf{R}^{k_i}/(\mathbf{Z}\cdot 2\pi)^{k_i}$ is the k_i-dimensional torus. Assume that the Fourier series

$$H(x,w) = \sum_m H_m(x)e^{im\cdot w} \tag{2.14}$$

is convergent. Let

$$\tilde{h}(x,y) = h(x,y) - H_0(x) = \sum_{m\neq 0} H_m(x)\exp\left(i\sum_{j=1}^{3} m_j\alpha_j y_j\right), \tag{2.15}$$

where $m_j = (m_{j1},\cdots,m_{jk_j}) \in \mathbf{Z}^{k_j}$, $\alpha_j \in \mathbf{R}^{k_j}$ and $y_j \in \mathbf{R}$. For any such almost periodic function $h(x,y)$, we construct a corresponding divergence representation by function $S_\sigma(x,y), \sigma = 1,\cdots,3$.

$$\tilde{h}(x,y) = \sum_{\sigma=1}^{3} \partial_{y_\sigma} S_\sigma(x,y). \tag{2.16}$$

We shall find $S_\sigma(x,y)$ of the form

$$S_\sigma(x,y) = \sum_{m\in\mathbf{Z}^k\setminus\{0\}} \eta_m^\sigma(x)\exp\left(i\sum_{j=1}^{3} m_j\alpha_j y_j\right). \tag{2.17}$$

From (2.15) – (2.17) we derive:

$$\sum_{m\neq 0} H_m(x)\exp\left(i\sum_{j=1}^{3} m_j\alpha_j y_j\right) = \tilde{h}(x,y) = \sum_{m\neq 0}\sum_{\sigma=1}^{3} m_\sigma\cdot\alpha_\sigma\eta_m(x)\exp\left(i\sum_{j=1}^{3} m_j\alpha_j y_j\right). \tag{2.18}$$

So (2.16) will hold if

$$\sum_{\sigma=1}^{3} m_\sigma \cdot \alpha_\sigma \eta_m^\sigma(x) = -iH_m(x), \tag{2.19}$$

for all $m \in \mathbf{Z}^k \backslash \{0\}$. Let the following assumptions be satisfied for some positive δ and δ':

$$\tilde{b}_l^j = \tilde{h}_l^j = \sum_{m \neq 0} H_{lm}^j(x) \exp\Big(i \sum_j m_j \alpha_j y_j\Big). \tag{2.20}$$

$$|m_\sigma \cdot \alpha_\sigma| \geq c|m_\sigma|^{-(k_\sigma - 1 + \delta)}, \quad \forall m_\sigma \in \mathbf{Z}\backslash\{0\}. \tag{2.21}$$

$$\Big\| H_{lm}^j(x) \Big\|_{C^0(\bar{\Omega})} \leq c(1 + |m_\sigma|)^{-(k_\sigma - 1 + \delta)}(1 + |m|)^{-(k+\delta')}. \tag{2.22}$$

$$\Big\| \partial_{x_\sigma} H_{lm}^j(x) \Big\|_{L^3(\Omega)} \leq c(1 + |m_\sigma|)^{-(k_\sigma - 1 + \delta)}(1 + |m|)^{-(k+\delta')}. \tag{2.23}$$

Now we can state the propositions as follows:

Proposition 2.2([3]) *Let the coefficients $b_l^j(x, y)$ of (2.1) satisfy the conditions as follows:*

(i) $b_l^j(x, y)$ *are almost-periodic in* y , $j = 1, \cdots, q$*;*

(ii) *the corresponding frequencies α_{ij} satisfy Diophantine condition* (2.21)*;*

(iii) *the coefficients $H_{lm}^j(x)$ in the series* (2.20) *of $\tilde{b}_l^j(x, y) = b_l^j(x, y) - b_l^{0j}(x)$ satisfy the decay conditions* (2.22), (2.23),

then we can represent $\tilde{b}_l^j(x, y)$ in the form

$$\tilde{b}_l^j(x, y) = \sum_{\sigma=1}^{3} \partial_{y_\sigma} S_{l\sigma}^j(x, y), \tag{2.24}$$

which satisfies

$$\Big| S_{l\sigma}^j(x, y) \Big| \leq C_0, \quad \Big\| \partial_{x_\sigma}^1 S_{l\sigma}^j(x, y) \Big\|_{L^3(\Omega)} \leq C_0, \tag{2.25}$$

here $\partial_{x_\sigma}^1$ indicates partial derivatives with respect to the first argument x of the function $S_{l\sigma}^j(x, y)$.

Proposition 2.3([3]) *Let the assumptions (1.2)-(1.7), (2.1), (2.2) and Proposition 2.2 hold. Then*

$$\Big| (f(x, \varepsilon^{-1}x, u^\varepsilon) - f_0(x, u^\varepsilon), u^\varepsilon - u^0) \Big| \leq \varepsilon C \|u^\varepsilon - u^0\|_V, \tag{2.26}$$

where the constant $C > 0$ depends only on $||u_0||_{F \cap V}$.

Denote (see [5]):

$$u_1^\varepsilon(t) = u^0(t) + \varepsilon \sum_{k=1}^{3} N_k(\varepsilon^{-1}x) \partial_{x_k} u^0(t), \tag{2.27}$$

where $N_k(\varepsilon^{-1}x)$, $k = 1,\ 2,\ 3$, are the solutions of the problem (2.8). Note that the function $u_1^\varepsilon(t)$ doesn't satisfy the 0-Dirichlet boundary condition. In order to avoid this difficulty, we introduce a family of cut-off functions $\tau^\varepsilon(x)$ satisfying two conditions as follows (see [5]): (1) $\tau^\varepsilon(x) \in C_0^\infty(\Omega)$, $0 \le \tau^\varepsilon \le 1$, $\tau^\varepsilon(x) \equiv 1$ off the ε-neighborhood of the boundary of Ω; (2) $\varepsilon|\bigtriangledown_x \tau^\varepsilon(x)| \le C$ in Ω, where the constant C is independent of ε. Thus we take

$$\begin{aligned} w^\varepsilon(t) &= u_1^\varepsilon(t) - \varepsilon(1-\tau^\varepsilon(x))\sum_{k=1}^{3} N_k(\varepsilon^{-1}x)\partial_{x_k}u^0(t) \\ &= u^0(t) + \varepsilon\tau^\varepsilon(x)\sum_{k=1}^{3} N_k(\varepsilon^{-1}x)\partial_{x_k}u^0(t). \end{aligned} \tag{2.28}$$

Then, obviously, $w^\varepsilon(t) \in V$. we need the proposition as follows:

Proposition 2.4([2]) *Let the assumption (1.4) hold, and let $w^\varepsilon(t)$, $A_\varepsilon u^\varepsilon$, A_0u^0 be defined by (2.28), (1.3), (2.7) respectively, $u^\varepsilon(t)$, $u^0(t)$ be the solution of the equation (1.1), (2.11) respectively. Then*

$$(A_\varepsilon u^\varepsilon(t) - A_0u^0(t), u^\varepsilon(t) - w^\varepsilon(t)) \le C\varepsilon^{\frac{2}{3}}\|u^0(t)\|_{2,2}^2, \tag{2.29}$$

where the constant $C > 0$ is independent of ε.

Proof of Theorem 1.3 Denote $v(x,t) = u^\varepsilon(x,t) - u^0(x,t)$. Subtracting (2.11) from (1.1) , we get

$$\partial_t v = A_\varepsilon u^\varepsilon - A_0u^0 - (f(x,\varepsilon^{-1}x,u^\varepsilon) - f_0(x,u^\varepsilon)) - (f_0(x,u^\varepsilon) - f_0(x,u^0)). \tag{2.30}$$

Multiplying both sides of (2.30) by v and integrating over Ω, we obtain

$$\begin{aligned} (\partial_t v, v) =&(A_\varepsilon u^\varepsilon - A_0u^0, v) - (f(x,\varepsilon^{-1}x,u^\varepsilon) - f_0(x,u^\varepsilon), v) \\ &- (f_0(x,u^\varepsilon) - f_0(x,u^0), v). \end{aligned} \tag{2.31}$$

To prove the theorem, we estimate each term of the right-hand side of (2.31) respectively. Using Proposition 2.4, we derive

$$\begin{aligned} \sum_{l=1}^{k}(A_\varepsilon^l u_\varepsilon^l(t) - A_0^l u^{0l}(t), v^l(t)) =& \sum_{l=1}^{k}(A_\varepsilon^l u_\varepsilon^l(t) - A_0^l u^{0l}(t), u_\varepsilon^l(t) - w^{\varepsilon l}(t)) \\ &+ \sum_{l=1}^{k}(A_\varepsilon^l u_\varepsilon^l(t) - A_0^l u^{0l}(t), v^l(t) - u_\varepsilon^l(t) + w^{\varepsilon l}(t)) \\ \le& C\varepsilon^{\frac{2}{3}}\|u^0\|_{2,2}^2 + \|A_\varepsilon u^\varepsilon - A_0u^0\|_H \cdot \|v - u^\varepsilon + w^\varepsilon\|_H. \end{aligned} \tag{2.32}$$

Note that the definitions (2.27), (2.28) and (2.5) imply the estimate

$$\|v(t) - u^\varepsilon(t) + w^\varepsilon(t)\|_H \le C\varepsilon\|u^0(t)\|_V. \tag{2.33}$$

Similar methods as in [2] for the equation (1.1) and (2.11) yield

$$\begin{aligned} &\int_T^{T+1}\|A_\varepsilon u^\varepsilon(t)\|_H^2 \mathrm{d}t + \int_T^{T+1}\|A_0u^0(t)\|_H^2 \mathrm{d}t + \int_T^{T+1}\|u^0(t)\|_V^2 \mathrm{d}t + \int_T^{T+1}\|u^0(t)\|_{2,2}^2 \mathrm{d}t \\ &\le Q(\|u_0\|_{F\cap V}), \end{aligned} \tag{2.34}$$

for the appropriate function Q independent of $T \geq 0$ (here we have implicitly used the elliptic regularity estimate $\|u^0\|_{2,2} \leq C\|A_0 u^0\|_H$). Inserting the estimate (2.33) to (2.32) and integrating over $t \in [0,T]$ then taking the estimate (2.34) into account, we have

$$\sum_{l=1}^{k} \int_0^T \left(A_\varepsilon^l u_\varepsilon^l(t) - A_0^l u_0^l(t), v^l \right) \mathrm{d}t \leq \varepsilon^{\frac{2}{3}} Q(\|u_0\|_{F\cap V})T. \tag{2.35}$$

Applying (2.26) to the second term of the right-hand side of (2.31), integrating over $t \in [0,T]$, using Minkowski-inequality and (2.34), we obtain

$$\int_0^T |f(x, \varepsilon^{-1}x, u^\varepsilon(t)) - f_0(x, u^\varepsilon(t)), v(t)|\mathrm{d}t \leq \varepsilon Q_1(\|u_0\|_{F\cap V})T. \tag{2.36}$$

Assumption (2.2) implies

$$\begin{aligned}\int_0^T |(f_0(x,u^\varepsilon) - f_0(x,u^0), v)|\mathrm{d}t &= \int_0^T \left| \left(\int_0^1 f'(su^\varepsilon + (1-s)u^0)\mathrm{d}s \cdot v, v \right) \right| \mathrm{d}t \\ &\leq C_2 \int_0^T \|v\|_H^2 \mathrm{d}t. \end{aligned} \tag{2.37}$$

Integrating (2.31) over $t \in [0,T]$ and taking account of (2.35)-(2.37), we get

$$\|v(T)\|_{0,2}^2 \leq \varepsilon^{\frac{2}{3}} Q(\|u_0\|_{F\cap V})T + 2\varepsilon Q_1(\|u_0\|_{F\cap V})T + 2C_2 \int_0^T \|v(T)\|_H^2 \mathrm{d}t. \tag{2.38}$$

Applying Gronwall's inequality to (2.38) proves Theorem 1.3 .

Now we are ready to derive the error's estimates for the global attractors $\mathcal{A}^\varepsilon$ and $\mathcal{A}^0$. To this end, we need some additional information about $\mathcal{A}^0$ which we in fact require to be exponentially attracting with exponential rate $\rho > 0$. We assume there exists a constant $C = C(\varepsilon_0)$ such that for all $t \geq 0$

$$d := \mathrm{dist}_H(u^0, \mathcal{A}^0) \leq Ce^{-\rho t}, \tag{2.39}$$

holds, uniformly for all $u_0 \in \bigcup_{0<\varepsilon\leq\varepsilon_0} \mathcal{A}^\varepsilon$, where $dist_H$ means the nonsymmetric Hausdorff distance (see [4]), i.e.

$$\mathrm{dist}_H(A,B) := \sup_{x\in A} \inf_{y \in B} \|x-y\|_H. \tag{2.40}$$

Proof of Theorem 1.4 Let

$$\mathcal{B} := \bigcup_{0<\varepsilon\leq\varepsilon_0} \mathcal{A}^\varepsilon. \tag{2.41}$$

Pick $0 < \varepsilon \leq \varepsilon_0$ and $u^\varepsilon \in \mathcal{A}^\varepsilon \subset \mathcal{B}$, arbitrarily. For $t \geq 0$ chosen below consider $u_0 \in \mathcal{A}^\varepsilon$ such that

$$S_t^\varepsilon u_0 = u^\varepsilon. \tag{2.42}$$

Then Theorem 1.3 and (2.39) imply

$$d(u^\varepsilon, \mathcal{A}^0) \le d(u^\varepsilon, u^0) + d(u^0, \mathcal{A}^0) \le C\varepsilon^{\frac{2}{3}} e^{\beta t} + Ce^{-\rho t}. \tag{2.43}$$

Choose $t \ge 0$, such that $\varepsilon^{\frac{2}{3}} e^{\beta t} = e^{-\rho t}$, thus $t = -\dfrac{\ln \varepsilon}{\beta + \rho}$. Substituting this choice of t back into (2.43), because of the arbitrariness of u^ε, we prove Theorem 1.4.

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