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Non-Existence of Global Solutions for a Fractional Wave-Diffusion Equation

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Abstract. We considered the Cauchy problem for the fractional wave-diffusion equation

$$D^{\alpha}u-\Delta|u|^{m-1}u+(-\Delta)^{\beta/2}D^{\gamma}|u|^{l-1}u=h(x,t)|u|^{p}+f(x,t)$$

with given initial data and where $p>1$, $1<\alpha<2$, $0<\beta<2$, $0<\gamma<1$. Nonexistence results and necessary conditions for global existence are established by means of the test function method. This results extend previous works.

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1 Introduction

In [1], Kirane and Tatar consider the Cauchy problem of the hyperbolic fractional equation

$$u_{tt}-\Delta u+D^{\beta}u=h(x,t)|u|^{p}, \tag{1.1}$$

where $p>1$ and $0<\beta<1$, this equation arises in the modeling of fast wave propagation in micro-inhomogeneous media see (see [2]). In [1], the authors established conditions on the initial data and the function $h(x,t)$ that are necessary for local and global existence. It is shown that if

$$1<p\leq 1+\frac{2\beta+\rho}{2+N-2\beta},$$

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(where ρ comes from the function h) then we have non-existence of global solutions.

When $m=l=1$, $\alpha=2$, $\beta=0$, $h=1$ and $\gamma=1$, this problem has been treated by a large number of researchers. Then we obtain the wave equation with the linear damping u_t. In this case Todorova and Yordanov [3], Mitidieri and Pohozahev [4] and Zhang [5] showed that the Fujita exponent is $p_c=1+2/N$. This result has been extended to solutions of the telegraph equation

$$D^{2\beta}u-\Delta u+D^{\beta}u=0,$$

by Cascaval et al. [6] this problem arises while studying some iterated Brownian motions (see [7]). We point out here that fractional derivatives serve, among other things, to model various anomalous damping such as noise attenuation and viscoelastic dissipations (see [8–12]). Indeed it has been shown by experiments (see [13]) that experiment data fit very well in the models involving fractional derivatives within a broad frequency range for several materials. This materials include synthetic polymers, electrochemistry, glassy materials and many other viscoelastic and hereditary mechanics.

In this paper, we consider the problem

$$\begin{cases} D^{\alpha}u-\Delta|u|^{m-1}u+(-\Delta)^{\beta/2}D^{\gamma}|u|^{l-1}u=h(x,t)|u|^p+f(x,t), \\ u(x,0)=u_0(x)\geq 0,\ \ u_t(x,0)=u_1(x)\geq 0, \qquad x\in \mathbb{R}^N. \end{cases} \tag{1.2}$$

We will generalize the results in [1] to problem (1.2) where $1<\alpha\leq 2$, $0<\beta<2$ and $0<\gamma<1$. Nonexistence results as well as necessary conditions for local and global existence will be established. In addition to this we can look at the equation in problem (1.2) as a generalization of the fractional diffusion-wave equation

$$D^{\alpha}u=\Delta u+h(x,t)|u|^p, \qquad 1<\alpha\leq 2. \tag{1.3}$$

This eq. (1.3) is now a special case of the eq. (1.2), we can consider the eq. (1.2) as the fractionally damped equation of (1.3). Eq. (1.3) serves as a model in the study of the thermal diffusion in fractal media. See Saichev and Zaslavsky [14], Mainardi [10, 11], Fujita [15] and references therein. Molz et al. in [16] discuss a physical interpretation of the fractional derivative in a Levy diffusion process. Our argument is based on the test-function method developed by Mitidieri and Pohozaev [4], Zhang [5] Kirane and Tatar [1] and others. The necessary conditions results are inspired by some arguments due to Baras and Kersner [17].

Now, we present two different definitions of fractional derivatives (see [13, 18]).

We define the fractional derivative in the Caputo sense of power μ by

$${}^{C}D^{\mu}_{+}u(t):=\frac{1}{\Gamma(n-\mu)}\int_0^t(t-\tau)^{n-\mu-1}u^{(n)}(\tau)\mathrm{d}\tau, \qquad n-1<\mu<n.$$

The fractional derivative in the Riemann-Liouville sense is given by

$${}^{RL}D^{\mu}_{+}u(t):=\frac{1}{\Gamma(n-\mu)}(\frac{\mathrm{d}}{\mathrm{d}t})^n\int_0^t(t-\tau)^{n-\mu-1}u(\tau)\mathrm{d}\tau, \qquad t>0.$$

The relationship between the Caputo derivative and the (left-handed) Riemann-Liouville derivative is given by the formula

$$ {}^{RL}D_+^{\mu}u(t)=\sum_{k=0}^{n-1}\frac{u^{(k)}(0)t^{k-\mu}}{\Gamma(1+k-\mu)}+\frac{1}{\Gamma(n-\mu)}\int_0^t(t-\tau)^{n-\mu-1}u^{(n)}(\tau)\mathrm{d}\tau, $$

$$ {}^{RL}D_+^{\mu}u(t)=\sum_{k=0}^{n-1}\frac{u^{(k)}(0)t^{k-\mu}}{\Gamma(1+k-\mu)}+{}^{C}D_+^{\mu}u(t). $$

For $T>0$, we define the right-handed Riemann-Liouville fractional derivative by

$$ {}^{RL}D_-^{\mu}u(t):=\frac{(-1)^n}{\Gamma(n-\mu)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\int_t^T(\tau-t)^{n-\mu-1}u(\tau)\mathrm{d}\tau, \qquad n-1<\mu\le n. $$

We have also the formula integration by parts (see [13], p.46).

$$ \int_0^T f(t)(D_{0|t}^{\alpha}g)(t)\mathrm{d}t=\int_0^T g(t)(D_{t|T}^{\alpha}f)(t)\mathrm{d}t, \qquad 0<\alpha<1. $$

2 Main results

The function $h(t,x)$ is assumed to be nonnegative and satisfies $h(\tau R^{4/\alpha},yT^{4/\mu})=R^{4\sigma/\alpha}\times T^{4\lambda/\mu}h(\tau,y)$ for some positive constants μ, σ, λ will be determined later and for R and T large.

Set $\rho:=(4\sigma/\alpha)+(4\lambda/\mu)$. Let us make clear first what we mean by a solution to problem (FDE). Q_T here will denote the set $Q_T:=(0,T)\times\mathbb{R}^N$, $L^p_{loc}(Q_T,h\mathrm{d}t\mathrm{d}x)$ will denote the space of all functions $v:\mathbb{R}_+\times\mathbb{R}^N\to\mathbb{R}$ such that $\int_K|v|^p h\mathrm{d}t\mathrm{d}x<\infty$ for any compact K in $\mathbb{R}_+\times\mathbb{R}^N$.

Definition 2.1. *A function u is a local weak solution to* (1.2) *defined on Q_T, $0<T<+\infty$, if $u\in L^p_{loc}(Q_T)$ such that*

$$ \begin{aligned} &\int_{Q_T}\varphi h|u|^p+\int_{Q_T}\varphi f(x,t)+\int_{\mathbb{R}^n}u_0D_-^{\alpha-1}\varphi(0)+\int_{Q_T}u_1D_-^{\alpha-1}\varphi \\ &=-\int_{Q_T}uD_-^{\alpha}\varphi-\int_{Q_T}|u|^{m-1}u\Delta\varphi+\int_{Q_T}|u|^{l-1}uD_-^{\gamma}(-\Delta)^{\beta/2}\varphi, \end{aligned} \tag{2.1} $$

for any test function $\varphi\in C^{2,2}_{x,t}(\mathbb{R}^n\times[0,T])$, such that $\varphi\ge 0$, $\varphi(T,x)=\varphi_t(T,x)=0$.

Now, we are in position to announce our results.

Theorem 2.1. *Let $p>1$ be such that $l/(1-\gamma)>p>\max(m,l)\geq 1$ and $\int_{\mathbb{R}^N\times\mathbb{R}_+} f(t,x)\mathrm{d}t\mathrm{d}x>0$. If*

$$1\leq N\leq \min\Big\{\Big[4\frac{p}{p-1}-\frac{4}{\alpha}+\rho\Big(\frac{p}{p-1}-1\Big)\Big]\frac{\alpha[2(p-l)-\beta(p-m)]}{4\gamma(p-m)(p-1)},$$
$$\frac{2p}{(p-m)}-\frac{1}{\gamma}\Big(\frac{2(p-l)}{(p-m)}-\beta\Big)+\frac{\alpha}{4\gamma}\Big(\frac{2(p-l)}{(p-m)}-\beta\Big)\rho\Big(\frac{p}{p-l}-1\Big),$$
$$\frac{2p}{(p-m)}-\frac{1}{\gamma}\Big(\frac{2(p-l)}{(p-m)}-\beta\Big)+\frac{\alpha}{4\gamma}\Big(\frac{2(p-l)}{(p-m)}-\beta\Big)\rho\Big(\frac{p}{p-m}-1\Big)\Big\}.$$

Then, problem (1.2) *does not admit global nontrivial solutions in time.*

Proof. The proof is by contradiction. So we assume that the solution is global.

Let $\Phi\in C_0^2(\mathbb{R}_+)$, $\Phi\geq 0$, Φ decreasing such that

$$\Phi(y):=\begin{cases}1, & \text{if } 0\leq y\leq 1,\\ 0, & \text{if } y\geq 2,\end{cases}$$

and $0\leq\Phi\leq 1$. We choose

$$\varphi(x,t):=\Phi\Big(\frac{t^\alpha+|x|^\mu}{R^4}\Big),$$

where R is a positive real number. The test function φ is chosen so that

$$\int_{Q_T}(h\varphi)^{1-q}|D_-^\alpha\varphi|^{\frac{p}{p-1}}<\infty,\qquad \int_{Q_T}(h\varphi)^{\frac{-m}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}}<\infty,$$
$$\int_{Q_T}(h\varphi)^{\frac{-l}{p-l}}|(-\Delta)^{\beta/2}D_-^\gamma\varphi|^{\frac{p}{p-l}}<\infty.$$

In order to estimate the right hand side of (2.1) on $Q_{TR^{4/\alpha}}$, we write by using the ε-Young inequality

$$\int_{Q_{TR^{4/\alpha}}}|u||D_-^\alpha\varphi|\leq\varepsilon\int_{Q_{TR^{4/\alpha}}}|u|^p h\varphi+C_\varepsilon\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-1}{p}}|D_-^\alpha\varphi|^{\frac{p}{p-1}}.\tag{2.2}$$

Similarly,

$$\int_{Q_{TR^{4/\alpha}}}|u|^l|D_-^\gamma(-\Delta)^{\beta/2}\varphi|\leq\varepsilon\int_{Q_{TR^{4/\alpha}}}|u|^p h\varphi+C_\varepsilon\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-l}{p-l}}|D_-^\gamma(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-l}},\tag{2.3}$$
$$\int_{Q_{TR^{4/\alpha}}}|u|^m|\Delta\varphi|\leq\varepsilon\int_{Q_{TR^{4/\alpha}}}|u|^p h\varphi+C_\varepsilon\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-m}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}}.\tag{2.4}$$

Gathering up, (2.2)-(2.4), with ε small enough, we infer that

$$\int_{Q_{TR^{4/\alpha}}}\varphi f+\int_{\mathbb{R}^n}u_0D_-^{\alpha-1}\varphi(0)+\int_{Q_{TR^{4/\alpha}}}u_1D_-^{\alpha-1}\varphi\tag{2.5}$$
$$\leq C_\varepsilon\Big(\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-1}{p}}|D_-^\alpha\varphi|^{\frac{p}{p-1}}+\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-m}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}}+\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-l}{p-l}}|D_-^\gamma(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-l}}\Big),$$

for some positive constant C_ε. Set $\Omega:=\left\{(\tau,y)\in\mathbb{R}_+\times\mathbb{R}^N;\ \tau^\alpha+|y|^\mu\leq 2\right\}$. Therefore, writing

$$\varphi(t,x)=\varphi\left(\tau R^{\frac{4}{\alpha}},R^{\frac{4}{\mu}}y\right):=\chi(\tau,y),$$
$$t=R^{\frac{4}{\alpha}}\tau,\qquad x=R^{\frac{4}{\mu}}y,\qquad \mathrm{d}x\mathrm{d}t=R^{\frac{4}{\mu}N+\frac{4}{\alpha}}\mathrm{d}y\mathrm{d}\tau,$$

we have

$$\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-1}{p-1}}|D^\alpha_{t|T}\varphi|^{\frac{p}{p-1}}\mathrm{d}x\mathrm{d}t\leq R^{-\alpha\frac{4}{\alpha}q+\frac{4}{\mu}N+\frac{4}{\alpha}+\rho(1-q)}\int_\Omega(h\chi)^{1-\frac{p}{p-1}}|D^\alpha_-\chi|^{\frac{p}{p-1}},$$
$$\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-m}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}}\mathrm{d}x\mathrm{d}t\leq R^{-\frac{8p}{(p-m)\mu}+\frac{4}{\mu}N+\frac{4}{\alpha}+\rho(1-\frac{p}{p-m})}\int_\Omega(h\chi)^{1-\frac{p}{p-m}}|\Delta\chi|^{\frac{p}{p-m}},$$
$$\int_{Q_{TR^{4/\alpha}}}(h\varphi)^{\frac{-l}{p-l}}|D^\gamma_-(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-l}}$$
$$\leq R^{-(\beta\frac{4}{\mu}+\gamma\frac{4}{\alpha})\frac{p}{p-l}+\frac{4}{\mu}N+\frac{4}{\alpha}+\rho(1-\frac{p}{p-l})}\int_\Omega(h\chi)^{\frac{-l}{p-l}}|D^\gamma_-(-\Delta)^{\beta/2}\chi|^{\frac{p}{p-l}}.$$

Now, we choose μ so that

$$\frac{8p}{(p-m)\mu}=\left(\beta\frac{4}{\mu}+\gamma\frac{4}{\alpha}\right)\frac{p}{p-l},$$

we get

$$\mu=\frac{\alpha}{\gamma}\left(\frac{2(p-l)}{(p-m)}-\beta\right)>0,\qquad \text{for } \beta<\frac{2(p-l)}{(p-m)}.$$

Now taking ε small enough, we obtain the estimate

$$\int_{Q_{TR^{4/\alpha}}}h|u|^p\leq R^\omega\left(\int_\Omega(h\chi)^{1-p'}\left(|D^\alpha_-\chi|^{\frac{p}{p-1}}+|\Delta\chi|^{\frac{p}{p-m}}+|D^\gamma_-(-\Delta)^{\beta/2}\chi|^{\frac{p}{p-l}}\right)\right),\tag{2.6}$$

where

$$\omega:=\max\left\{-4\frac{p}{p-1}+\frac{4\gamma(p-m)(p-1)}{\alpha[2(p-l)-\beta(p-m)]}N+\frac{4}{\alpha}+\rho\left(1-\frac{p}{p-1}\right),\right.$$
$$\left.-\frac{8p}{(p-m)\mu}+\frac{4}{\mu}N+\frac{4}{\alpha}+\rho\left(1-\frac{p}{p-m}\right),\ -\frac{8p}{(p-m)\mu}+\frac{4}{\mu}N+\frac{4}{\alpha}+\rho\left(1-\frac{p}{p-l}\right)\right\}.$$

In the estimate (2.6), we have to distinguish two cases:

Either $\omega<0$, that is $p<p_c$: In this case, passing to the limit as $R\to\infty$ in (2.6), we obtain

$$\lim_{R\to\infty}\left\{\int_{Q_{TR^{4/\alpha}}}\varphi f+\int_{Q_{TR^{4/\alpha}}}h\varphi|u|^p\right\}=\int_{\mathbb{R}^N\times\mathbb{R}_+}f+\int_{\mathbb{R}^N\times\mathbb{R}_+}h|u|^p\leq 0.$$

This contradicts the requirement $\int_{\mathbb{R}^N\times\mathbb{R}_+}f>0$.

Or $\omega=0$ (i. e. $p=p_c$), in this case, we modify the test function by introducing a new parameter $0<S<R$.

$$\varphi(x,t):=\Phi\left(\frac{t^\alpha}{(SR)^4}+\frac{|x|^\mu}{R^4}\right),$$

let us perform the change of variables $t=(RS)^{4/\alpha}\tau$, $x=R^{4/\mu}y$. Moreover, we obtain via (2.6)

$$\int_{\mathbb{R}^N\times R_+} h|u|^p\leq C. \tag{2.7}$$

Observe that because of the convergence of the integral in (2.7) if

$$C_{RS}=\left\{(x,t):\ R^4\leq\frac{t^\alpha}{S^4}+|x|^\mu\leq 2R^4\right\},$$

then

$$\lim_{R\to\infty}\int_{C_{RS}} h|u|^p\varphi=0. \tag{2.8}$$

Using the estimates (2.1), (2.2) and (2.4), we may write

$$\begin{aligned}&\int_{Q_{T(RS)^{4/\alpha}}} f(t,x)+(1-2\varepsilon)\int_{Q_{T(RS)^{4/\alpha}}} h|u|^p\varphi\\ \leq& C(\varepsilon)S^{-\frac{4p}{p-1}+\frac{4}{\alpha}-\frac{4\sigma}{\alpha(p-1)}}\int_{\Omega_1}(h\chi)^{\frac{-1}{p-1}}|D^\alpha_-\chi|^{\frac{p}{p-1}}+\left(\int_{C_{RS}} h|u|^p\varphi\right)^{\frac{m}{p}}\left(\int_{\Omega_1}(h\chi)^{\frac{-m}{p-m}}|\Delta\chi|^{\frac{p}{p-m}}\right)^{1-\frac{m}{p}}\\ &+C(\varepsilon)S^{-\frac{4\gamma p}{(p-l)\alpha}+\frac{4}{\alpha}-\frac{l}{p-l}\frac{4\sigma}{\alpha}}\int_{\Omega_1}(h\chi)^{\frac{-l}{p-l}}|D^\gamma_-(-\Delta)^{\beta/2}\chi|^{\frac{p}{p-l}},\end{aligned} \tag{2.9}$$

where

$$\Omega_1:\ =\{(y,\tau):1\leq\tau^\alpha+|y|^\mu\leq 2\}.$$

Thus, passing to the limit in (2.9) as $R\to\infty$, and taking account of (2.8), we obtain

$$\begin{aligned}&\int_{\mathbb{R}^N\times\mathbb{R}_+} f+\int_{\mathbb{R}^N\times\mathbb{R}_+} h|u|^p\\ \leq& C(\varepsilon)S^{-\frac{4p}{p-1}+\frac{4}{\alpha}-\frac{4\sigma}{\alpha(p-1)}}\int_{\Omega_1}(h\chi)^{\frac{-1}{p-1}}|D^\alpha_-\chi|^{\frac{p}{p-1}}\\ &+C(\varepsilon)S^{-\frac{4\gamma p}{(p-l)\alpha}+\frac{4}{\alpha}-\frac{l}{p-l}\frac{4\sigma}{\alpha}}\int_{\Omega_1}(h\chi)^{\frac{-l}{p-l}}|D^\gamma_-(-\Delta)^{\beta/2}\chi|^{\frac{p}{p-l}}.\end{aligned} \tag{2.10}$$

Then, taking the limit when S goes to infinity in (2.10) because the left hand side is independent of S, we obtain $u=0$, which is contradiction with the fact that $\int_{\mathbb{R}^N\times\mathbb{R}_+} f>0$. The proof is complete. □

Remark 2.1. Observe that when $m=l=1$, $\alpha\to 2$, $\beta=0$ and the critical exponent is $p_c=1+(\rho+2\gamma)/(2+N-2\gamma)$. This in agreement with the one found in Kirane-Tatar [1].

Remark 2.2. When $m=l=1$, $\alpha\to 2$, $\beta=0$ and $\gamma\to 1$, the critical exponent is $p_c=1+2/N$ (Todorova-Yordanov [3]).

Remark 2.3. When $m=l=1$, $\alpha\to 1$, $\beta=0$ and $\gamma\to 1$ the critical exponent coincides with with the well know Fujita exponent $p_c=1+2/N$ (Fujita [19]).

3 Necessary conditions for local and global existence

In this section, we assume that $\inf_{t\in R^+} h(t,x)>0$ and $\inf_{t\in R^+} f(t,x)>0$. Our first results in this section are the following

Theorem 3.1. *Let u be a local solution to* (2.1) *where $T<+\infty$, m, $l>1$ and $p>\max(m,l)$, assume that $\int_{\mathbb{R}^N} u_0>0$. Then, there exist constants K_1, K_2 and K_3 such that*

$$\liminf_{|x|\to+\infty}\left(u_0(x)\min\left((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\right)\right)\le K_1T^{\alpha(1-q)},$$
$$\liminf_{|x|\to+\infty}\left(u_1(x)\min\left((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\right)\right)\le K_2T^{\alpha-1-\frac{\alpha p}{p-1}},$$
$$\liminf_{|x|\to+\infty}\left((\inf_{t\in\mathbb{R}_+} f)\min\left((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\right)\right)\le K_3T^{-\frac{\alpha p}{p-1}}.$$

Proof. Let $\Phi\in H^\beta(\Omega)$, $\Phi\ge 0$, be such that$-\Delta\Phi=k'\Phi$, in $\Omega=\{1<|x|<2\}$, $\Phi=0$ in $\partial\Omega$ and $(-\Delta)^{\beta/2}\Phi=k\Phi$, for some positive constants k, k'.

We consider

$$\varphi(x,t)=\Phi\left(\frac{x}{R}\right)\left(1-\frac{t^2}{T^2}\right)^{2q},\qquad q=\frac{p}{p-1}. \tag{3.1}$$

By the definition of the weak solution, we have

$$\begin{aligned}&\int_Q\varphi f+\int_{\Omega_R}u_0D_-^{\alpha-1}\varphi(0)+\int_Q u_1D_-^{\alpha-1}\varphi\\ &\le C_\varepsilon\left(\int_Q(h\varphi)^{\frac{-1}{p}}|D_-^\alpha\varphi|^{\frac{p}{p-1}}+\int_Q(h\varphi)^{\frac{-m}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}}+\int_Q(h\varphi)^{\frac{-l}{p-l}}|D_-^\gamma(-\Delta)^{\beta/2}\varphi|^{\frac{p}{p-l}}\right).\end{aligned} \tag{3.2}$$

Here $\Omega_R:=\{R<|x|<2R\}$, $Q:=[0,T]\times\Omega_R$. We set

$$\mathcal{A}=\int_Q(h\varphi)^{1-\frac{p}{p-1}}|D_-^\alpha\varphi|^{\frac{p}{p-1}},$$
$$\mathcal{B}=\int_Q(h\varphi)^{\frac{-m}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}},$$
$$\mathcal{C}=\int_Q(h\varphi)^{\frac{-l}{p-l}}|(-\Delta)^{\beta/2}D_-^\gamma\varphi|^{\frac{p}{p-l}}.$$

It is clear, from our choice of φ that the requirements

$$\varphi(x,T)=\varphi_t(x,0)=\varphi_t(x,T)=0, \tag{3.3}$$

are satisfied. We remark from (3.3) that

$$^{RL}D^{\alpha}_{t|T}\varphi = {}^{C}D^{\alpha}_{t|T}\varphi \quad \text{and} \quad {}^{RL}D^{\alpha-1}_{t|T}\varphi = {}^{C}D^{\alpha-1}_{t|T}\varphi.$$

Now, we estimate $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in terms of T and R.

Let us making use the change of variables $t=T\tau$. Using this and the assumptions on φ, we find

$$\begin{aligned}\mathcal{B}&=\int_Q (h\varphi)^{1-\frac{p}{p-m}}|\Delta\varphi|^{\frac{p}{p-m}}\\&\le CTR^{-\frac{2p}{p-m}}\int_{\Omega_R} h^{1-\frac{p}{p-m}}\Phi\Big(\frac{x}{R}\Big).\end{aligned} \tag{3.4}$$

For the term $\mathcal{A}$, it is easy to see that

$$\begin{aligned}\mathcal{A}&=\int_Q (h\varphi)^{1-\frac{p}{p-1}}|D^{\alpha}_{-}\varphi|^{\frac{p}{p-1}}\\&=\int_Q h^{1-q}\Big(1-\frac{t^2}{T^2}\Big)^{2q(1-q)}\Phi\left|D^{\alpha}_{-}\Big(1-\frac{t^2}{T^2}\Big)^{2q}\right|^{q}.\end{aligned}$$

Now, we compute $D^{\alpha}_{-}(1-t^2/T^2)^{2q}$ and obtain

$$\begin{aligned}&D^{\alpha}_{-}\Big(1-\frac{t^2}{T^2}\Big)^{2q}\\&=\frac{-4q}{T^2\Gamma(2-\alpha)}\int_t^T\left[\Big(1-\frac{\sigma^2}{T^2}\Big)^{2q-1}-2\frac{\sigma^2}{T^2}(2q-1)\Big(1-\frac{\sigma^2}{T^2}\Big)^{2q-2}\right](\sigma-t)^{1-\alpha}\mathrm{d}\sigma,\end{aligned}$$

and set

$$\begin{aligned}I&\equiv\frac{-4qT^{-4q}}{\Gamma(2-\alpha)}\int_t^T (T^2-\sigma^2)^{2q-1}(\sigma-t)^{1-\alpha}\mathrm{d}\sigma,\\J&\equiv\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\int_t^T \sigma^2(T^2-\sigma^2)^{2q-2}(\sigma-t)^{1-\alpha}\mathrm{d}\sigma.\end{aligned}$$

Using the Euler's change of variable

$$y=\frac{\sigma-t}{T-t}\Rightarrow \sigma-t=(T-t)y, \qquad 0<y\le 1,$$

we see that

$$1-y=\frac{T-\sigma}{T-t} \qquad \text{and} \qquad 1-y^2=\frac{T^2-\sigma^2}{(T-t)^2}-2t\frac{1-y}{T-t},$$
$$T^2-\sigma^2=(1-y^2)(T-t)^2+2t(1-y)(T-t).$$

Therefore

$$I=-\frac{4qT^{-4q}}{\Gamma(2-\alpha)}(T-t)^{1-\alpha+2q}\int_0^1(1-y)^{2q-1}((T-t)(1+y)+2t)^{2q-1}y^{1-\alpha}\mathrm{d}y.$$

Observe that

$$(T-t)(1+y)+2t=(T+t)+y(T-t),$$

and

$$y(T-t)<(T-t)\le(T+t), \quad \text{for} \quad y<1, \quad t\ge 0.$$

Then applying the Binomial formula for non integer power to the term

$$((T-t)(1+y)+2t)^{2q-1},$$

we find that

$$I=\frac{-4qT^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-1}(T-t)^{1-\alpha+2q+k}(T+t)^{2q-k-1}\int_0^1(1-y)^{2q-1}y^{1-\alpha+k}\mathrm{d}y,$$

where

$$C_k^{2q-1}=\frac{(2q-1)(2q-2)\cdots(2q-k)}{k\times(k-1)\times(k-2)\times\cdots\times 3\times 2\times 1}.$$

Using the formula

$$\int_0^1(1-\tau)^{u-1}\tau^{v-1}\mathrm{d}\tau=\frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \qquad u,\, v>0, \tag{3.5}$$

we obtain

$$I=\frac{-4qT^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-1}(T-t)^{1-\alpha+2q+k}(T+t)^{2q-k-1}\frac{\Gamma(2q)\Gamma(2-\alpha+k)}{\Gamma(2q+2-\alpha+k)}.$$

Similarly, for J

$$J=\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\int_0^1(t+(T-t)y)^2(1-y)^{2q-2}((T+t)+y(T-t))^{2q-2}(T-t)^{-\alpha+2q}y^{1-\alpha}\mathrm{d}y.$$

Developing in entire series

$$J=\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\int_0^1 (t+(T-t)y)^2(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q}\times(1-y)^{2q-2}y^{k+1-\alpha}\mathrm{d}y.$$

Now writing J as following

$$\begin{aligned}J=&\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{+\infty}C_k^{2q-2}\bigg[t^2(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q}\int_0^1(1-y)^{2q-2}y^{k+1-\alpha}\mathrm{d}y\\&+2t(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+1}\int_0^1(1-y)^{2q-2}y^{k+2-\alpha}\mathrm{d}y\\&+(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+2}\int_0^1(1-y)^{2q-2}y^{k+3-\alpha}\mathrm{d}y\bigg].\end{aligned}$$

The formula (3.5) yields

$$\begin{aligned}J=&\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\bigg[t^2(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q}\frac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}\\&+2t(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+1}\frac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\\&+(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+2}\frac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\bigg].\end{aligned}$$

Consequently,

$$\begin{aligned}&D_{t|T}^{\alpha}\left(1-\frac{t^2}{T^2}\right)^{2q}\\=&\frac{-4qT^{-4q}}{\Gamma(2-\alpha)}\left[\sum_{k=0}^{\infty}C_k^{2q-1}(T-t)^{1-\alpha+2q+k}\times(T+t)^{2q-k-1}\frac{\Gamma(l)\Gamma(2-\alpha+r)}{\Gamma(l+2-\alpha+r)}\right]\\&+\frac{8q(2q-1)T^{-4q}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\bigg[t^2(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q}\frac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}\\&+2t(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+1}\frac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\\&+(T+t)^{2q-2-k}(T-t)^{k-\alpha+2q+2}\frac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\bigg].\end{aligned}$$

By setting $t=\tau T$, we find

$$
\begin{aligned}
&D^{\alpha}_{t|T}\left(1-\frac{t^2}{T^2}\right)^{2q}\\
=&\frac{-4qT^{-\alpha}}{\Gamma(2-\alpha)}\left[\sum_{k=0}^{\infty}C_k^{2q-1}(1-\tau)^{1-\alpha+2q+k}\times(1+\tau)^{2q-k-1}\frac{\Gamma(l)\Gamma(2-\alpha+r)}{\Gamma(l+2-\alpha+r)}\right]\\
&+\frac{8q(2q-1)}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_k^{2q-2}\left[\tau^2(1+\tau)^{2q-2-k}(1-\tau)^{k-\alpha+2q}\frac{\Gamma(2q-1)\Gamma(2-\alpha+k)}{\Gamma(2q+1-\alpha+k)}\right.\\
&+2\tau(1+\tau)^{2q-2-k}(1-\tau)^{k-\alpha+2q+1}\frac{\Gamma(2q-1)\Gamma(3-\alpha+k)}{\Gamma(2q+2-\alpha+k)}\\
&\left.+(1+\tau)^{2q-2-k}(1-\tau)^{k-\alpha+2q+2}\frac{\Gamma(2q-1)\Gamma(4-\alpha+k)}{\Gamma(2q+3-\alpha+k)}\right].
\end{aligned}\tag{3.6}
$$

Thus

$$
D^{\alpha}_{t|T}\left(1-\frac{t^2}{T^2}\right)^{2q}\le\frac{C_1}{\Gamma(2-\alpha)}T^{-\alpha},\tag{3.7}
$$

where C_1 is constant depending of q and α. Using (3.7) we gives

$$
\begin{aligned}
&\int_Q(h\varphi)^{1-\frac{p}{p-1}}|D^{\alpha}_{-}\varphi|^{\frac{p}{p-1}}\\
\le&\frac{C_1T^{1-\frac{\alpha p}{p-1}}}{\Gamma(2-\alpha)}\int_{\Omega_R}(\inf_{t\in R_+}h)^{1-\frac{p}{p-1}}\Phi\left(\frac{x}{R}\right)\int_0^1(1-\tau)^{2q(1-\frac{p}{p-1})}\tau^{2q(1-\frac{p}{p-1})}\mathrm{d}\tau.
\end{aligned}
$$

Consequently,

$$
\mathcal{A}\le C_2T^{1-\frac{\alpha p}{p-1}}\int_{\Omega_R}(\inf_{t\in R_+}h)^{1-\frac{p}{p-1}}\Phi,\tag{3.8}
$$

where

$$
C_2=\frac{C_1\Gamma(2q(1-\frac{p}{p-1})+1)^2}{\Gamma(2-\alpha)\Gamma(4q(1-\frac{p}{p-1})+2)}.
$$

Similarly we compute $D^{\alpha-1}_{t|T}(1-\frac{t^2}{T^2})^{2q}$:

$$
D^{\alpha-1}_{t|T}\left(1-\frac{t^2}{T^2}\right)^{2q}=\frac{-1}{\Gamma(2-\alpha)}\int_t^T(\sigma-t)^{1-\alpha}\left(\left(\frac{T^2-\sigma^2}{T^2}\right)^{2q}\right)'\mathrm{d}\sigma.
$$

We set

$$
I:=\int_t^T\sigma(T^2-\sigma^2)^{2q-1}(\sigma-t)^{1-\alpha}\mathrm{d}\sigma,
$$

or

$$I=(T-t)^{2q-\alpha+1}\int_t^T ((T-t)y+t)(1-y)^{2q-1}((T+t)+y(T-t))^{2q-1}y^{1-\alpha}\mathrm{d}y.$$

Call the generalized binomial formula we may write

$$I=(T-t)^{2q-\alpha+1}\int_0^1 ((T-t)y+t)(1-y)^{2q-1}\sum_{k=0}^{\infty}C_{2q-1}^{k}(T+t)^{2q-1-k}(T-t)^k y^{k+1-\alpha}\mathrm{d}y<+\infty.$$

Then

$$\begin{aligned}I=&(T-t)^{2q-\alpha+2}\sum_{k=0}^{\infty}C_{2q-1}^{k}(T+t)^{2q-1-k}(T-t)^k\int_0^1 y^{k+2-\alpha}(1-y)^{2q-1}\mathrm{d}y\\&+(T-t)^{2q-\alpha+1}t\sum_{k=0}^{\infty}C_{2q-1}^{k}(T+t)^{2q-1-k}(T-t)^k\int_0^1 y^{k+1-\alpha}(1-y)^{2q-1}\mathrm{d}y.\end{aligned}$$

The formula (3.5) again, gives

$$\begin{aligned}&D_{t|T}^{\alpha-1}\left(1-\frac{t^2}{T^2}\right)^{2q}\\=&\frac{4qT^{-4q}}{\Gamma(2-\alpha)}\Bigg((T-t)^{2q-\alpha+2}\sum_{k=0}^{\infty}C_{2q-1}^{k}\times(T+t)^{2q-1-k}(T-t)^k\frac{\Gamma(k+3-\alpha)\Gamma(2q)}{\Gamma(k+3-\alpha+2q)}\\&+(T-t)^{2q-\alpha+1}t\sum_{k=0}^{\infty}C_{2q-1}^{k}(T+t)^{2q-1-k}(T-t)^k\frac{\Gamma(k+2-\alpha)\Gamma(2q)}{\Gamma(k+2-\alpha+2q)}\Bigg).\end{aligned}\tag{3.9}$$

In particular we have

$$D_{t|T}^{\alpha-1}\varphi(0)=\frac{4qT^{-\alpha+1}}{\Gamma(2-\alpha)}\sum_{k=0}^{\infty}C_{2q-1}^{k}\frac{\Gamma(k+3-\alpha)\Gamma(2q)}{\Gamma(k+3-\alpha+2q)}.\tag{3.10}$$

Substituting expression of $D_{-}^{\alpha-1}\varphi$ in $\int_Q u_1 D_{-}^{\alpha-1}\varphi$, we get

$$\begin{aligned}&\int_Q u_1 D_{-}^{\alpha-1}\varphi\\=&\frac{4qT^{-4q}}{\Gamma(2-\alpha)}\int_{\Omega_R}u_1(x)\Phi\left(\frac{x}{R}\right)\\&\times\int_0^T\Bigg[(T-t)^{2q-\alpha+2}\sum_{k=0}^{\infty}C_{2q-1}^{k}(T+t)^{2q-1-k}(T-t)^k\int_0^1 y^{k+2-\alpha}(1-y)^{2q-1}\mathrm{d}y\\&+(T-t)^{2q-\alpha+1}t\sum_{k=0}^{\infty}C_{2q-1}^{k}(T+t)^{2q-1-k}(T-t)^k\int_0^1 y^{k+1-\alpha}(1-y)^{2q-1}\mathrm{d}y\Bigg]\mathrm{d}t.\end{aligned}$$

Consequently,

$$\int_Q u_1 D_-^{\alpha-1}\varphi = \frac{C_3 T^{-\alpha+2}}{\Gamma(2-\alpha)} \int_{\Omega_R} u_1(x)\Phi\left(\frac{x}{R}\right). \tag{3.11}$$

It is easy to see that

$$\begin{aligned}\mathcal{C} &= \int_Q (h\varphi)^{\frac{-l}{p-l}} |(-\Delta)^{\beta/2} D_-^{\gamma}\varphi|^{\frac{p}{p-l}} \\ &= CR^{-\beta\frac{p}{p-l}} \int_Q \left(h\left(1-\frac{t^2}{T^2}\right)^{2q}\right)^{\frac{-l}{p-l}} \Phi\left(\frac{x}{R}\right) \left| D_-^{\gamma}\left(1-\frac{t^2}{T^2}\right)^{2q}\right|^{\frac{p}{p-l}}.\end{aligned}$$

Using (3.9), we find

$$\mathcal{C} \le C_4 R^{-\beta\frac{p}{p-l}} T^{\frac{-\gamma p}{p-l}} \int_{\Omega_R} (\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}} \Phi\left(\frac{x}{R}\right) \mathrm{d}x \int_0^T \left(1-\frac{t^2}{T^2}\right)^{2q(1-\frac{p}{p-l})} \mathrm{d}t.$$

Hence

$$\mathcal{C} \le C_5 R^{-\beta\frac{p}{p-l}} T^{1-\frac{\gamma p}{p-l}} \int_{\Omega_R} (\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}} \Phi\left(\frac{x}{R}\right) \mathrm{d}x, \tag{3.12}$$

where

$$C_5 = \frac{C_4 \Gamma(1+2q'(1-\frac{p}{p-l}))^2}{\Gamma(2+4q'(1-\frac{p}{p-l}))}.$$

Gathering the estimates (3.4), (3.8), (3.11) and (3.12), we obtain

$$\begin{aligned} & C_1 T \int_{\Omega_R} \inf_{t\in\mathbb{R}^+} f(x,t)\Phi\left(\frac{x}{R}\right) + C_2 T^{-\alpha+1} \int_{\Omega_R} u_0(x)\Phi\left(\frac{x}{R}\right) + C_3 T^{-\alpha+2} \int_{\Omega_R} u_1(x)\Phi\left(\frac{x}{R}\right) \\ & \le \left[C_4 T^{1-\frac{\alpha p}{p-1}} + C_5 T R^{-\frac{2p}{p-m}} + C_6 R^{-\beta\frac{p}{p-l}} T^{\frac{-\gamma p}{p-l}+1} \right] \\ & \times \int_{\Omega_R} \max\left((\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-1}}, (\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-m}}, (\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}} \right) \Phi. \end{aligned} \tag{3.13}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\Omega_R} u_0(x)\Phi\Big(\frac{x}{R}\Big)\\
\geq& \inf_{|x|>R}\bigg(u_0(x)\min\Big((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\Big)\bigg)\\
&\times\int_{\Omega_R}\max\Big((\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big), \qquad (3.14a)\\
&\int_{\Omega_R}(\inf_{t\in\mathbb{R}_+} f)\Phi\Big(\frac{x}{R}\Big)\\
\geq& \inf_{|x|>R}\bigg((\inf_{t\in\mathbb{R}_+} f)\min\Big((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\Big)\bigg)\\
&\times\int_{\Omega_R}\max\Big((\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big), \qquad (3.14b)\\
&\int_{\Omega_R} u_1(x)\Phi\Big(\frac{x}{R}\Big)\\
\geq& \inf_{|x|>R}\bigg(u_1(x)\min\Big((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\Big)\bigg)\\
&\times\int_{\Omega_R}\max\Big((\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big). \qquad (3.14c)
\end{aligned}$$

Combining the estimates (3.13), (3.14) and dividing the result by the term

$$\int_{\Omega_R}\max\Big((\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+} h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big)>0,$$

we obtain

$$\begin{aligned}
&C_1T\inf_{|x|>R}\bigg((\inf_{t\in\mathbb{R}_+} f)\min\Big((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\Big)\bigg)\\
\leq&\Big[C_4T^{1-\frac{\alpha p}{p-1}}+C_5TR^{-\frac{2p}{p-m}}+C_6R^{-\beta\frac{p}{p-l}}T^{\frac{-\gamma p}{p-l}+1}\Big], \qquad (3.15a)\\
&C_2T^{-\alpha+1}\inf_{|x|>R}(u_0(x)\bigg(\min\Big((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\Big)\bigg)\\
\leq&\Big[C_4T^{1-\frac{\alpha p}{p-1}}+C_5TR^{-\frac{2p}{p-m}}+C_6R^{-\beta\frac{p}{p-l}}T^{\frac{-\gamma p}{p-l}+1}\Big], \qquad (3.15b)\\
&C_3T^{-\alpha+2}\inf_{|x|>R}\bigg(u_1(x)\min\Big((\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+} h)^{\frac{p}{p-l}-1}\Big)\bigg)\\
\leq&\Big[C_4T^{1-\frac{\alpha p}{p-1}}+C_5TR^{-\frac{2p}{p-m}}+C_6R^{-\beta\frac{p}{p-l}}T^{\frac{-\gamma p}{p-l}+1}\Big]. \qquad (3.15c)
\end{aligned}$$

Now, passing to the limit as $R\to+\infty$ in (3.15) yields

$$\liminf_{|x|\to+\infty}\left(u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le C_7T^{\alpha(1-q)}, \tag{3.16a}$$

$$\liminf_{|x|\to+\infty}\left(u_1(x)(\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le C_8T^{\alpha(1-q)-1}, \tag{3.16b}$$

$$\liminf_{|x|\to+\infty}\left(\inf_{t\in\mathbb{R}^+}f\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le C_9T^{-\alpha q}. \tag{3.16c}$$

This completes the proof. □

Corollary 3.1. *Assume that problem* (2.1) *has a nontrivial global weak solution. Then one at least of the following is satisfied*

$$\liminf_{|x|\to+\infty}\left(u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)=0,$$

$$\liminf_{|x|\to+\infty}\left(u_1(x)(\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)=0,$$

$$\liminf_{|x|\to+\infty}\left((\inf_{t\in\mathbb{R}^+}f)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)=0.$$

Corollary 3.2. *If one of the following limits is infinity*

$$\liminf_{|x|\to+\infty}\left(u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right),$$

$$\liminf_{|x|\to+\infty}\left(u_1(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right),$$

$$\liminf_{|x|\to+\infty}\left((\inf_{t\in\mathbb{R}^+}f)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right),$$

then problem (2.1) *cannot have any local weak solution.*

Corollary 3.3. *If*

$$A=\lim_{|x|\to+\infty}\inf\left(u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)>0,$$

$$B=\lim_{|x|\to+\infty}\inf\left(u_1\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)>0,$$

$$C=\lim_{|x|\to+\infty}\left(\inf_{t\in\mathbb{R}_+}f\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)>0,$$

then

$$T\le\min\left\{\frac{C_7}{A^{\alpha(q-1)}},\frac{C_8}{B^{-\alpha+1+\frac{\alpha p}{p-1}}},\frac{C_9}{C^{\frac{\alpha p}{p-1}}}\right\}.$$

The next theorem give another necessary conditions for nonexistence of global weak solution to (2.1).

Theorem 3.2. *Suppose the problem* (2.1) *has a nontrivial global weak solution. Then, there are positive constants* K_1, K_2 *and* K_3 *such that*

$$\liminf_{|x|\to+\infty}\left(|x|^{2q(1-\frac{\alpha}{\gamma q})}u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le K_1,$$
$$\liminf_{|x|\to+\infty}\left(|x|^{2q(1-\frac{(\alpha-1)}{\gamma q})}u_1(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le K_2,$$

provided that $q>\alpha/\gamma$, *and*

$$\liminf_{|x|\to+\infty}\left(\inf_{t\in\mathbb{R}_+}f(x,t)|x|^{q\min\{\alpha,(\beta+\gamma)\}}\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le K_3.$$

Proof. In the relation

$$\begin{aligned}&C_2\int_{\Omega_R}u_0(x)\Phi\left(\frac{x}{R}\right)\\ \le&[C_4T^{\alpha(1-q)}+C_5R^{-\frac{2p}{p-m}}T^{\alpha}+C_6R^{-\beta\frac{p}{p-l}}T^{\alpha-\gamma q}]\\ &\times\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi,\end{aligned}\tag{3.17}$$

as $\gamma<\alpha$ and $\alpha-\gamma q<0$, we have $T^{\alpha(1-q)}<T^{\alpha-\gamma q}$ for $T>1$. Then

$$\begin{aligned}&C_2\int_{\Omega_R}u_0(x)\Phi(\frac{x}{R})\\ \le&[(C_4+C_6R^{-\beta q})T^{\alpha-\gamma q}+C_5R^{-2q}T^{\alpha}]\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi,\end{aligned}\tag{3.18}$$

It is easy to see that the minimum attaint at

$$T=\left(\frac{\gamma q-\alpha}{C_5\alpha}\right)^{\frac{1}{\gamma q}}\left(C_4R^{2q}+C_6R^{(2-\beta)q}\right)^{\frac{1}{\gamma q}}.$$

By substitution in (3.18), we find

$$\begin{aligned}&C_2\int_{\Omega_R}u_0(x)\Phi\left(\frac{x}{R}\right)\\ \le&C_7R^{(\frac{\alpha}{\gamma q}-1)2q}(C_4+C_6R^{-\beta q})^{\frac{\alpha}{\gamma q}}\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi,\end{aligned}\tag{3.19}$$

where

$$C_7=\left(\frac{\gamma q-\alpha}{C_5\alpha}\right)^{\frac{\alpha-\gamma q}{\gamma q}}+C_5\left(\frac{\gamma q-\alpha}{C_5\alpha}\right)^{\frac{\alpha}{\gamma q}}.$$

Therefore,

$$\begin{aligned}&C_2\int_{\Omega_R}|x|^{2q(1-\frac{\alpha}{\gamma q})}u_0(x)\Phi\left(\frac{x}{R}\right)\\ \le&C_7(C_4+C_6R^{-\beta q})^{\frac{\alpha}{\gamma q}}\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi.\end{aligned}\tag{3.20}$$

We obtain from the definition of $\Omega_R=\{x:R<|x|<2R\}$,

$$\begin{aligned}&\inf_{|x|>R}\left(|x|^{2q(1-\frac{\alpha}{\gamma q})}u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\\ &\times\int_{\Omega_R}|x|^{-2q(1-\frac{\alpha}{\gamma q})}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi\\ \le&(K_1+C_6R^{-\beta q})^{\frac{\alpha}{\gamma q}}\int_{\Omega_R}|x|^{-2q(1-\frac{\alpha}{\gamma q})}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi.\end{aligned}\tag{3.21}$$

Now, dividing both sides of (3.21) by

$$\int_{\Omega_R}|x|^{-2q(1-\frac{\alpha}{\gamma q})}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi>0,$$

we find

$$\begin{aligned}&\inf_{|x|>R}\left(|x|^{2q(1-\frac{\alpha}{\gamma q})}u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\\ \le&(K_1+C_6R^{-\beta q})^{\frac{\alpha}{\gamma q}}.\end{aligned}\tag{3.22}$$

Thus

$$\liminf_{|x|\to+\infty}\left(|x|^{2q(1-\frac{\alpha}{\gamma q})}u_0(x)\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\le K.\tag{3.23}$$

From (3.13), we may write

$$\begin{aligned}&C_3\int_{\Omega_R}u_1(x)\Phi\left(\frac{x}{R}\right)\\ \le&\left[(C_4+C_6R^{-\beta q})T^{\alpha-\gamma q-1}+C_5T^{\alpha-1}R^{-2q}\right]\\ &\times\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi.\end{aligned}\tag{3.24}$$

By taking

$$T=\left(\frac{-\alpha+\gamma q+1}{C_5(\alpha-1)}\right)^{\frac{1}{\gamma q}}\left(C_4R^{2q}+C_6R^{(2-\beta)q}\right)^{\frac{1}{\gamma q}},$$

the relation (3.24) gives us

$$\begin{aligned}&C_3\int_{\Omega_R}u_1(x)\Phi\Big(\frac{x}{R}\Big)\\ \le&(K_3+K_4R^{-\beta q})^{\frac{(\alpha-1)}{\gamma q}}R^{2q(\frac{(\alpha-1)}{\gamma q}-1)}\\ &\times\int_{\Omega_R}\max\Big((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big).\end{aligned}$$

Using the definition of Ω_R, we find

$$\begin{aligned}&\int_{\Omega_R}u_1(x)\Phi\Big(\frac{x}{R}\Big)\\ \le&(K_3+K_4R^{-\beta q})^{\frac{(\alpha-1)}{\gamma q}}\\ &\times\int_{\Omega_R}|x|^{2q(\frac{(\alpha-1)}{\gamma q}-1)}\max\Big((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big).\end{aligned}\tag{3.25}$$

We use the estimate

$$\begin{aligned}&\int_{\Omega_R}u_1(x)\Phi\Big(\frac{x}{R}\Big)\\ \ge&\inf_{|x|>R}\Big(\min\Big((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\Big)|x|^{2q(1-\frac{(\alpha-1)}{\gamma q})}u_1(x)\Big)\\ &\times\int_{\Omega_R}|x|^{2q(\frac{(\alpha-1)}{\gamma q}-1)}\max\Big((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big).\end{aligned}$$

Dividing the both sides of (3.25) by

$$\int_{\Omega_R}|x|^{2q(\frac{(\alpha-1)}{\gamma q}-1)}\max\Big((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\Big)\Phi\Big(\frac{x}{R}\Big)$$

and passing to the limit, we get

$$\liminf_{|x|\to+\infty}\Big(\min\Big((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\Big)|x|^{2q(1-\frac{(\alpha-1)}{\gamma q})}u_1(x)\Big)\le K.$$

Similarly, by (3.17) we have

$$\begin{aligned}&\int_{\Omega_R}(\inf_{t\in\mathbb{R}^+}f(x,t))\Phi\Big(\frac{x}{R}\Big)\\ \le&[C_4T^{-\alpha q}+C_5R^{-2q}+C_6R^{-\beta q}T^{-\gamma q}]\int_{\Omega_R}\max\Big((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\Big)\Phi.\end{aligned}$$

Taking $T=R$ gives

$$\begin{aligned}&\int_{\Omega_R}(\inf_{t\in\mathbb{R}^+} f(x,t))\Phi\left(\frac{x}{R}\right)\le [C_4R^{-\alpha q}+C_5R^{-2q}+C_6R^{-(\beta+\gamma)q}]\\&\times\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi\\&\le K_3R^{-q\min\{\alpha,(\beta+\gamma)\}}\int_{\Omega_R}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi.\end{aligned}$$

If follows from the definition of $\Omega_R=\{x:R<|x|<2R\}$ that

$$\begin{aligned}&\inf_{|x|>R}\left((\inf_{t\in\mathbb{R}_+}f(x,t))|x|^{q\min\{\alpha,(\beta+\gamma)\}}\min\left((\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-1}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-m}-1},(\inf_{t\in\mathbb{R}_+}h)^{\frac{p}{p-l}-1}\right)\right)\\&\times\int_{\Omega_R}|x|^{-q\min\{\alpha,(\beta+\gamma)\}}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi\left(\frac{x}{R}\right)\\&\le K_3\int_{\Omega_R}|x|^{-q\min\{\alpha,(\beta+\gamma)\}}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi.\end{aligned}\tag{3.26}$$

The conclusion follows by dividing both sides of (3.26) by

$$\int_{\Omega_R}|x|^{-q\min\{\alpha,(\beta+\gamma)\}}\max\left((\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-1}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-m}},(\inf_{t\in\mathbb{R}_+}h)^{1-\frac{p}{p-l}}\right)\Phi>0$$

and by passing to the limit when $R\to+\infty$. □

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