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# Box Dimension of Weyl Fractional Integral of Continuous Functions with Bounded Variation

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**Abstract.** We know that the Box dimension of $f(x)\in C^1[0,1]$ is 1. In this paper, we prove that the Box dimension of continuous functions with bounded variation is still 1. Furthermore, Box dimension of Weyl fractional integral of above function is also 1.

**Key Words**: Fractional calculus, box dimension, bounded variation.

**AMS Subject Classifications**: 28A80, 26A33, 26A30

## 1 Introduction

As a new branch in mathematics, fractal geometry has proved its value with many applications over many fields. Many initial and conclusive results on fractals were done in [2,3,7]. If $f(x)$ has continuous derivative, it is not difficult to see that Box dimension of $f(x)$ is 1, indeed a regular $1-$set. We want to know whether this result still holds for the function $f(x)$ with bounded variation? What about their fractional integral? Firstly, we give definitions of Hausdorff dimension and Box dimension.

**Definition 1.1** (see [1]). Let a Borel set $F\in\mathscr{R}^n$ is as follows. For $s\geq 0$ and $\delta>0$, define

$$\mathscr{H}^s_\delta(F)=\inf\Big\{\sum_{i=1}^{\infty}|U_i|^s:\{U_i\}\text{ is a }\delta\text{-cover of }F\Big\},$$

where $|U|=\sup\{|x-y|:x,y\in U\}$ denotes the diameter of a nonempty set $U$ and the infimum is taken over all countable collections $\{U_i\}$ of sets for which $F\subset\bigcup_i^\infty U_i$ and

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$0<|U_i|\le\delta$. As $\delta$ decreases, $\mathscr{H}^s_\delta(F)$ can not decrease, and therefore it has a limit (possibly infinite) as $\delta\to 0$; define

$$\mathscr{H}^s(F)=\lim_{\delta\to 0}\mathscr{H}^s_\delta(F).$$

The quantity $\mathscr{H}^s(F)$ is known as $s$-dimensional Hausdorff measure of $F$. For a given $F$ there is a value $\dim_H(F)$ for which $\mathscr{H}^s(F)=\infty$ for $s<\dim_H(F)$ and $\mathscr{H}^s(F)=0$ for $s>\dim_H(F)$. Hausdorff dimension $\dim_H(F)$ is defined to be this value, that is:

$$dim_H(F)=\inf\{s:\mathscr{H}^s(F)=0\}=\sup\{s:\mathscr{H}^s(F)=\infty\}.$$

**Definition 1.2** (see [1]). Let $F$ be a any non-empty bounded subset of $R^2$ and let $N_\delta(F)$ be the smallest number of sets of diameter at most $\delta$ which can cover $F$. The lower and upper Box dimensions of $F$ respectively are defined as

$$\underline{\dim}_B(F)=\underline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \tag{1.1}$$

and

$$\overline{\dim}_B(F)=\overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}. \tag{1.2}$$

If (1.1) and (1.2) are equal, we refer to the common value as the Box dimension of $F$:

$$\dim_B(F)=\lim_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}.$$

**Definition 1.3** (see [6]). Let $f(x)$ be a finite function on $I$, $I=[0,1]$. Let $\{x_i\}_{i=1}^n$ be arbitrary points which satisfy

$$0=x_0<x_1<x_2<\cdots<x_n=1.$$

Write

$$V_f:=\sup_{(x_0,x_1,\cdots,x_n)}\sum_{k=1}^n|f(x_k)-f(x_{k-1})|. \tag{1.3}$$

If (1.3) is finite, then $f(x)$ is of bounded variation on $I$. Let $BV_I$ denote the set of functions of bounded variation on $I$. Meanwhile, Let $C(I)$ denote the set of functions which are continuous on $I$.

**Definition 1.4** (see [4]). Let $f(x)\in C(I)$ and $0<v<1$. If $f(x)$ is piecewise integrable, we define the Weyl fractional integral of $f(x)$ of order $v$ as

$$D^{-v}f(x)=\frac{1}{\Gamma(v)}\int_x^\infty\frac{f(t)}{(t-x)^{1-v}}dt.$$

In this paper, let $G(f,I)$ denote the graph of $f(x)$ on $I$, and $\dim_B G(f,I)$ denote the Box dimension of $f(x)$ on $I$.

## 2 Lemmas

We first derive some simple but widely applicable estimates for the Box dimension of graphs. Given a function $f(x)$ and an interval $[a,b]$, we write $R_f$ the maximum range of $f(x)$ over $[a,b]$, i.e.,

$$R_f[a,b]=\sup_{a<x,y<b}|f(x)-f(y)|.$$

To prove theorems of Section 3, we need some lemmas.

**Lemma 2.1** (see [1]). *Let $f(x)\in C(I)\bigcap BV_I$. Suppose that $0<\delta<1$, and $m$ be the least integer greater than or equal to $\delta^{-1}$ then, if $N_\delta$ is the number of squares of the $\delta-$mesh that intersect $G(f,I)$, then*

$$\delta^{-1}\sum_{i=o}^{m-1}R_f[i\delta,(i+1)\delta]\leq N_\delta\leq 2m+\delta^{-1}\sum_{i=0}^{m-1}R_f[i\delta,(i+1)\delta].$$

*Proof.* The number of mesh squares of side $\delta$ in the column above the interval $[i\delta,(i+1)\delta]$ that intersect $G(f,I)$ is at least $R_f[i\delta,(i+1)\delta]/\delta$ and at most $2+(R_f[i\delta,(i+1)\delta]/\delta)$. $f(x)$ is continuous. By summing all such intervals together, we can get Lemma 2.1. □

If $f(x)$ is a continuous function and Box dimension of $f(x)$ exists, we know that Box dimension of any continuous functions is no less than 1. Then we give the following lemmas.

**Lemma 2.2.** *If $f(x)$ is a continuous function on $I$, we have $\underline{\dim}_B G(f,I)\geq 1$.*

*Proof.* By using Definition 1.2, we have

$$\underline{\dim}_B G(f,I)\geq \underline{\lim}_{\delta\to 0}\frac{\log\frac{C}{\delta}}{-\log\delta}=1.$$

So we get the conclusion that $\underline{\dim}_B G(f,I)\geq 1$. □

**Lemma 2.3.** *If $f(x)$ is a continuous function on $I$, then $\overline{\dim}_B G(f,I)\leq 2$.*

*Proof.* By using Definition 1.2, we have

$$\overline{\dim}_B G(f,I)\leq \overline{\lim}_{\delta\to 0}\frac{\log\frac{C}{\delta^2}}{-\log\delta}=2.$$

Then $\overline{\dim}_B G(f,I)\leq 2$. □

## 3 Theorems

**Theorem 3.1.** *If* $f(x)\in C(I)\bigcap BV_I$, *we have*

$$\dim_B G(f,I)=1.$$

Let $\{x_i\}_{i=1}^n$ be arbitrary points satisfying

$$0=x_0<x_1<x_2<\cdots<x_n=1.$$

We have

$$\sup_{(x_0,x_1,\cdots,x_n)}\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|<C.$$

Let $m$ be the least integer greater than or equal to $1/\delta$. $N_\delta$ is the number of squares of the $\delta-$mesh that intersect $G(f,I)$. By Lemma 2.1, we know

$$N_\delta\le 2m+\delta^{-1}\sum_{i=1}^{m}R_f[(i-1)\delta,i\delta].$$

For $1\le i\le m-1$ and $x_{i,0}=i\delta$, $x_{i,3}=(i+1)\delta$, $x_{i,1}$, $x_{i,2}\in(i\delta,(i+1)\delta)$,

$$R_f[i\delta,(i+1)\delta]\le\sup_{x_{i,0}<x_{i,1}<x_{i,2}<x_{i,3}}\sum_{k=1}^{3}|f(x_i,k)-f(x_i,k-1)|.$$

So there exists a certain absolutely positive constant $C$ such that

$$N_\delta\le C\delta^{-1}.$$

From Definition 1.1, it holds that

$$\overline{\dim}_B G(f,I)\le 1,\quad 0<v<1. \tag{3.1}$$

The topolopy dimension of a continuous function $f(x)$ is no less than 1. From Definition 1.1, we get

$$\underline{\dim}_B G(f,I)\ge 1,\quad 0<v<1. \tag{3.2}$$

Combining (3.1) and (3.2), we get the result of Theorem 3.1.

**Remark 3.1.** If $f(x)\in LipM$, then $\dim_B G(f,I)=1$.

*Proof.* $f(x)\in LipM$, $\forall x,y\in I$, we have

$$|f(x)-f(y)|\le M|x-y|,$$

where $M$ is a positive constant. Let $\{x_i\}_{i=1}^n$ be arbitrary points satisfying

$$0=x_0<x_1<x_2<\cdots<x_n=1.$$

We have

$$\sup_{(x_0,x_1,\cdots,x_n)}\sum_{k=1}^{n}|f(x_k)-f(x_{k-1})|\le M\sum_{k=1}^{n}|x_k-x_{k-1}|\le M.$$

Then $f(x)\in BV_I$. From Theorem 3.1, we can prove $\dim_B G(f,I)=1$. ☐

**Theorem 3.2.** *If $f(x)\in C(I)\bigcap BV_I$, we have*

$$\dim_B G(D^{-v}f,I)=1. \tag{3.3}$$

*Proof.* Since $f(x)\in C(I)$ and $f(x)$ is of bounded variation on $I$, by Theorem 6.6 of [6], we know that $f(x)$ can be replaced by the difference of two monotone increasing and continuous functions $g_1(x)$ and $g_2(x)$. Then

$$f(x)=g_1(x)-g_2(x),$$

where $g_1(x)=h_1(x)-c$, $g_2(x)=h_2(x)-c$, $h_1(1)=h_2(1)=c$ on $[1,+\infty)$. $h_1(x)$ and $h_2(x)$ are also monotone increasing and continuous functions.

(1) If $f(0)\ge 0$, we can choose $g_1(0)\ge 0$ and $g_2(0)=0$. By Definition 1.3, if

$$G_1(x)=D^{-v}g_1(x)=\frac{1}{\Gamma(v)}\int_x^\infty\frac{h_1(t)-c}{(t-x)^{1-v}}dt,\quad 0<v<1,$$

we know that $G_1(x)$ still is continuous on $I$ when $g_1(x)$ is continuous on $I$. Let $0\le x_1\le x_2\le 1$ and $0<v<1$. We have

$$\begin{aligned}
G_1(x_2)-G_1(x_1)=&\frac{1}{\Gamma(v)}\int_{x_2}^{\infty}(t-x_2)^{v-1}(h_1(t)-c)dt-\frac{1}{\Gamma(v)}\int_{x_1}^{\infty}(t-x_1)^{v-1}(h_1(t)-c)dt\\
=&\frac{1}{\Gamma(v)}\int_{x_2}^{1}(t-x_2)^{v-1}(h_1(t)-c)dt-\frac{1}{\Gamma(v)}\int_{x_1}^{1}(t-x_1)^{v-1}(h_1(t)-c)dt\\
=&\frac{1}{\Gamma(v)}\Big(\int_{x_2}^{1}(t-x_2)^{v-1}h_1(t)dt-\int_{x_1}^{1}(t-x_1)^{v-1}h_1(t)dt\Big)\\
&+\frac{1}{\Gamma(v)}\Big(\int_{x_1}^{1}(t-x_1)^{v-1}cdt-\int_{x_2}^{1}(t-x_2)^{v-1}cdt\Big)\\
=&\frac{1}{\Gamma(v)}\int_{x_1}^{1-x_2+x_1}(t-x_1)^{v-1}h_1(t-x_1+x_2)dt-\frac{1}{\Gamma(v)}\int_{x_1}^{1}(t-x_1)^{v-1}h_1(t)dt\\
&+\frac{1}{\Gamma(v)}\Big(\int_{x_2}^{1-x_1+x_2}(t-x_2)^{v-1}cdt-\int_{x_2}^{1}(t-x_2)^{v-1}cdt\Big)\\
=&\frac{1}{\Gamma(v)}\int_{x_1}^{1-x_2+x_1}(t-x_1)^{v-1}(h_1(t-x_1+x_2)-h_1(t))dt\\
&+\frac{1}{\Gamma(v)}\int_{1+x_1-x_2}^{1}(t-x_1)^{v-1}(c-h_1(t))dt\\
\ge&0.
\end{aligned}$$

Thus function $G_1(x)$ still is a monotone increasing and continuous function on $I$. If

$$G_2(x)=D^{-v}g_2(x)=\frac{1}{\Gamma(v)}\int_x^\infty \frac{h_2(t)-c}{(t-x)^{1-v}}dt, \quad 0<v<1,$$

$G_2(x)$ is also a monotone increasing and continuous function on $I$.

(2) If $f(0)<0$, we can choose $g_1(x)=0$ and $g_2(x)>0$. Using a similar argument, we can get that both $D^{-v}g_1(x)$ and $D^{-v}g_2(x)$ are monotone increasing and continuous functions on $I$. From Theorem 6.6 of [6], we know $D^{-v}f(x)$ still is a continuous function of bounded variation on $I$. $\square$

# 4 Conclusions

The result that Box dimension of a continuous function of bounded variation on $I$ is 1 has been proved in Theorem 3.1. Theorem 3.2 shows that Box dimension of Weyl fractional integral of such function still is 1. As we know that Box dimension of classical integral of a one-dimensional continuous function on $I$ is 1. We consider whether Box dimension of Weyl fractional integral of a one-dimensional continuous function on $I$ is 1 or not. From this paper, we know that Box dimension of Weyl fractional integral of a continuous function of bounded variation still is 1 on $I$.

However, we still can not calculate Hausdorff dimension of Weyl fractional integral of continuous with bounded variation. The calculation of Hausdorff dimension is difficult. Furthermore, we only discuss fractal dimension of continuous function of bounded variation on $I$ in this paper. We hardly know about fractal dimension of continuous function with unbounded variation.

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### References

[1] K. Falconer, Fractal Geometry: Mathematical Foundations and Application, J. Wiley Sons, New York, 1990.

[2] Y. S. Liang, Box dimension of Riemann-Liouvile fractional integrals of continuous functions of bounded variation, Nonlinear Anal., 72 (2010), 4304–4306.

[3] B. B. Mandelbrot, The Fractal Geometry of Nature, W. H. Freeman and Company, New York, 1983.

[4] K. B. Oldham, and J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.

[5] Z. Y. Wen, Mathematical Foundations of Fractal Geometry, Shanghai Sci. Tech. Edu. Publ. House, Shanghai, 2000, in Chinese.

[6] W. X. Zheng and S. W. Wang, Real Function and Functional Analysis, High Education Publication, Beijing, 1980.

[7] S. P. Zhou, K. Yao and W. Y. Su, Fractional integrals of the weierstrass functions: the exact Box dimension, Anal. Theory Appl., 20 (2004), 332–341.