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C^p Condition and the Best Local Approximation

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Abstract. In this paper, we introduce a condition weaker than the L^p differentiability, which we call C^p condition. We prove that if a function satisfies this condition at a point, then there exists the best local approximation at that point. We also give a necessary and sufficient condition for that a function be L^p differentiable. In addition, we study the convexity of the set of cluster points of the net of best appoximations of f, $\{P_\epsilon(f)\}$ as $\epsilon \to 0$.

Key Words: Best L^p approximation, local approximation, L^p differentiability.

AMS Subject Classifications: 41A50, 41A10

1 Introduction

Let x_1, $a \in \mathbb{R}$, $a>0$, and let $\mathcal{L}$ be the space of equivalence class of Lebesgue measurable real functions defined on $I_a := (x_1-a, x_1+a)$. For each Lebesgue measurable set $A \subset I_a$, with $|A|>0$, we consider the semi-norm on $\mathcal{L}$,

$$\|h\|_{p,A} := \Big(|A|^{-1}\int_A |h(x)|^p dx\Big)^{1/p}, \quad 1<p<\infty,$$

where $|A|$ denotes the measure of the set A. As usual, we denote by $L^p(I_a)$ the space of functions $h\in\mathcal{L}$ with $\|h\|_{p,I_a}<\infty$. If $0<\epsilon\le a$, $I_{-\epsilon} := (x_1-\epsilon, x_1)$, $I_{+\epsilon} := (x_1, x_1+\epsilon)$, we write $\|h\|_{p,\pm\epsilon} = \|h\|_{p,I_{\pm\epsilon}}$, and $\|h\|_{p,\epsilon} = \|h\|_{p,I_\epsilon}$. For a non negative integer s, we denote by Π^s the linear space of polynomials of degree at most s. Henceforward, we consider $n\in\mathbb{N}\cup\{0\}$. If $h\in L^p(I_a)$, it is well known that there exists a unique best $\|\cdot\|_{p,\epsilon}$-approximation of h from Π^n, say $P_\epsilon(h)$, i.e., $P_\epsilon(h)\in\Pi^n$ satisfies

$$\|h-P_\epsilon(h)\|_{p,\epsilon} \le \|h-P\|_{p,\epsilon} \quad \text{for all } \ P\in\Pi^n.$$

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$P_\epsilon(h)$ is the unique polynomial in Π^n, which verifies

$$\int_{I_\epsilon} |(h-P_\epsilon(h))(x)|^{p-1}\mathrm{sgn}((h-P_\epsilon(h))(x))(x-x_1)^j dx=0, \quad 0\le j\le n, \tag{1.1}$$

see [2].

If $\lim_{\epsilon\to 0} P_\epsilon(h)$ exists, say $P_0(h)$, it is called the *best local approximation of h at x_1 from* Π^n (b.l.a.). In general, we shall also denote by $P_0(h)$ the set

$$\Big\{P\in\Pi^n : P=\lim_{k\to\infty} P_{\epsilon_k}(h) \text{ for some } \epsilon_k\downarrow 0\Big\}.$$

The problem of best local approximation was formally introduced and studied in a paper by Chui, Shisha and Smith [3]. However, the initiation of this could be dated back to results of J. L. Walsh [10], who proved that the Taylor polynomial of an analytic function h over a domain is the limit of the net of polynomial best approximations of a given degree, by shrinking the domain to a single point. Later, several authors studied the existence of the b.l.a. assuming a certain order of differentiability. In [8] and [12], this problem was considered when h is L^p differentiable. Recently, in [7] and [5] the authors proved the existence of the b.l.a. under weaker conditions, more precisely they assumed existence of lateral L^p derivatives of order n and L^p differentiability of order $n-1$. In [4] it was proved that if $p=2$ and h is differentiable up to order $n-1$, then $P_0(h)$ is either empty or convex. Later, in [11] using interpolation properties of the best approximation, the author extended this result for $1<p<\infty$. The main purpose of this paper is to give more general conditions on a function h so that there exists the b.l.a., and to study its connection with the L^p differentiability. Further, we study the convexity of $P_0(h)$. The following definition is motivated by the characterization (1.1).

Definition 1.1. We shall say that $f\in L^p(I_a)$ satisfies the C^p condition of order n at x_1, if there exists $Q\in\Pi^n$ such that

$$\int_{I_\epsilon} |(f-Q)(x)|^{p-1}\mathrm{sgn}((f-Q)(x))(x-x_1)^j dx=o(\epsilon^{n(p-1)+j+1}), \tag{1.2}$$

$0\le j\le n$, as $\epsilon\to 0$.

Analogously, we shall say that f satisfies the left (right) C^p condition of order n at x_1, if there exists $Q\in\Pi^n$ verifying (1.2) with $I_{-\epsilon}(I_{+\epsilon})$ instead of I_ϵ.

We denote with $c_n^p(x_1)$ the class of functions in $L^p(I_a)$ which satisfy the C^p condition of order n at x_1. We recall that a function $f\in L^p(I_a)$ is *L^p differentiable of order n at x_1* (i.e., $f\in t_n^p(x_1)$) if there exists $Q\in\Pi^n$ such that $\|f-Q\|_{p,\epsilon}=o(\epsilon^n)$. This concept was introduced by Calderón and Zygmund in [1]. Using the Hölder inequality, it is easy to see that $t_n^p(x_1)\subset c_n^p(x_1)$, moreover the inclusion is strict. In fact, if $h(x)=\sin(1/x)$, $x\neq 0$, then $h\in c_0^2(0)$, however a straightforward computation shows that $h\notin t_0^2(0)$. It immediately follows from Definition 1.1 that $c_n^p(x_1)$ satisfies: a) If $f\in c_n^p(x_1)$, then $f+P\in c_n^p(x_1)$ for all $P\in\Pi^n$, and b) If $f\in c_n^p(x_1)$, then $\lambda f\in c_n^p(x_1)$, for all $\lambda\in\mathbb{R}$. In the second section of this paper, we prove that if $f\in c_n^p(x_1)$, $2\le p<\infty$, then there exists the b.l.a., and it is the unique $Q\in\Pi^n$ satisfying (1.2). We also prove that $f\in t_n^p(x_1)$ if and only if $f\in c_n^p(x_1)$ and $\|f-P_\epsilon(f)\|_{p,\epsilon}=o(\epsilon^n)$. In the case $p=2$, we show that Definition (1.1) allows us to introduce a new concept of differentiation. In the third section of this paper we prove that if $f\in c_{n-1}^p(x_1)$, then $P_0(f)$ is either empty or convex. It extends, for $p\ge 2$ and a broader class of functions, a similar result established in [11]. Henceforward, without loss of generality, we shall establish our results at the point $x_1=0$. We shall write K for a positive constant not necessarily the same in each occurrence.

2 The main results

In this section we shall prove a theorem of existence of the best local approximation for $p\ge 2$. Given a function $f\in L^p(I_a)$, $Q\in\Pi^n$, and $0<\epsilon\le a$, we define the following sets.

$$A_\epsilon=\{f\ge P_\epsilon(f)>Q\}\cap I_\epsilon,\qquad B_\epsilon=\{Q<f<P_\epsilon(f)\}\cap I_\epsilon,\tag{2.1a}$$
$$C_\epsilon=\{f\le Q<P_\epsilon(f)\}\cap I_\epsilon,\qquad D_\epsilon=\{P_\epsilon(f)<f<Q\}\cap I_\epsilon,\tag{2.1b}$$
$$E_\epsilon=\{f\ge Q>P_\epsilon(f)\}\cap I_\epsilon,\qquad F_\epsilon=\{f\le P_\epsilon(f)<Q\}\cap I_\epsilon.\tag{2.1c}$$

Suppose that $P_\epsilon(f)-Q$ has m zeros in I_ϵ, according to their multiplicity counting, for a net $\epsilon\downarrow 0$, say $x_i=x_i(\epsilon)$. We write $(P_\epsilon(f)-Q)(x)=\prod_{i=1}^{s(\epsilon)}(x-x_i)^{r_i(\epsilon)}H_\epsilon(x)$, with $H_\epsilon(x)\neq 0$, $x\in I_\epsilon$, and $\sum_{i=1}^{s(\epsilon)}r_i(\epsilon)=m$.

Let $R_\epsilon(x):=\eta(\epsilon)\prod_{i=1}^{s(\epsilon)}(x-x_i)^{r_i(\epsilon)}$ be with $\eta(\epsilon)=\pm 1$ such that $R_\epsilon(x)(P_\epsilon(f)-Q)(x)\ge 0$, $x\in I_\epsilon$. We put $R_\epsilon(x)=\sum_{j=0}^m b_jx^j$, $b_j=b_j(\epsilon)$. With this notation we establish the following lemma.

Lemma 2.1. *Suppose that $f\in c_l^p(0)$, $0\le l\le n$. If $Q\in\Pi^l$ verifies* (1.2) *and $m\le l$, then*

1.
$$\int_{M_\epsilon}|(|(f-P_\epsilon(f))(x)|^{p-1}-|(f-Q)(x)|^{p-1})R_\epsilon(x)|\frac{dx}{\epsilon}=o(\epsilon^{l(p-1)})\sum_{j=0}^m|b_j|\epsilon^j,$$

where M_ϵ is equal to A_ϵ, C_ϵ, E_ϵ or F_ϵ.

2.
$$\int_{N_\epsilon}|(|(f-P_\epsilon(f))(x)|^{p-1}+|(f-Q)(x)|^{p-1})R_\epsilon(x)|\frac{dx}{\epsilon}=o(\epsilon^{l(p-1)})\sum_{j=0}^m|b_j|\epsilon^j,$$

where N_ϵ is equal to B_ϵ or D_ϵ.

Proof. Clearly, the sets defined in (2.1) are pairwise disjoint and

$$A_\epsilon\cup B_\epsilon\cup C_\epsilon\cup D_\epsilon\cup E_\epsilon\cup F_\epsilon=I_\epsilon,\tag{2.2}$$

except by the set of zeros of R_ϵ.

By hypothesis we have

$$\begin{aligned}&\int_{I_\epsilon}|(f-Q)(x)|^{p-1}\mathrm{sgn}((f-Q)(x))x^j dx\\ =&o(\epsilon^{l(p-1)+j+1})=o_j(\epsilon^{l(p-1)+1})\epsilon^j,\quad 0\le j\le l,\quad \text{as } \epsilon\to 0.\end{aligned}\tag{2.3}$$

From (1.1) we have

$$\int_{I_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}\mathrm{sgn}((f-P_\epsilon(f))(x))x^j dx=0,\quad 0\le j\le l.\tag{2.4}$$

Multiplying (2.3) member to member by b_j and adding on j from 0 to m, we obtain

$$\begin{aligned}&\int_{A_\epsilon}|(f-Q)(x)|^{p-1}|R_\epsilon(x)|dx+\int_{B_\epsilon}|(f-Q)(x)|^{p-1}|R_\epsilon(x)|dx\\ &-\int_{C_\epsilon}|(f-Q)(x)|^{p-1}|R_\epsilon(x)|dx+\int_{D_\epsilon}|(f-Q)(x)|^{p-1}|R_\epsilon(x)|dx\\ &-\int_{E_\epsilon}|(f-Q)(x)|^{p-1}|R_\epsilon(x)|dx+\int_{F_\epsilon}|(f-Q)(x)|^{p-1}|R_\epsilon(x)|dx\\ =&\sum_{j=0}^{m}o_j(\epsilon^{l(p-1)+1})b_j\epsilon^j=o(\epsilon^{l(p-1)+1})\sum_{j=0}^{m}|b_j|\epsilon^j.\end{aligned}\tag{2.5}$$

In fact, if

$$w=w(\epsilon):=\sum_{j=0}^{m}|b_j|\epsilon^j\neq 0,$$

the last equality is a consequence of

$$\Big|w^{-1}\sum_{j=0}^{m}o_j(\epsilon^{l(p-1)+1})b_j\epsilon^j\Big|\le\sum_{j=0}^{m}|o_j(\epsilon^{l(p-1)+1})|=o(\epsilon^{l(p-1)+1}).$$

In a similar way, from (2.4) we get

$$\begin{aligned}&\int_{A_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}|R_\epsilon(x)|dx-\int_{B_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}|R_\epsilon(x)|dx\\ &-\int_{C_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}|R_\epsilon(x)|dx-\int_{D_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}|R_\epsilon(x)|dx\\ &-\int_{E_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}|R_\epsilon(x)|dx+\int_{F_\epsilon}|(f-P_\epsilon(f))(x)|^{p-1}|R_\epsilon(x)|dx=0.\end{aligned}\tag{2.6}$$

Subtracting the Eq. (2.5) from (2.6), we get

$$\begin{aligned}
&-\int_{A_\epsilon}(|(f-Q)(x)|^{p-1}-|(f-P_\epsilon(f))(x)|^{p-1})|R_\epsilon(x)|dx\\
&-\int_{B_\epsilon}(|(f-P_\epsilon(f))(x)|^{p-1}+|(f-Q)(x)|^{p-1})|R_\epsilon(x)|dx\\
&-\int_{C_\epsilon}(|(f-P_\epsilon(f))(x)|^{p-1}-|(f-Q)(x)|^{p-1})|R_\epsilon(x)|dx\\
&-\int_{D_\epsilon}(|(f-P_\epsilon(f))(x)|^{p-1}+|(f-Q)(x)|^{p-1})|R_\epsilon(x)|dx\\
&-\int_{E_\epsilon}(|(f-P_\epsilon(f))(x)|^{p-1}-|(f-Q)(x)|^{p-1})|R_\epsilon(x)|dx\\
&-\int_{F_\epsilon}(|(f-Q)(x)|^{p-1}-|(f-P_\epsilon(f))(x)|^{p-1})|R_\epsilon(x)|dx\\
=&o(\epsilon^{l(p-1)+1})\sum_{j=0}^{m}|b_j|\epsilon^j. \qquad (2.7)
\end{aligned}$$

Now, we observe that the six integrals in (2.7) are nonnegative. Thus, each term in (2.7) is equal to $o(\epsilon^{l(p-1)+1})\sum_{j=0}^{m}|b_j|\epsilon^j$. This proves the lemma. $\square$

Next, we prove one of our main results.

Theorem 2.1. *Let $p\geq 2$, $0\leq l\leq n$, and $f\in c_l^p(0)$. If $Q\in\Pi^l$ verifies* (1.2) *then $P_0(f)$ is either empty or for each j, $0\leq j\leq l$, and for each $P\in P_0(f)$,*

$$P^{(j)}(0)=Q^{(j)}(0). \qquad (2.8)$$

Proof. We suppose $P_0(f)\neq\emptyset$. Let $P\in P_0(f)$ and let $\epsilon_k\downarrow 0$ be such that $\lim_{k\to\infty}P_{\epsilon_k}(f)=P$. Without loss of generality, we can assume that $P_{\epsilon_k}(f)\neq Q$ for all k. Suppose that there exists a sequence (which we do not relabel) such that $P_{\epsilon_k}(f)-Q$ has m zeros, according to their multiplicity counting, in I_{ϵ_k}, say $x_{i,k}$, $0\leq i\leq m-1$. As above of Lemma 2.1, we consider $R_{\epsilon_k}(x)=\sum_{j=0}^{m}b_jx^j$ such that $R_{\epsilon_k}(x)(P_{\epsilon_k}(f)-Q)(x)\geq 0$, $x\in I_{\epsilon_k}$. The proof is divided in two parts: (a) $m\geq l+1$ and (b) $m\leq l$.

We assume (a). Clearly, the divided differences $P_{\epsilon_k}[x_{0,k},\cdots,x_{j,k}]$ and $Q[x_{0,k},\cdots,x_{j,k}]$, $0\leq j\leq l$, are equals. On the other hand, $P_{\epsilon_k}[x_{0,k},\cdots,x_{j,k}]=(j!)^{-1}P_{\epsilon_k}^{(j)}(\eta_{j,k})$ and $Q[x_{0,k},\cdots,x_{j,k}]=(j!)^{-1}Q^{(j)}(\nu_{j,k})$, where $\eta_{j,k},\nu_{j,k}\in I_{\epsilon_k}$. Thus, $P^{(j)}(0)=Q^{(j)}(0)$, $0\leq j\leq l$.

Now, we assume (b). Let M_{ϵ_k} and N_{ϵ_k} be the sets introduced in Lemma 2.1. For $a\geq 0$ and $b\geq 0$ there exists a constant $K>0$ such that $(a+b)^{p-1}\leq K(a^{p-1}+b^{p-1})$. If $x\in N_{\epsilon_k}$, $a=|(f-P_{\epsilon_k}(f))(x)|$, and $b=|(f-Q)(x)|$, we have

$$|(P_{\epsilon_k}(f)-Q)(x)|^{p-1}\leq K(|(f-P_{\epsilon_k}(f))(x)|^{p-1}+|(f-Q)(x)|^{p-1}).$$

Therefore

$$\begin{aligned}&\int_{N_{\epsilon_k}}|R_{\epsilon_k}(x)||(P_{\epsilon_k}(f)-Q)(x)|^{p-1}dx\\ \le& K\int_{N_{\epsilon_k}}|(f-P_{\epsilon_k}(f))(x)|^{p-1}|R_{\epsilon_k}(x)|dx+K\int_{N_{\epsilon_k}}|(f-Q)(x)|^{p-1})|R_{\epsilon_k}(x)|dx\\ \le& o(\epsilon_k^{l(p-1)+1})\sum_{j=0}^{m}|b_j|\epsilon_k^j. \end{aligned}\tag{2.9}$$

Since $p-1\ge 1$, for $a\ge 0$ and $b\ge 0$ it verifies $a^{p-1}+b^{p-1}\le(a+b)^{p-1}$. If $x\in M_{\epsilon_k}$, $a=|(f-P_{\epsilon_k}(f))(x)|$, and $b=|(P_{\epsilon_k}(f)-Q)(x)|$ we get, $a+b=|(f-Q)(x)|$, therefore

$$|(P_{\epsilon_k}(f)-Q)(x)|^{p-1}\le|(f-Q)(x)|^{p-1}-|(f-P_{\epsilon_k}(f))(x)|^{p-1}.\tag{2.10}$$

From (2.10) we obtain

$$\begin{aligned}&\int_{M_{\epsilon_k}}|R_{\epsilon_k}(x)||(P_{\epsilon_k}(f)-Q)(x)|^{p-1}dx\\ \le&\int_{M_{\epsilon_k}}\big||(f-Q)(x)|^{p-1}-|(f-P_{\epsilon_k}(f))(x)|^{p-1}\big||R_{\epsilon_k}(x)|dx\\ \le& o(\epsilon_k^{l(p-1)+1})\sum_{j=0}^{m}|b_j|\epsilon_k^j.\end{aligned}\tag{2.11}$$

Adding the two inequalities of type (2.9) for the sets B_{ϵ_k} and D_{ϵ_k}, and the four inequalities of type (2.11) for the sets A_{ϵ_k}, C_{ϵ_k}, E_{ϵ_k} and F_{ϵ_k}, we have

$$\int_{I_{\epsilon_k}}|R_{\epsilon_k}(x)||(P_{\epsilon_k}(f)-Q)(x)|^{p-1}\frac{dx}{2\epsilon_k}\le o({\epsilon_k}^{l(p-1)})\sum_{j=0}^{m}|b_j|{\epsilon_k}^j.\tag{2.12}$$

Now, we consider the norm ρ on Π^n defined by $\rho(T)=\sum_{j=0}^{n}|c_j|$ if $T(x)=\sum_{j=0}^{n}c_jx^j$, and we define $T^{\epsilon}(x):=T(\epsilon x)$. By means of the change of variable $x=\epsilon_k t$, (2.12) can be written

$$\int_{I_1}|R_{\epsilon_k}^{\epsilon_k}(x)||(P_{\epsilon_k}(f)-Q)^{\epsilon_k}(x)|^{p-1}\frac{dx}{2}\rho^{-1}(R_{\epsilon_k}^{\epsilon_k})\le o({\epsilon_k}^{l(p-1)}).\tag{2.13}$$

Let

$$W_{\epsilon_k}=\frac{R_{\epsilon_k}^{\epsilon_k}}{\rho(R_{\epsilon_k}^{\epsilon_k})}.$$

Since $\rho(W_{\epsilon_k})=1$, there exists a subsequence, which we denote in the same way, such that and $W_{\epsilon_k}\to W_0\in\Pi^m$. Let $S\subset I_1$ be a compact set of positive measure, which does not contain any zero of W_0, and let $\beta=\min_{t\in S}|W_0(t)|>0$. There exists k_0 such that $|W_{\epsilon_k}(t)|>\beta/2$ for all $k\ge k_0$ and for all $t\in S$. As a consequence, we have

$$\frac{\beta}{2}\int_S|(P_{\epsilon_k}(f)-Q)^{\epsilon_k}(x)|^{p-1}dx\le\int_{I_1}|(P_{\epsilon_k}(f)-Q)^{\epsilon_k}(x)|^{p-1}|W_{\epsilon_k}(t)|dx=o(\epsilon_k^{l(p-1)}),$$

i.e.,

$$\|(P_{\epsilon_k}(f)-Q)^{\epsilon_k}\|_{p-1,S}=o(\epsilon_k^l). \tag{2.14}$$

Now, we recall a Pólya type inequality (see [6, Lemma 2.1]) There exists a constant $K>0$ such that

$$|(P_\epsilon(f)-Q)^{(j)}(0)|\le\frac{K}{\epsilon^j}\|P_\epsilon(f)-Q\|_{p-1,\epsilon},\quad 0\le j\le n,\quad 0<\epsilon\le a. \tag{2.15}$$

From (2.14), (2.15), and the equivalence two norms on Π^n, we obtain

$$|(P_{\epsilon_k}(f)-Q)^{(j)}(0)|\le\frac{K}{\epsilon_k^j}\|(P_{\epsilon_k}(f)-Q)^{\epsilon_k}\|_{p,1}=o(\epsilon_k^{l-j}), \tag{2.16}$$

so

$$(P_{\epsilon_k}(f)-Q)^{(j)}(0)\to 0,\quad 0\le j\le l\quad \text{as}\ \ k\to\infty. \tag{2.17}$$

Therefore, since $\lim_{k\to\infty}P_{\epsilon_k}(f)=P$, we get (2.8). □

Remark 2.1. We observe that the constraint $p\ge 2$, only was used to obtain the inequality (2.11).

As a consequence of the proof of Theorem 2.1 we obtain

Theorem 2.2. *If $p\ge 2$ and $f\in c_n^p(0)$, then there exists the best local approximation of f at 0 from Π^n, and it is the unique polynomial in Π^n which satisfies* (1.2).

Proof. Since $m\le n$, the theorem analogously follows as in the proof of Theorem 2.1, (b), for $l=n$. In fact, (2.17) implies $P_{\epsilon_k}(f)\to Q$, as $k\to\infty$. Finally, as $\{\epsilon_k\}$ is arbitrary we get $P_\epsilon(f)\to Q$, as $\epsilon\to 0$. Now, the uniqueness of Q verifying (1.2) is clear. □

The next theorem gives a characterization of L^p differentiable functions.

Theorem 2.3. *Let $p\ge 2$ and $f\in L^p(I_a)$. Then $f\in t_n^p(0)$ if and only if $f\in c_n^p(0)$ and $\|f-P_\epsilon(f)\|_{p,\epsilon}=o(\epsilon^n)$.*

Proof. Suppose $f\in t_n^p(0)$. Since we have mentioned in Introduction $t_n^p(0)\subset c_n^p(0)$ and clearly $\|f-P_\epsilon(f)\|_{p,\epsilon}=o(\epsilon^n)$. Now, assume $f\in c_n^p(0)$ and $\|f-P_\epsilon(f)\|_{p,\epsilon}=o(\epsilon^n)$. Let $Q\in\Pi^n$ be verifying (1.2). From the equivalence two norms on Π^n and (2.14), we have $\|P_\epsilon(f)-Q\|_{p,\epsilon}=o(\epsilon^n)$. Therefore, we get

$$\|f-Q\|_{p,\epsilon}\le\|f-P_\epsilon(f)\|_{p,\epsilon}+\|P_\epsilon(f)-Q\|_{p,\epsilon}=o(\epsilon^n),\ \text{i.e.,}\ \ f\in t_n^p(0).$$

So, we complete the proof. □

Given $Q_1,Q_2\in\Pi^n$, let S_ϵ be one of the following sets $\{f>Q_i>Q_j\}\cap I_\epsilon$, $\{f<Q_i<Q_j\}\cap I_\epsilon$, $i,j=1,2$, $i\ne j$.

Lemma 2.2. *Let f be a bounded function on I_a, and let $1<p<\infty$.*

(a) Let $Q_1,Q_2\in\Pi^n$ be such that $Q_1(0)\neq Q_2(0)$. Then there exist $0<\epsilon_0\leq a$ and $K>0$ such that

$$\big||(f-Q_1)(x)|^{p-1}-|(f-Q_2)(x)|^{p-1}\big|\geq K|(Q_1-Q_2)(x)|^{p-1} \tag{2.18}$$

for all $x\in S_\epsilon$, and for all $0<\epsilon\leq\epsilon_0$,.

(b) Let $Q\in\Pi^0$, and let $P_\epsilon(f)$ be the best constant approximation of f. Suppose that for a sequence $\epsilon_k\downarrow 0$, $|Q-P_{\epsilon_k}(f)|\geq\alpha>0$, then there exist $K>0$ and $k_0\in\mathbb{N}$ such that

$$\big||(f-Q)(x)|^{p-1}-|(f-P_{\epsilon_k}(f))(x)|^{p-1}\big|\geq K|Q-P_{\epsilon_k}(f)|^{p-1} \tag{2.19}$$

for all $x\in M_{\epsilon_k}$, $k\geq k_0$, where M_{ϵ_k} was introduced in Lemma 2.1.

Proof. (a) If (2.18) is not true, then there exist a sequence $\epsilon_m\downarrow 0$ and $x_m\in S_{\epsilon_m}$ such that

$$0\leq\big||(f-Q_1)(x_m)|^{p-1}-|(f-Q_2)(x_m)|^{p-1}\big|\leq\frac{1}{m}|(Q_1-Q_2)(x_m)|^{p-1}. \tag{2.20}$$

Since f is bounded on I_a, the sequences $\{(f-Q_1)(x_m)\}$ and $\{(f-Q_2)(x_m)\}$ are bounded. Therefore, for some subsequence which we denote in the same way, it follows from (2.20)

$$|(Q_1-Q_2)(x_m)|=||(f-Q_1(x_m)|-|(f-Q_2)(x_m)||\to 0.$$

The last equality follows from definition of the set S_{ϵ_m}. Since $x_m\to 0$, we have $Q_1(0)=Q_2(0)$, a contradiction.

(b) Since f is bounded and $P_{\epsilon_k}(f)$ is constant, it is easy to see that $\{P_{\epsilon_k}(f)\}$ is uniformly bounded. Then there exists a subsequence, which we denote in the same way, and $T\in\Pi^0$ such that $P_{\epsilon_k}(f)\to T$. If (2.19) is not true, a similar argument to the proof of (a) yields $Q-T=0$. On the other hand, $|Q-T|\geq\alpha>0$, a contradiction. □

Theorem 2.4. *Let $1<p<\infty$, and let f be a bounded function on I_a. Then*

(a) If $Q_1,Q_2\in\Pi^n$ satisfy (1.2) *then $Q_1(0)=Q_2(0)$. In particular, for $n=0$ there exists at most a constant polynomial verifying* (1.2).

(b) If $f\in c_0^p(0)$ then there exists the best local approximation of f at 0, *and it is the unique constant polynomial verifying* (1.2).

Proof. (a) Suppose that $Q_1(0)\neq Q_2(0)$. By Lemma 2.2, there exist ϵ_0 and $K>0$ verify (2.18). Proceeding as in Theorem 2.1 with Q_1 instead of Q and Q_2 instead of $P_\epsilon(f)$ we obtain that $Q_1-Q_2=0$, a contradiction. In fact, we observe that (2.11) remains valid for all p, $1<p<\infty$, $\epsilon_k\leq\epsilon_0$ and $S_\epsilon=M_\epsilon$.

(b) Let $Q\in\Pi^0$ be verifying (1.2) and $P_\epsilon(f)$ the best constant approximant. If $P_{\epsilon_k}(f)\nrightarrow Q$, for some sequence $\epsilon_k\downarrow 0$, using Lemma 2.2 and proceeding as in Theorem 2.1, we have that $P_{\epsilon_k}(f)\to Q$, which is a contradiction. □

Remark 2.2. We observe that all the theorems proved in this Section hold, with the obvious modifications, if f satisfies the left (right) C^p condition of order n at 0, and we consider $\|\cdot\|_{p,-\epsilon}(\|\cdot\|_{p,+\epsilon})$ instead of $\|\cdot\|_{p,\epsilon}$.

If $f\in c_n^p(0)$, and $p\geq 2$, let $T_{n,p}(f)$ be the unique polynomial in Π^n satisfying (1.2). The next theorem can be easily proved.

Theorem 2.5. *The operator $T_{n,2}:c_n^2(0)\to\Pi^n$ is linear. Further, $c_n^2(0)\subset c_{n-1}^2(0)$, and if $f\in c_n^2(0)$, then $T_{n,2}(f)(x)=T_{n-1,2}(f)(x)+\alpha(f)x^n$, $\alpha(f)\in\mathbb{R}$.*

If $f\in c_n^2(0)$, the Theorem 2.5 allows us to define the k-th derivative in the C^2 sense by $f^{(k)}(0):=(T_{n,2}(f))^{(k)}(0)$, $0\leq k\leq n$. Clearly, if f has a k-th derivative in the L^2 sense, it coincides with the k-th derivative in the C^2 sense.

3 Convexity of $P_0(f)$

We begin this section by proving the continuity of the function $F:(0,a)\to\Pi^n$ defined by $F(\epsilon)=P_\epsilon(f)$, with $f\in L^p(I_a)$, $1<p<\infty$.

Lemma 3.1. *F is a continuous function.*

Proof. Fix $\epsilon_0\in(0,a)$, and let $\epsilon_m\in(0,a)$ be such that $\epsilon_m\to\epsilon_0$. There exists $m_0\in\mathbb{N}$ such that for all $m\geq m_0$ we have $\epsilon_m\geq\epsilon_0/2$. Then,

$$\|f-P_{\epsilon_m}(f)\|_{p,\frac{\epsilon_0}{2}}^p\leq\frac{2\epsilon_m}{\epsilon_0}\|f-P_{\epsilon_m}(f)\|_{p,\epsilon_m}^p\leq\frac{2\epsilon_m}{\epsilon_0}\|f\|_{p,\epsilon_m}^p\leq K. \tag{3.1}$$

Thus, the sequence $\{P_{\epsilon_m}\}$ is uniformly bounded. Consequently, there exists a subsequence which denote in the same way, such that $P_{\epsilon_m}(f)$ converges to $Q\in\Pi^n$. In addition, by (1.1) we have

$$\int_{I_a}|(f-P_{\epsilon_m}(f))(x)|^{p-1}\text{sgn}((f-P_{\epsilon_m}(f))(x))x^j\chi_{I_{\epsilon_m}}dx=0,\quad 0\leq j\leq n, \tag{3.2}$$

where χ_A is the characteristic function of the set A. It is easy to see that the integrands in (3.2) are bounded by an integrable function, so from (3.2) and Lebesgue Dominated Convergence Theorem, we get

$$\int_{I_a}|(f-Q)(x)|^{p-1}\text{sgn}((f-Q)(x))x^j\chi_{I_{\epsilon_0}}dx=0,\quad 0\leq j\leq n. \tag{3.3}$$

Therefore $Q=P_{\epsilon_0}(f)$, i.e., $F(\epsilon_m)\to F(\epsilon_0)$. □

Using the same technique that in [4], Proposition 3.1, and Lemma 3.1, we can prove the following theorem.

Theorem 3.1. *Let $f \in L^p(I_a)$, $1<p<\infty$, be such that its best $\|\cdot\|_{p,\epsilon}$-approximation from Π^n, is $P_\epsilon(f)=\sum_{i=0}^{n}\alpha_i(\epsilon)x^i$, where $\alpha_i(\epsilon)\to\alpha_i$, as $\epsilon\to 0$, $0\le i\le n-1$. Then $P_0(f)$ is either empty or convex.*

As a consequence of Theorem 2.1 for $l=n-1$, and Theorem 3.1, we have the next result, which extends Corollary 3 in [11] for $p\ge 2$.

Theorem 3.2. *Let $p\ge 2$ and $f\in c^p_{n-1}(0)$. Then $P_0(f)$ is either empty or convex.*

Remark 3.1. In [9], the author gave an example of a function $f\in L^2(I_a)$, continuous at 0 such that the set of cluster points of the best $\|\cdot\|_{2,\epsilon}$-approximation from Π^2 is not empty and is not convex. Since f is continuous at 0, $f\in c^2_0(0)$. Therefore, we cannot assume the weaker condition $f\in c^2_{n-2}(0)$ in Theorem 3.2.

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