Anal. Theory Appl.
Vol. 27, No. 2 (2011), 181–186
DOI10.1007/s10496-011-0181-9

BOUNDS FOR COMMUTATORS OF MULTILINEAR FRACTIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS

Jinhui Wang, Jing Lan

(*Jiangxi University of Science and Technology, China*)

and

Jun Li

(*Fuzhou Institute of Education, China*)

Received Dec. 9, 2010; Revised Apr. 28, 2011

Abstract. We will show bounds for commutators of multilinear fractional integral operators with some homogeneous kernels.

Key words: *multilinear operator, fractional integral, commutator, multiple weight, homogeneous kernel*

AMS (2010) subject classification: 42B20

In 1999, C. E. Kenig and E. M. Stein[8] initiated the study of multilinear fractional integral operators defined as

$$I_\alpha(\vec{f})(x)=\int_{(\mathbf{R}^n)^m}\frac{1}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}}\prod_{k=1}^m f_k(y_k)\mathrm{d}\vec{y}$$

(See [6] or [10] for more about fractional integral). Recently, K. Moen [11]m X. Chen and Q. Xue[3] developed the weighted theory for it, which was motivated by related research for multilinear singular integral in [7] and [9]. In their work the following of weights the for multilinear fractional integral was established.

Definition 1[11], [3]. Let $1\leqslant p_1,\cdots,p_m<\infty$, $\dfrac{1}{p}=\dfrac{1}{p_1}+\cdots+\dfrac{1}{p_m}$, and $q>0$. Suppose that $\vec{\omega}=(\omega_1,\cdots,\omega_m)$ and each ω_i $(i=1,\cdots,m)$ is a nonnegative function on $\mathbf{R}^n$. Then $\vec{\omega}\in A_{(\vec{p},q)}$

if it satisfies

$$\sup_Q \left(\frac{1}{|Q|}\int_Q \nu_{\vec{\omega}}{}^q\right)^{\frac{1}{q}} \prod_{i=1}^m \left(\frac{1}{|Q|}\int_Q \omega_i{}^{-p_i'}\right)^{\frac{1}{p_i'}} < \infty,$$

where $\nu_{\vec{\omega}} = \prod_{i=1}^m \omega_i$. If $p_i = 1$, $\left(\frac{1}{|Q|}\int_Q \omega_i^{-p_i'}\right)^{\frac{1}{p_i'}}$ is understood as $(\inf_Q \omega_i)^{-1}$.

Furthermore, a weighted norm inequality for multilinear fractional integral operators as below is proved.

Theorem A([11], [3])**.** *Let* $0 < \alpha < mn$, $1 < p_1, \cdots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $\vec{\omega} \in A_{(\vec{p},q)}$ *if and only if* I_α *can be extended to a bounded operator*

$$\left\|I_\alpha(\vec{f})\right\|_{L^q(\nu_{\vec{\omega}}{}^q)} \leqslant C\prod_{i=1}^m \left\|f_i\right\|_{L^{p_i}(\omega_i^{p_i})}. \tag{1}$$

In [3], besides the above, the authors proved another two results such as Theorem B and C, by the way of contemplating weighted norm inequalities for multilinear fractional integral with some homogeneous kernels and Coifman-Rochberg-Weiss commutators of multilinear fractional integral.

Theorem B[3]**.** *Let* $0 < \alpha < mn$, $1 \leqslant s' < p_1, \cdots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Denote* $\vec{\omega}^{s'} = (\omega_1^{s'}, \cdots, \omega_m^{s'})$ *and* $\frac{\vec{p}}{s'} = (\frac{p_1}{s'}, \cdots, \frac{p_m}{s'})$. *Assume* $\vec{\omega}^{s'} \in A_{(\frac{\vec{p}}{s'}, \frac{q}{s'})} \bigcap A_{(\frac{\vec{p}}{s'}, \frac{q_\varepsilon}{s'})} \bigcap A_{(\frac{\vec{p}}{s'}, \frac{q_{-\varepsilon}}{s'})}$, *where* $\frac{1}{q_\varepsilon} = \frac{1}{p} - \frac{\alpha+\varepsilon}{n}$ *and* $\frac{1}{q_{-\varepsilon}} = \frac{1}{p} - \frac{\alpha-\varepsilon}{n}$. *Then, there exists a constant* $C > 0$ *independent of* $\vec{f}$ *such that*

$$\left\|I_{\Omega,\alpha}(\vec{f})\right\|_{L^q(\nu_{\vec{\omega}}{}^q)} \leqslant C\prod_{i=1}^m \left\|f_i\right\|_{L^{p_i}(\omega_i^{p_i})}, \tag{2}$$

where

$$I_{\Omega,\alpha}\vec{f}(x) = \int_{(\mathbf{R}^n)^m} \frac{\prod_{i=1}^m \Omega_i(x-y_i) f_i(y_i)}{|(x-y_1, \cdots, x-y_m)|^{mn-\alpha}}\, \mathrm{d}\vec{y}$$

and each $\Omega_i(x) \in L^s(\mathbf{S}^{n-1})$ $(i = 1, \cdots, m)$ *for some* $s > 1$ *is a homogeneous function with degree zero on* $\mathbf{R}^n$, *i.e. for any* $\lambda > 0$ *and* $x \in \mathbf{R}^n$, $\Omega_i(\lambda x) = \Omega_i(x)$.

Theorem C[3]**.** *Let* $0 < \alpha < mn$, $1 < p_1, \cdots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *For* $r > 1$ *with* $0 < r\alpha < mn$, *if* $\vec{\omega}^r \in A_{(\frac{\vec{p}}{r}, \frac{q}{r})}$ *and* $\nu_{\vec{\omega}}{}^q \in A_\infty$, *then there exists a constant* $C > 0$ *independent of* $\vec{b}$ *and* $\vec{f}$ *such that*

$$\left\|I_{\vec{b},\alpha}(\vec{f})\right\|_{L^q(\nu_{\vec{\omega}}{}^q)} \leqslant C\sup_i \|b_i\|_{\mathrm{BMO}} \prod_{i=1}^m \left\|f_i\right\|_{L^{p_i}(\omega_i^{p_i})}, \tag{3}$$

where the commutators of I_α is defined as

$$I_{\vec{b},\alpha}(\vec{f})(x)=\sum_{i=1}^{m} I^i_{b_i,\alpha}(\vec{f})(x)$$

and each term of the right-hand side is the commutator of I_α in the i-th entry with b_i, that is

$$I^i_{b_i,\alpha}(\vec{f})(x)=b_i(x)I_\alpha(f_1,\cdots,f_i,\cdots,f_m)(x)-I_\alpha(f_1,\cdots,b_if_i,\cdots,f_m)(x).$$

Now, what we concern about is studying the commutators of locally integrable function b and multilinear fractional integral with homogeneous kernels in the j-th entry

$$\begin{aligned}[b,I_{\Omega,\alpha}]_j(\vec{f})(x) &= I^j_{b,\Omega,\alpha}(\vec{f})(x)\\ &= \int_{(\mathbf{R}^n)^m}\frac{b(x)-b(y_j)}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}}\prod_{k=1}^{m}\Omega_k(x-y_k)f_k(y_k)\mathrm{d}\vec{y},\end{aligned}$$

where $d\vec{y}=dy_1\cdots dy_m$ and $|(y_1,\cdots,y_m)|=|y_1|+\cdots+|y_m|$.

In 1989, J. O. Strömberg and A. Torchinsky[13] concluded that an appropriate weighted inequality for operators could provide an unweighted inequality for its commutators. In 1993, J. Alvarez, R. J. Bagby, D. S. Kurtz and C. Pérez[1] exploited this idea further to prove the boundedness of commutators of general linear operators on weighted L^p spaces by estimates for linear operators. Additionally, this idea also appeared in [4] [10] for fractional and singular integral operators with homogeneous kernels, and in [2] for multilinear singular integral operators with applications to non-smooth kernel. Thus we get the result as below inspired by these works.

Theorem 1. *Let $0<\alpha<mn$, $1<p_1,\cdots,p_m<\infty$, $\dfrac{1}{p}=\dfrac{1}{p_1}+\cdots+\dfrac{1}{p_m}$ and $\dfrac{1}{q}=\dfrac{1}{p}-\dfrac{\alpha}{n}$. Besides, the assumption on Ω is the same as in Theorem B. If $b\in$BMO, then there exists a constant $C>0$ independent of b and $\vec{f}$ such that*

$$\big\|[b,I_{\Omega,\alpha}]_j(\vec{f})\big\|_{L^q}\leqslant C\prod_{i=1}^{m}\|f_i\|_{L^{p_i}}.$$

Proof. Obviously, we can set $j=1$ in the proof. Because we can see that $g(z)=e^{z(b(x)-b(y))}$ with $z=x+iy$ is analytic on $\mathbf{C}$, and it's easy to get

$$b(x)-b(y)=g'(0)=\frac{1}{2\pi i}\int_{|z|=1}\frac{g(z)}{|z|^2}\mathrm{d}z=\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta}(b(x)-b(y))}e^{-i\theta}\mathrm{d}\theta$$

by the Cauchy integral formula. Consequently, it makes sure that

$$\begin{aligned}
[b,I_{\Omega,\alpha}]_1(\vec{f})(x) &= \int_{(\mathbf{R}^n)^m} \frac{b(x)-b(y_1)}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}} \prod_{k=1}^{m} \Omega_k(x-y_k)f(y_k)\mathrm{d}y_k \\
&= \int_{(\mathbf{R}^n)^m} \left(\frac{1}{2\pi} \int_0^{2\pi} e^{e^{i\theta}(b(x)-b(y_1))} e^{-i\theta}\, d\theta \right) \frac{\prod_{k=1}^{m} \Omega_k(x-y_k)f(y_k)\mathrm{d}y_k}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}} \\
&= \frac{1}{2\pi} \int_0^{2\pi} I_{\Omega,\alpha}(f_1 e^{-be^{i\theta}}, f_2,\cdots,f_m)(x) e^{b(x)e^{i\theta}} e^{-i\theta}\mathrm{d}\theta \\
&\le \frac{1}{2\pi} \int_0^{2\pi} I_{|\Omega|,\alpha}(|f_1|e^{-b\cos\theta}, |f_2|,\cdots,|f_m|)(x) e^{b(x)\cos\theta}\mathrm{d}\theta.
\end{aligned}$$

Next, we prepare two lemmas. The first one due to J. García-Cuerva, J. L. Rubio de Francia[5] is similar to the classical result on Muckenhoupt's A_p weights.

Lemma 1[10]**.** *Let $0<\alpha<n$, $1<p<n/\alpha$ and $1/q = 1/p-\alpha/n$. For $\lambda>0$, then there exists $\eta>0$ such that if $b\in \mathrm{BMO}$ and $\|b\|_{\mathrm{BMO}}<\eta$, then $e^{\lambda b(x)} \in A(p,q)$.*

We can generalize Lemma 1 for the weights $A_{(\vec{p},q)}$ to multilinear settings by a remark in [11].

Lemma 2[11]**.** *If $p_k \le q_k$ with $1/q = 1/q_1+,\cdots,+1/q_m$, then $\bigcup_{q_k} \prod_{k=1}^{m} A_{(p_k,q_k)} \subset A_{(\vec{p},q)}$.*

Therefore, when $\|b\|_{\mathrm{BMO}}$ is assuming sufficient small, by the above two lemmas as above and Hölder's inequality, we have $(e^{b(x)\cos\theta},1,\cdots,1)\in A_{\vec{p},q}$ which meets the condition of weights in Theorem B for any θ. Applying the weighted boundedness of $I_{\alpha,\Omega}$ and Minkowski's inequality simply, as a result, we have

$$\begin{aligned}
\left\|[b,I_{\Omega,\alpha}]_1(\vec{f})\right\|_{L^q} &\le \frac{1}{2\pi} \int_0^{2\pi} \left\| I_{|\Omega|,\alpha}(|f_1|e^{-b\cos\theta},|f_2|,\cdots,|f_m|)(x) \right\|_{L^q(e^{qb\cos\theta})} \mathrm{d}\theta \\
&\le \frac{C}{2\pi} \int_0^{2\pi} \left\| f_1 e^{-b\cos\theta} \right\|_{L^{p_1}(e^{p_1 b\cos\theta})} \prod_{k=2}^{m} \|f_k\|_{L^{p_k}} \mathrm{d}\theta \\
&\le C \prod_{k=1}^{m} \|f_k\|_{L^{p_k}}.
\end{aligned}$$

Moreover, C. Pérez, G. Pradolini, R.H. Torres and R. Trujillo-González [12] studied iterated commutators $T_{\prod b}$ for a multilinear Calderón-Zygmund singular integral operator T defined as

$$T_{\prod \vec{b}}(\vec{f}) = [b_1,[b_2,\cdots[b_{m-1},[b_m,T]_m]_{m-1}\cdots]_2]_1(\vec{f}).$$

So the iterated commutator of multilinear fractional integral operators with homogeneous kernels Has the following form

$$I_{\prod \vec{b},\Omega,\alpha}(\vec{f})(x) = \int_{(\mathbf{R}^n)^m} \frac{\prod_{k=1}^{m}(b_k(x)-b_k(y_k))\Omega_k(x-y_k)f_k(y_k)}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}} \mathrm{d}\vec{y}.$$

Here and now, the bounds for $I_{\prod\vec{b},\Omega,\alpha}$ can be concluded by similar methods. In fact, we can see

$$\begin{aligned} I_{\prod\vec{b},\Omega,\alpha}(\vec{f})(x) &= \int_{(\mathbf{R}^n)^m}\prod_{j=1}^{m}\left(\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta_j}(b_j(x)-b_j(y_j))}e^{-i\theta_j}\mathrm{d}\theta_j\right)\frac{\prod_{k=1}^m\Omega_k(x-y_k)f(y_k)\mathrm{d}y_k}{|(x-y_1,\cdots,x-y_m)|^{mn-\alpha}} \\ &= \frac{1}{(2\pi)^m}\int_{[0,2\pi]^m}I_{\Omega,\alpha}(f_1e^{-b_1e^{i\theta_1}},\cdots,f_me^{-b_me^{i\theta_m}})(x)\prod_{k=1}^{m}e^{b_k(x)e^{i\theta_k}}e^{-i\theta_k}\mathrm{d}\theta_k \\ &\le \frac{1}{(2\pi)^m}\int_{[0,2\pi]^m}I_{|\Omega|,\alpha}(|f_1|e^{-b_1\cos\theta_1},\cdots,|f_m|e^{-b_m\cos\theta_m})(x)\prod_{k=1}^{m}e^{b_k(x)\cos\theta_k}\mathrm{d}\theta_k. \end{aligned}$$

Also note that $(e^{-b_1\cos\theta_1},\cdots,e^{-b_m\cos\theta_m})$ satisfies the condition of weights in Theorem B for any $\vec{\theta}$ immediately, we get the boundedness of iterated commutators by a similar calculation of weighted L^p norms and weighted estimates for $I_{\Omega,\alpha}$.

Theorem 2. *Let $0<\alpha<mn$, $1<p_1,\cdots,p_m<\infty$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Besides, the assumption on Ω is the same as in Theorem B. If $b_k\in\mathrm{BMO}$ with $k=1,2,\cdots,m$, then there exists a constant $C>0$ independent of $\vec{b}$ and $\vec{f}$ such that*

$$\big\|I_{\prod\vec{b},\Omega,\alpha}(\vec{f})\big\|_{L^q}\leqslant C\prod_{i=1}^{m}\big\|f_i\big\|_{L^{p_i}}.$$

Acknowledgement. Prof. Qingying Xue at Beijing Normal University called our attention to this topic and gave us useful suggestions. Mr. Xi Chen at Australian National University provided us with much material on related theory, while visiting Jiangxi University of Science and Technology in 2010. We would like to express our deep gratitude to them.

References

[1] Alvarez, H,m Bagby, R. J., Kurtz, D. S. and Pérez, C., Weighted Estimates for Commutators of Linear Operators, Studia Math., 104(1993), 195-209.

[2] Chen, X., On the Commutators of Multilinear Singular Integral, in preparation.

[3] Chen, C. and Xue, Q., Weighted Estimates for a Class of Multilinear Fractional type Operators, J. Math. Anal. Appl., 362:2(2010), 355-373.

[4] Ding, Y., Weighted Boundedness for Commutators of Fractional Integral Operators with Rough Kernels, J. Beijing Normal Univ. (Natural Sci.), 32(1996), 157-161. (In Chinese)

[5] García-Cuerva, J. and Rubio de Francia, J. L., Weighted Norm Inequalities and Related Topics, North-Holland Math. Studies 116, North-Holland, Amsterdam, 1985.

[6] Grafakos, L., Classical and Modern Fourier Analysis, Prentice Hall, New Jersey, 2004.

[7] Grafakos, L. and Torres, R. H., Multilinear Calderón-Zygmund theory, Adv. Math., 165:1 (2002), 124-164.

[8] Kenig, C. E. and Stein, E. M., Multilinear Estimates and Fractional Integration, Math. Res. Lett., 6(1999), 1-15.

[9] Lerner, A. K., Ombrosi, C., Pérez, C., Torres, R. H. and Trujillo-González, R., New Maximal Functions and Multiple Weights for the Multilinear Calderón-Zygmund theory, Adv. Math., 220(2009), 1222-1264.

[10] Lu, S., Ding, Y. and Yan, D., Singular Integral and Related Topics, World Scientific, Singapore, 2007.

[11] Moen, K., Weighted Inequalities for Multilinear Fractional Integral Operators, Collect. Math., 60(2009), 213-238.

[12] Pérez, C., Pradolini, G., Torres, R. H. and Trujillo-González, R., End-Points estimates for iterated conmutators of multilinear singular integrals, preprint.

[13] Strömberg, J. O. and Torchinsky, A., Weighted Hardy Spaces, Lecture Notes Math., 1381, Springer, 1989.

School of Science
Jiangxi University of Science and Technology
Ganzhou, 341000
P. R. China

J. H. Wang
E-mail: lxywjh@mail.jxust.cn

J. Lan
E-mail: chinavl231@126.com

J. Li
Department of Primary Education (Mathematics)
Fuzhou Institute of Education
Fuzhou, 350001
P. R. China

E-mail: garcia.lijun@gmail.com