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SOME NEW ITERATED FUNCTION SYSTEMS CONSISTING OF GENERALIZED CONTRACTIVE MAPPINGS

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Abstract. Iterated function systems (IFS) were introduced by Hutchinson in 1981 as a natural generalization of the well-known Banach contraction principle. In 2010, D. R. Sahu and A. Chakraborty introduced K-Iterated Function System using Kannan mapping which would cover a larger range of mappings. In this paper, following Hutchinson, D. R. Sahu and A. Chakraborty, we present some new iterated function systems by using the so-called generalized contractive mappings, which will also cover a large range of mappings. Our purpose is to prove the existence and uniqueness of attractors for such class of iterated function systems by virtue of a Banach-like fixed point theorem concerning generalized contractive mappings.

Key words: *iterated function system, attractor, generalized contractive mapping, complete metric space, fixed point*

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1 Introduction

As a natural generalization of the well-known Banach contraction principle, iterated function systems (IFS) were introduced by Hutchinson (see [8]) and popularized by Barnsley (see

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[4]). They represent one way of defining fractals as attractors of certain discrete dynamical systems. Moreover, they can be effectively applied to fractal image compressions. Therefore, it is no doubt that they have been attracting considerable attention of mathematicians and computer experts (see e.g. [3, 5-7, 10-13]). As observed recently in Andres[1], the same approach can be naturally extended to iterated multifunction systems (IMS) with resulting objects called multivalued fractals.

In this paper, the existence of new fractals is proved by means of the Banach-like theorem for so-called generalized contractive mappings. As a consequence, some new iterated function systems are founded, which are an important addition to Hutchinson's Iterated Function System and K-Iterated Function System.

2 Iterated Function Systems

In this section we recall some well known aspects of iterated function system used in the sequel (more complete and rigorous treatments may be found in [4] or [8]).

Let X denote a complete metric space with a distance function d and T be a mapping from X into itself. Then T is called a contraction mapping if there is a constant $0 \le s < 1$ such that

$$d(T(x), T(y)) \le sd(x, y).$$

Polish mathematician S. Banach proved a very important result, regarding contraction mapping in 1922, known as Banach Contraction Principle (see [2]).

Theorem 2.1[2]. *Let $T : X \to X$ be a contraction mapping on a complete metric space (X, d). Then T possesses exactly one fixed point $x^* \in X$. Moreover, for any point x in X, the sequence $\{T_n(x) : n = 0, 1, 2, \cdots\}$ converges to $x^* \in X$. That is $\lim_{n\to\infty} T_n(x) = x^*$ for each $x \in X$.*

In the famous paper [8], J.E. Hutchinson proved that, given a set of contractions in a complete metric space X, there exists a unique nonempty compact set $A \subset X$, named the attractor or fractal of the iterated functions system (IFS).

IFS generally employs contractive maps over a complete metric space (X, d), where the Banach's celebrated result mentioned above guarantees the existence and uniqueness of the fixed point known as "attractor" or "fractal". This can be done since the Hutchinson-Barnsley operator is also a contraction mapping over $H(X)$, where $H(X)$ denotes the space whose points are the compact subsets of X.

We now give some basic definitions and theorems concerning iterated function system, which are used in the proof below. Most of notations and results here is taken from [12].

Let (X,d) be a complete metric space and $H(X)$ denote the space whose points are the compact subsets of X known as Hausdorff space, other than the empty set. Let $x,y\in X$ and let $A,B\in H(X)$. Then

(1) the distance from the point x to the set B is defined as

$$d(x,B)=\min\{d(x,y):y\in B\},$$

(2) the distance from the set A to the set B is defined as

$$d(A,B)=\max\{d(x,B):x\in A\},$$

(3) the Hausdorff distance from the set A to the set B is defined as

$$h(A,B)=d(A,B)\vee d(B,A).$$

Then the function h is the metric defined on the space $H(X)$. Note that throughout this paper the notation $u\vee v$ means the maximum of the pair of real numbers u and v. In IFS, the contractive maps act on the members of Hausdorff space, i.e., the compact subsets of X. Thus, an iterated function system is defined as follows:

A Hutchinson's iterated function system consists of a complete metric space (X,d) together with a finite set of contraction mappings $T_n:X\to X$ with respect to contractivity factor s_n, for $n=1,2,3,\cdots,N$. Thus, the following theorem was given by Hutchinson or Barnsley(see [4, 8]):

Theorem 2.2[4,8]**.** *Let $\{X:T_n,n=1,2,3,\cdots,N\}$ be an iterated function system with contractivity factor s. Then the transformation $W:H(X)\to H(X)$ defined by $W(B)=\cup_{n=1}^{N}T_n(B)$ for all $B\in H(X)$, is a contraction mapping on the complete metric space $(H(X),h(d))$ with contractivity factor s. That is*

$$h(W(B),W(C))\le sh(B,C).$$

Its unique fixed point, which is also called an attractor, $A\in H(X)$, obeys

$$A=W(A)=\cup_{n=1}^{N}T_n(A),$$

and is given by $A=\lim_{n\to\infty}W^{\circ n}(B)$ for any $B\in H(X)$, where $W^{\circ n}$ denotes the n-fold composition of W.

3 New Iterated Function Systems

In this section, we shall try to explore the possibility of improvement in IFS where we replace the contraction condition by a more general condition which is called, for convenience, generalized contraction condition. One can see that such a condition is similar not only to the contraction condition, but also to Kannan condition. In 1969, Kannan [9] introduced a mapping, known as Kannan mapping defined as follows:

If there exists a number α, $0 < \alpha < \frac{1}{2}$, such that, for all $x, y \in X$,

$$d(T(x), T(y)) \leq \alpha[d(x, T(x)) + d(y, T(y))].$$

Then T is called a Kannan mapping.

In stead of Kannan mapping, we investigate another contractive mapping, that is, the so-called generalized contractive mapping, as follows.

Definition 3.1. Let (X,d) be a complete metric space. The mapping $T : X \to X$ is called a generalized contractive mapping if it satisfies the following generalized contraction condition

$$d(Tx, Ty) \leq a(d(x,y) + d(x,Ty) + d(y,Tx)) \tag{3.1}$$

for all $x, y \in X$, where a is a constant satisfying $0 < a < \frac{1}{3}$.

Now let us first present the fixed point theorem for such class of generalized contractive mappings.

Proposition 3.1. *Let (X,d) be a complete metric space. Suppose the mapping $T : X \to X$ is a generalized contractive mapping satisfying (3.1). Then T has a unique fixed point p in X. Moreover, for any $x_0 \in X$, the successive sequence $\{x_n\}$ where $x_n = Tx_{n-1} (n = 1, 2, \cdots)$ converges to the fixed point p and satisfies $d(x_n, p) \leq \frac{(2a)^n}{(1-a)^{n-1}(1-3a)} d(Tx_0, x_0)$.*

Proof. Suppose x_0 is an arbitrary point in X. Set

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \cdots.$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence. From (3.1), we know

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Tx_n, Tx_{n-1}) \\ &\leq ad(x_n, x_{n-1}) + ad(x_n, Tx_{n-1}) + ad(Tx_n, x_{n-1}) \\ &\leq 2ad(x_n, x_{n-1}) + ad(x_{n+1}, x_n) \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \\ &\leq ad(x_{n-1}, x_n) + ad(x_{n-1}, Tx_n) + ad(Tx_{n-1}, x_n) \\ &\leq 2ad(x_n, x_{n-1}) + ad(x_{n+1}, x_n) \end{aligned}$$

So, it follows that

$$\begin{aligned} 2d(x_{n+1},x_n) &= d(x_{n+1},x_n)+d(x_n,x_{n+1}) \\ &\le 4ad(x_n,x_{n-1})+2ad(x_{n+1},x_n), \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1},x_n) &\le \frac{2a}{1-a}d(x_n,x_{n-1}) = bd(x_n,x_{n-1}) \\ &\le b^2d(x_{n-1},x_{n-2}) \le \cdots \le b^n d(Tx_0,x_0), \end{aligned}$$

where $b=\dfrac{2a}{1-a}$.

Now, for any $n>m$, we have

$$\begin{aligned} d(x_{n+m},x_n) &\le d(x_{n+m},x_{n+m-1})+d(x_{n+m-1},x_{n+m-2})+\cdots+d(x_{n+1},x_n) \\ &\le (b^{n+m-1}+b^{n+m-2}+\cdots+b^n)d(Tx_0,x_0) \\ &\le \frac{b^n}{1-b}d(Tx_0,x_0). \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence in (X,d). So there exists $p\in X$ such that $x_n\to p$ as $n\to\infty$. From (3.1), we have

$$\begin{aligned} d(x_n,Tp) &= d(Tx_{n-1},Tp) \\ &\le ad(x_{n-1},p)+ad(x_{n-1},Tp)+ad(Tx_{n-1},p) \\ &= ad(x_{n-1},p)+ad(x_{n-1},Tp)+ad(x_n,p) \end{aligned}$$

Taking $n\to\infty$, we get

$$d(Tp,p)\le ad(p,Tp).$$

Then, we get $(1-a)d(q,Tp)\le 0$ which implies that $d(p,Tp)=0$ since $1-a>0$. So we have $p=Tp$.

Next, we show that T has a unique point in X. Assume there exists another point $q\in X$ such that $Tq=q$, then by (3.1) we see

$$\begin{aligned} d(p,q) &= d(Tp,Tq) \\ &\le a(d(p,q)+d(p,Tq)+d(Tp,q)) \\ &= a(d(p,q)+d(p,q)+d(p,q)) \end{aligned}$$

which gives $(1-3a)d(p,q)\le 0$, thus $d(p,q)=0$, So, $p=q$. That is to say, p is a unique fixed point of T in X.

Finally, we show that the formula for the error estimate of successive sequence. From a chain of inequalities presented before we see

$$d(x_{m+n},x_n) \le \frac{b^n}{1-b} d(Tx_0,x_0), \tag{3.2}$$

Letting $m \to \infty$ in (3.2), we get

$$d(x_n,p) \le \frac{b^n}{1-b} d(Tx_0,x_0)$$

which completes the proof of Proposition 3.1.

Similarly, we can also get the following proposition. We will omit its proof.

Proposition 3.2. *Let (X,d) be a complete metric space. Suppose the mapping $T: X \to X$ satisfies*

$$d(Tx,Ty) \le a(d(Tx,y)+d(x,Ty))$$

for all $x,y \in X$, where $0 < a < \frac{1}{2}$. Then T has a unique fixed point p in X. Moreover, for any $x_0 \in X$, the successive sequence $\{x_n\}$ where $x_n = Tx_{n-1} (n = 1,2,\cdots)$ converges to the fixed point p and satisfies $d(x_n,p) \le \frac{(2a)^n}{(1-a)^{n-1}} d(Tx_0,x_0)$.

In order to present the new iterated function systems, we need the following lemmas.

Lemma 3.1. *Let (X,d) be a complete metric space. Suppose the mapping $T: X \to X$ is continuous and satisfies*

$$d(Tx,Ty) \le a(d(x,y)+d(Tx,y)+d(x,Ty))$$

for all $x,y \in X$, where $0 < a < \frac{1}{3}$. Then $T: H(X) \to H(X)$ defined by $T(B) = \{T(X) : x \in B\}$ for each $B,C \in H(X)$ also satisfies

$$h(T(B),T(C)) \le a(h(B,C)+h(T(B),C)+h(B,T(C))). \tag{3.3}$$

Proof. It is easy to see that $T: H(X) \to H(X)$ satisfies

$$d(T(B),T(C)) \le a(d(B,C)+d(T(B),C)+d(B,T(C))), \forall B,C \in H(X)$$

and

$$d(T(C),T(B)) \le a(d(C,B)+d(T(C),B)+d(C,T(B))), \forall B,C \in H(X).$$

So, for all $B,C \in H(X)$, we have

$$\begin{aligned} h(T(B),T(C)) &= d(T(B),T(C)) \vee d(T(C),T(B)) \\ &\le a\{[d(B,C)+d(T(B),C)+d(B,T(C))] \vee [d(C,B)+d(T(C),B)+d(C,T(B))]\} \\ &\le a\{[d(B,C) \vee d(C,B)] + [d(T(B),C) \vee d(C,T(B))] + [d(B,T(C)) \vee d(T(C),B)] \\ &\le a[h(B,C)+h(T(B),C)+h(B,T(C))], \end{aligned}$$

which completes the proof of Lemma 3.1.

Lemma 3.2. *Let (X,d) be a complete metric space. Let $T_n: n=1,2,\cdots,N$ be mappings which map $(H(X),h)$ into $(H(X),h)$. Suppose the mappings T_n satisfy*

$$h(T_n(B),T_n(C)) \le \alpha_n(h(B,C)+h(T_n(B),C)+h(B,T_n(C)))$$

for all $B,C\in H(X)$, where $0<\alpha_n<\frac{1}{3}$. Define $T: H(X)\to H(X)$ by $T(B)=T_1(B)\cup T_2(B)\cup\cdots\cup T_N(B)=\cup_{n=1}^{N}T_n(B)$ for each $B\in H(X)$. Then T also satisfies

$$h(T(B),T(C)) \le \alpha(h(B,C)+h(T(B),C)+h(B,T(C))),$$

where $B,C\in H(X), \alpha=\max\{\alpha_n: n=1,2,\cdots,N\}$.

Proof. We shall prove the lemma by using mathematical induction method. For $N=1$, the statement is obviously true. Now for $N=2$, we see that

$$\begin{aligned}
h(T(B),T(C)) &= h(T_1(B)\cup T_2(B),T_1(C)\cup T_2(C))\\
&\le h(T_1(B),T_1(C))\vee h(T_2(B),T_2(C))\\
&\le \alpha_1[h(B,C)+h(T_1(B),C)+h(B,T_1(C))]\vee\alpha_2[h(B,C)+h(T_2(B),C)+h(B,T_2(C))]\\
&\le \max\{\alpha_1,\alpha_2\}[h(B,C)+h(T_1(B),C)\vee h(T_2(B),C)+h(B,T_1(C))\vee h(B,T_2(C))]\\
&= \alpha[h(B,C)+h(C,T_1(B)\cup T_2(B))+h(B,T_1(C)\cup T_2(C))]
\end{aligned}$$

where $\alpha=\max\{\alpha_1,\alpha_2\}$. Therefore,

$$h(T(B),T(C)) \le \alpha(h(B,C)+h(T(B),C)+h(B,T(C))).$$

By induction we see Lemma 3.2 is proved.

Thus, from all the above results, we are in the position to present the following theorem for the new iterated function systems $\{T_n\}_{n=1}^{N}$ consisting of generalized contractive mappings defined as

$$d(T_nx,T_ny) \le \alpha_n(d(x,y)+d(T_nx,y)+d(x,T_ny)) \tag{3.4}$$

for all $x,y\in X$, where α_n are constants with $0<\alpha_n<\frac{1}{3}$.

Theorem 3.1. *Let (X,d) be a complete metric space. Suppose the mappings $T_n: X\to X, n=1,\cdots,N$ are continuous and satisfy the generalized contraction condition as* (3.4). *Then the transformation $T: H(X)\to H(X)$ defined by $T(B)=\cup_{n=1}^{N}T_n(B)$ for all $B\in H(X)$ also satisfies the generalized contraction condition* (3.3). *Its unique fixed point in (H(X), h(d)), which is also called an attractor, $A\in H(X)$, obeys $A=T(A)=\cup_{n=1}^{N}T_n(A)$ and is given by $A=\lim_{n\to\infty}T^{\circ n}(B)$ for any $B\in H(X)$.*

Similar to Theorem 3.1, the following Theorem 3.2 can be deduced by the same method, we omits its proof.

Theorem 3.2. *Let (X,d) be a complete metric space. Suppose the mappings $T_n : X \to X, n = 1,\cdots,N$ are continuous and satisfy the following condition*

$$d(T_nx,T_ny) \le \alpha_n(d(T_nx,y)+d(x,T_ny)) \tag{3.5}$$

for all $x,y \in X$, where α_n are constants with $0 < \alpha_n < \frac{1}{2}$. Then the transformation $T : H(X) \to H(X)$ defined by $T(B) = \cup_{n=1}^{N} T_n(B)$ for all $B \in H(X)$ also satisfies the condition

$$h(T(B),T(C)) \le \alpha(h(T(B),C)+h(B,T(C))),$$

where $B,C \in H(X), \alpha = \max\{\alpha_n : n = 1,2,\cdots,N\}$. Its unique fixed point in (H(X), h), which is also called an attractor, $A \in H(X)$, obeys $A = T(A) = \cup_{n=1}^{N} T_n(A)$ and is given by $A = \lim_{n\to\infty} T^{\circ n}(B)$for any $B \in H(X)$.

Remark 3.1. Theorem 3.1 and Theorem 3.2 are a valuable addition to the main results of the literatures [3, 8, 12].

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