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GENERALIZATIONS OF THE SUZUKI AND KANNAN FIXED POINT THEOREMS IN G-CONE METRIC SPACES

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Abstract. In this paper we develop the Banach contraction principle and Kannan fixed point theorem on generalized cone metric spaces. We prove a version of Suzuki and Kannan type generalizations of fixed point theorems in generalized cone metric spaces.

Key words: *fixed point, D-metric space, 2-metric space, generalized cone metric space, Kannan mapping, generalized Kannan mapping, contractive mapping*

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1 Introduction

Recently, non-convex analysis has found some applications in optimization theory, and so there have been some investigations about non-convex analysis, especially ordered normed spaces, normal cones and topical functions. Huang and Zhang[5] have replaced the real numbers by an ordering Banach space and define cone metric spaces. They have proved some fixed point theorems of contractive mappings on cone metric spaces.

Let $(E,\|.\|)$ be a real Banach space and P be a subset of E, then P is called a cone whenever

(1) P is closed, non-empty and $P \neq \{0\}$.

(2) $ax+by \in P$ for all $x,y \in P$ and non-negative real numbers a,b.

(3) $P \cap (-P) = \{0\}$.

For a given cone $P \subseteq E$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y-x \in P$. We write $x \prec y$ if $x \preceq y$ and $x \neq y$ and write $x \ll y$ if $y-x \in$ int P, where int$P$ denotes the interior of P. The cone P is normal if there is a number $M>0$ such that for all $x,y \in E$

$$0 \preceq x \preceq y \Longrightarrow \|x\| \leq M\|y\|.$$

The least positive number M satisfying the above inequality is called the normal constant of P. Rezapour and Hamalbarani[8] proved that there are no normal cones with normal constant $M<1$ and for each $k>1$ there are cones with normal constant $M>k$. The cone P is called regular if every increasing sequence which is bounded from above is convergent, that is, if $\{x_n\}_{n\in\mathbf{N}}$ is sequence in E such that $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq y$ for some $y\in E$, then there is $x\in E$ such that $\lim_{n\to\infty}\|x_n-x\|=0$. Equivalently, the cone P is called regular if every decreasing sequence which is bounded from below is convergent.

A subset $F\subseteq E$ is called bounded if there exists $u\in E$ such that $x\preceq u$ for every $x\in F$, in this case we say u is an upper bound of F. Similarly the lower bounds can be defined for F. The set F is said to be bounded if F is bounded from both below and above. We say that E has the supremum property, if every non-empty above bounded subset of E has a least upper bound in E. It is easy to see that if E has the supremum property, then E is regular.

The concept of 2-Metric space as a generalization of metric space was first defined by Gahler[4] in 1963, but some other authors proved that there is no relation between these two concepts. Dhage[3] defined D-metric space as a generalization of the metric space, later Mustafa and Sims[7] showed that most of the claims concerning fundamental topological properties of D-metric spaces are incorrect and reintroduced a new structure of generalized metric spaces, called G-metric space. Recently Beg et, al. [2] introduced the G-cone metric space and proved some common fixed point theorems in this space.

Throughout this paper we suppose that $(E,\|.\|)$ is a real Banach space and P is a normal cone with a constant M and the supremum property in E and $\mathrm{int}P\neq\emptyset$. The purpose of this paper is to generalize and unify a version of Banach contraction principle and Kannan fixed point theorem on complete generalized cone metric spaces. First we briefly recall the definitions and basic properties of generalized cone metric spaces. For more information we refer to the articles by Mustafa and Sims[7] and Beg et al[2].

Definition 1.1. Let X be a non-empty set, then the mapping $G: X\times X\times X\to E$ is called a generalized cone metric or simply a G-cone metric on X if it satisfies the following axioms:

(G_1) $G(x,y,z)=0$ if $x=y=z$.

(G_2) $G(x,x,y)\succ 0$, whenever $x\neq y$, for all $x,y\in X$.

(G_3) $G(x,x,y)\preceq G(x,y,z)$, whenever $y\neq z$.

(G_4) G is a symmetry function of its three variables.

(G_5) $G(x,y,z)\preceq G(x,y,u)+G(u,u,z)$ for all $x,y,z,u\in X$.

If G is a G-cone metric on X, then (X,G) is called G-cone metric space. A G-cone metric space (X,G) is called symmetric if $G(x,y,y) = G(x,x,y)$, for all $x,y \in X$.

In the following we give examples of the symmetric and nonsymmetric G-cone metric space.

Example 1.2. Let (X,d) be a cone metric space. For all $x,y,z \in X$ define

(1) $G_1 : X \times X \times X \to E$ by $G_1(x,y,z) = d(x,y) + d(x,z) + d(y,z)$.

(2) $G_2 : X \times X \times X \to E$ by $G_2(x,y,z) = \max\{d(x,y), d(x,z), d(y,z)\}$.

Clearly (X,G_i) for all $i = 1,2$ is a symmetric G-cone metric space.

Example 1.3. Let $X = \{a,b\}$, $E = \mathbf{R}^3$ and $P = \{(x,y,z) \in \mathbf{R}^3 | \, x,y,z \geq 0\}$. Define $G : X^3 \to E$ by

$$G(a,a,a) = G(b,b,b) = (0,0,0),$$
$$G(a,b,b) = G(b,b,a) = G(b,a,b) = (0,1,1),$$
$$G(a,a,b) = G(b,a,a) = G(a,b,a) = (0,1,0).$$

Then (X,G) is a nonsymmetric G-cone metric space.

Definition 1.4. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in G-cone metric space (X,G). Then

(1) We say that $\{x_n\}_{n\in\mathbf{N}}$ converges to $x \in X$, if for every $c \in E$ with $c \gg 0$ there is $N \in \mathbf{N}$ such that for all $m,n > N$, $G(x_m,x_n,x) \ll c$. In this case we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

(2) $\{x_n\}_{n\in\mathbf{N}}$ is said to be a Cauchy sequence if for every $c \in E$ with $c \gg 0$ there is $N \in \mathbf{N}$ such that for all $m,n,k > N$, $G(x_m,x_n,x_k) \ll c$.

(3) A G-cone metric space (X,G) is said to be complete if every Cauchy sequence in X is convergent in X.

The following results are studied by Beg et al. in [2].

Proposition 1.5. *Let (X,G) be a G-cone metric space and $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in X. Then the following conditions are equivalent*

(1) *$\{x_n\}_{n\in\mathbf{N}}$ is converges to x.*

(2) *$G(x_n,x_n,x) \to 0$ as $n \to \infty$.*

(3) *$G(x_n,x,x) \to 0$ as $n \to \infty$.*

(4) *$G(x_m,x_n,x) \to 0$ as $n,m \to \infty$.*

Proposition 1.6. *Let (X,G) be a G-cone metric space and $\{x_n\}_{n\in\mathbf{N}}$ be a sequence in X. Then*

(1) *If $x,y \in X$ and if $\{x_n\}_{n\in\mathbf{N}}$ converges to x,y, then $x = y$.*

(2) *$\{x_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence if and only if $G(x_n,x_m,x_k) \to 0$ as $n,m,k \to \infty$.*

(3) *Every convergent sequence in X is a Cauchy sequence.*

Proposition 1.7. *Let (X,G) be a G-cone metric space and let $\{x_n\}_{n\in\mathbf{N}}, \{y_n\}_{n\in\mathbf{N}}$, $\{z_n\}_{n\in\mathbf{N}}$ be sequences in X such that $x_n \to x$, $y_n \to y$, $z_n \to z$. Then $G(x_n,y_m,z_k) \to G(x,y,z)$ as $n,m,k \to \infty$.*

From the definition we have

$$\frac{1}{2}G(x,y,y) \preceq G(x,x,y) \preceq 2G(x,y,y), \qquad \forall x,y \in X. \tag{1}$$

2 Fixed Point Theorems

In this section we study some fixed point theorems which are generalizations of the Banach contraction principle and Suzuki fixed point theorem[9]. The following theorem is referred to as the Banach contraction principle[1].

Theorem 2.1. *Let (X,d) be a complete metric space and let T be a contraction on X, i.e., there exists $r \in [0,1)$ such that $d(Tx,Ty) \leq rd(x,y)$ for all $x,y \in X$. Then T has a unique fixed point.*

Let (X,G) be a G-cone metric space, then a mapping $T : X \to X$ is called a G-contraction mapping if there exists a real number $r \in [0,1)$ such that

$$G(Tx,Ty,Tz) \preceq rG(x,y,z), \qquad \forall x,y,z \in X.$$

The following theorem is a generalization of Theorem 2.1 in G-cone metric spaces.

Theorem 2.2. *Let (X,G) be a complete G-cone metric space and let $T : X \to X$ be a G-contraction mapping on X, that is, there exists $r \in [0,1)$ such that*

$$G(Tx,Ty,Tz) \preceq rG(x,y,z)$$

for all $x,y,z \in X$. Then there exists a unique fixed point $z \in X$ for T. Moreover,

$$\lim_{n\to\infty} T^n x = z$$

for all $x \in X$.

Proof. For each $u \in X$ define the sequence $\{u_n\}_{n\in\mathbf{N}}$ in X by $u_n = T^n u$, then for all $n > m$ we have

$$\begin{aligned}
G(u_m,u_m,u_n) &\preceq G(u_m,u_m,u_{m+1}) + G(u_{m+1},u_{m+1},u_n) \\
&\preceq \cdots \preceq \sum_{k=m}^{n-1} G(u_k,u_k,u_{k+1}) \preceq G(u,u,Tu)\sum_{k=m}^{n-1} r^k \\
&\preceq \frac{r^m}{1-r}G(u,u,Tu) \to 0, \qquad \text{as} \qquad m \to \infty.
\end{aligned}$$

This shows that $\{u_n\}_{n\in\mathbf{N}}$ is a Cauchy sequence in (X,G). Since (X,G) is complete, there exists $z\in X$ such that $u_n\to z$. Thus for all $c\in E$ with $c\gg 0$ and $m\in\mathbf{N}$ there exists $N\in\mathbf{N}$ such that $G(u_n,u_n,z)\ll \frac{c}{3m}$, for all $n\geq N$. Hence we have

$$G(z,z,Tz)\preceq G(z,z,u_n)+G(u_n,u_n,Tz)\preceq 2G(u_n,u_n,z)+rG(u_{n-1},u_{n-1},z)\ll \frac{c}{m}.$$

This shows that $\frac{c}{m}-G(z,z,Tz)\in P$. Since $\frac{c}{m}-G(z,z,Tz)\to -G(z,z,Tz)$ and P is closed, $-G(z,z,Tz)\in P$, but $G(z,z,Tz)\in P$. Therefore $G(z,z,Tz)=0$ and so $Tz=z$. Now if x is another fixed point of T, then

$$G(z,z,x)=G(Tz,Tz,Tx)\preceq rG(z,z,x).$$

Hence $G(z,z,x)=0$ which implies $z=x$. Therefore the fixed point of T is unique.

The following theorem introduces a new type of mappings which generalizes Theorem 2.2.

Theorem 2.3. *Let (X,G) be a complete G-cone metric space and let $\lambda:[0,1)\to(\frac{1}{6},1]$ be a non-increasing and onto function defined by*

$$\lambda(r)=\begin{cases} 1, & 0\leq r\leq \frac{\sqrt{3}-1}{2},\\ \frac{1-2r}{2r^2}, & \frac{\sqrt{3}-1}{2}\leq r<\frac{1}{\sqrt{5}},\\ \frac{1}{4+2r}, & \frac{1}{\sqrt{5}}\leq r<1.\end{cases}$$

Suppose that $T:X\to X$ is a mapping on X and there exists $r\in[0,1)$ such that

$$\lambda(r)G(x,x,Tx)\preceq G(x,y,z)\quad\Longrightarrow\quad G(Tx,Ty,Tz)\preceq rG(x,y,z)$$

for all $x,y,z\in X$. Then there exists a unique fixed point $z\in X$ of T. Moreover,

$$\lim_{n\to\infty}T^n x=z$$

for all $x\in X$.

Proof. Since $\lambda(r)\leq 1$ hence $\lambda(r)G(x,x,Tx)\preceq G(x,x,Tx)$. Using the hypothesis we have

$$G(Tx,Tx,T^2x)\preceq rG(x,x,Tx),$$

so $G(T^nx,T^nx,T^{n+1}x)\preceq r^nG(x,x,Tx)$ for all $n\in\mathbf{N}$, $x\in X$. Choose $u\in X$ and define the sequence $\{u_n\}_{n\in\mathbf{N}}$ in X by $u_n=T^nu$. As in the proof of Theorem 2.2, we can prove that $\{u_n\}_{n\in\mathbf{N}}$ converges to some $z\in X$. Thus for all $x\in X-\{z\}$ there exists $N\in\mathbf{N}$ such that for all $n\geq N$

$$G(u_n,u_n,z)\preceq\frac{G(x,x,z)}{4}.$$

We also have

$$\begin{aligned}\lambda(r)G(u_n,u_n,Tu_n) &\preceq G(u_n,u_n,u_{n+1}) \preceq G(u_n,u_n,z)+G(z,z,u_{n+1})\\ &\preceq \frac{3}{4}G(x,x,z)=G(x,x,z)-\frac{1}{4}G(x,x,z)\\ &\preceq G(x,x,z)-G(u_n,u_n,z)\preceq G(x,x,u_n)\end{aligned}$$

which implies $G(Tx,Tx,u_{n+1})\preceq rG(x,x,u_n)$. If $n\to\infty$ we obtain

$$G(Tx,Tx,z)\preceq rG(x,x,z) \tag{2}$$

for all $x\neq z$. Arguing by contradiction, assume $T^kz\neq z$ for all $k\in\mathbf{N}$. By (2) we have

$$G(T^{k+1}z,T^{k+1}z,z)\preceq r^kG(Tz,Tz,z), \qquad \forall k\in\mathbf{N}. \tag{3}$$

By the hypothesis for r have three cases: If $0\le r\le\dfrac{\sqrt{3}-1}{2}$, then $2r^2+2r-1\le 0$ and $3r^2<1$. Also if we assume $G(T^2z,T^2z,z)\prec G(T^2z,T^2z,T^3z)$, then we have

$$\begin{aligned}G(Tz,Tz,z) &\preceq G(Tz,Tz,T^2z)+G(T^2z,T^2z,z)\prec rG(z,z,Tz)+G(T^2z,T^2z,T^3z)\\ &\preceq rG(z,z,Tz)+r^2G(z,z,Tz)\\ &=(r^2+r)G(z,z,Tz)\preceq(2r^2+2r)G(Tz,Tz,z)\preceq G(Tz,Tz,z)\end{aligned}$$

which is a contradiction. Hence we have

$$\lambda(r)G(T^2z,T^2z,T^3z)=G(T^2z,T^2z,T^3z)\preceq G(T^2z,T^2z,z).$$

By the hypothesis and (3) we obtain

$$\begin{aligned}G(Tz,Tz,z) &\preceq G(Tz,Tz,T^3z)+G(T^3z,T^3z,z)\preceq 2G(T^3z,T^3z,Tz)+G(T^3z,T^3z,z)\\ &\preceq 2rG(T^2z,T^2z,z)+r^2G(Tz,Tz,z)\preceq 3r^2G(Tz,Tz,z)\prec G(Tz,Tz,z).\end{aligned}$$

This is a contradiction. In the second case, if $\dfrac{\sqrt{3}-1}{2}\le r<\dfrac{1}{\sqrt{5}}$, then $3r^2<1$. If we assume $G(T^2z,T^2z,z)\prec\lambda(r)G(T^2z,T^2z,T^3z)$, then we obtain

$$\begin{aligned}G(Tz,Tz,z) &\preceq G(Tz,Tz,T^2z)+G(T^2z,T^2z,z)\prec rG(z,z,Tz)+\lambda(r)G(T^2z,T^2z,T^3z)\\ &\preceq rG(z,z,Tz)+r^2\lambda(r)G(z,z,Tz)\preceq(2r+2r^2\lambda(r))G(Tz,Tz,z)=G(Tz,Tz,z),\end{aligned}$$

which is a contradiction. Thus $\lambda(r)G(T^2z,T^2z,T^3z) \preceq G(T^2z,T^2z,z)$. As in the previous case we obtain

$$G(Tz,Tz,z) \preceq 3r^2G(Tz,Tz,z) \prec G(Tz,Tz,z).$$

This is a contradiction. In the third case, let $\frac{1}{\sqrt{5}} \leq r < 1$, then for all $x,y \in X$, either

$$\lambda(r)G(x,x,Tx) \preceq G(x,x,y) \quad \text{or} \quad \lambda(r)G(Tx,Tx,T^2x) \preceq G(Tx,Tx,y).$$

Because if

$$G(x,x,y) \prec \lambda(r)G(x,x,Tx) \quad \text{and} \quad G(Tx,Tx,y) \prec \lambda(r)G(Tx,Tx,T^2x),$$

then we get

$$\begin{aligned} G(Tx,Tx,x) &\preceq G(Tx,Tx,y)+G(y,y,x) \\ &\preceq G(Tx,Tx,y)+2G(x,x,y) \prec \lambda(r)G(Tx,Tx,T^2x)+2\lambda(r)G(x,x,Tx) \\ &\preceq r\lambda(r)G(x,x,Tx)+2\lambda(r)G(x,x,Tx) \preceq (2r+4)\lambda(r)G(Tx,Tx,x) = G(Tx,Tx,x) \end{aligned}$$

which is a contradiction. Therefore for all $n \in \mathbf{N}$ either

$$\lambda(r)G(u_{2n},u_{2n},u_{2n+1}) \preceq G(u_{2n},u_{2n},z) \quad \text{or} \quad \lambda(r)G(u_{2n+1},u_{2n+1},u_{2n+2}) \preceq G(u_{2n+1},u_{2n+1},z).$$

This yields

$$G(u_{2n+1},u_{2n+1},Tz) \preceq rG(u_{2n},u_{2n},z) \quad \text{or} \quad G(u_{2n+2},u_{2n+2},Tz) \preceq rG(u_{2n+1},u_{2n+1},z).$$

Since $u_n \to z$, there exists a subsequence $\{u_{n_k}\}_{k\in\mathbf{N}}$ such that $u_{n_k} \to Tz$. This shows $Tz = z$ which is a contradiction. Thus there is some $j \in \mathbf{N}$ so that $T^jz = z$. Using the Cauchy property of $\{u_n\}_{n\in\mathbf{N}}$ we obtain $Tz = z$. That is, z is a fixed point of T. The uniqueness of fixed point follows from (2). This completes the proof.

Theorem 2.4. *Let (X,G) be a complete G-cone metric space and let $\lambda : [0,1) \to (\frac{1}{3},1]$ be a non-increasing and onto function defined by*

$$\lambda(r) = \begin{cases} 1, & 0 \leq r \leq \sqrt{2}-1, \\ \frac{1-2r}{r^2}, & \sqrt{2}-1 \leq r < \frac{1}{\sqrt{5}}, \\ \frac{1}{2+r}, & \frac{1}{\sqrt{5}} \leq r < 1. \end{cases}$$

Suppose that $T : X \to X$ is a mapping on X and there exists $r \in [0,1)$ such that

$$\lambda(r)G(Tx,Tx,x) \preceq G(x,y,z) \quad \Longrightarrow \quad G(Tx,Ty,Tz) \preceq rG(x,y,z)$$

for all $x,y,z \in X$. Then there exists a unique fixed point $z \in X$ of T. Moreover,

$$\lim_{n\to\infty} T^n x = z$$

for all $x \in X$.

Proof. Since $\lambda(r) \le 1$, we have $\lambda(r)G(Tx,Tx,x) \preceq G(Tx,Tx,x)$ for all $x \in X$. Using the hypothesis we have

$$G(T^2x,T^2x,Tx) \preceq rG(Tx,Tx,x).$$

This shows $G(T^{n+1}x,T^{n+1}x,T^nx) \preceq r^nG(Tx,Tx,x)$ for all $n \in \mathbf{N}$. Choose $u \in X$ and define the sequence $\{u_n\}_{n\in\mathbf{N}}$ in X by $u_n = T^nu$. As in the proof of Theorem 2.2, we can prove that $\{u_n\}_{n\in\mathbf{N}}$ converges to some $z \in X$. Thus for all $x \in X - \{z\}$ there exists $N \in \mathbf{N}$ such that for all $n \ge N$

$$G(u_n,u_n,z) \preceq \frac{G(x,x,z)}{4}.$$

We also have

$$\begin{aligned}\lambda(r)G(Tu_n,Tu_n,u_n) &\preceq G(u_{n+1},u_{n+1},u_n) \preceq G(u_{n+1},u_{n+1},z) + G(z,z,u_n)\\ &\preceq G(u_{n+1},u_{n+1},z) + 2G(u_n,u_n,z)\\ &\preceq \frac{3}{4}G(x,x,z) = G(x,x,z) - \frac{1}{4}G(x,x,z)\\ &\preceq G(x,x,z) - G(u_n,u_n,z) \preceq G(x,x,u_n).\end{aligned}$$

Using the hypothesis we have

$$G(Tx,Tx,Tu_n) \preceq rG(x,x,u_n).$$

Let $n \to \infty$, we get

$$G(Tx,Tx,z) \preceq rG(x,x,z) \tag{4}$$

for all $x \neq z$. Arguing by contradiction, assume $T^kz \neq z$ for all $k \in \mathbf{N}$. By (4) we have

$$G(T^2z,T^2z,z) \preceq rG(Tz,Tz,z).$$

Hence for all $k \in \mathbf{N}$ we obtain

$$G(T^{k+1}z,T^{k+1}z,z) \preceq r^kG(Tz,Tz,z).$$

By the hypothesis for r have three cases. In the first case, let $0 \le r \le \sqrt{2}-1$, then $r^2+2r \le 1$ and $3r^2 < 1$. Also if we assume

$$G(T^2z,T^2z,z) \prec G(T^3z,T^3z,T^2z).$$

Then we have

$$\begin{aligned} G(Tz,Tz,z) &\preceq G(Tz,Tz,T^2z)+G(T^2z,T^2z,z) \prec 2G(T^2z,T^2z,Tz)+G(T^3z,T^3z,T^2z) \\ &\preceq 2rG(Tz,Tz,z)+r^2G(Tz,Tz,z) \\ &= (2r+r^2)G(Tz,Tz,z) \preceq G(Tz,Tz,z) \end{aligned}$$

which is a contradiction. Hence we have

$$G(T^3z,T^3z,T^2z) \preceq G(T^2z,T^2z,z).$$

Since $\lambda(r)=1$, we have

$$\lambda(r)G(T^3z,T^3z,T^2z) = G(T^3z,T^3z,T^2z) \preceq G(T^2z,T^2z,z).$$

By the hypothesis we obtain

$$G(T^3z,T^3z,Tz) \preceq rG(T^2z,T^2z,z).$$

This yields

$$\begin{aligned} G(Tz,Tz,z) &\preceq G(Tz,Tz,T^3z)+G(T^3z,T^3z,z) \\ &\preceq 2G(T^3z,T^3z,Tz)+G(T^3z,T^3z,z) \\ &\preceq 2rG(T^2z,T^2z,z)+r^2G(Tz,Tz,z) \\ &\preceq 2r^2G(Tz,Tz,z)+r^2G(Tz,Tz,z) \\ &= 3r^2G(Tz,Tz,z) \prec G(Tz,Tz,z). \end{aligned}$$

This is a contradiction. In the second case if

$$\sqrt{2}-1 \le r < \frac{1}{\sqrt{5}},$$

then $\lambda(r) = \dfrac{1-2r}{r^2}$ and $3r^2 < 1$. Let $G(T^2z,T^2z,z) \prec \lambda(r)G(T^3z,T^3z,T^2z)$, then we obtain

$$\begin{aligned} G(Tz,Tz,z) &\preceq G(Tz,Tz,T^2z)+G(T^2z,T^2z,z) \\ &\prec 2G(T^2z,T^2z,Tz)+\lambda(r)G(T^3z,T^3z,T^2z) \\ &\preceq 2rG(Tz,Tz,z)+r^2\lambda(r)G(Tz,Tz,z) \\ &= (2r+r^2\lambda(r))G(Tz,Tz,z) = G(Tz,Tz,z). \end{aligned}$$

Which is a contradiction. Thus $\lambda(r)G(T^3z,T^3z,T^2z)\preceq G(T^2z,T^2z,z)$. By the hypothesis we have

$$G(T^3z,T^3z,Tz)\preceq rG(T^2z,T^2z,z),$$

consequently

$$\begin{aligned} G(Tz,Tz,z) &\preceq G(Tz,Tz,T^3z)+G(T^3z,T^3z,z)\\ &\preceq 2G(^3z,T^3z,Tz)+r^2G(Tz,Tz,z)\\ &\preceq 2rG(T^2z,T^2z,z)+r^2G(Tz,Tz,z)\\ &\preceq 2r^2G(Tz,Tz,z)+r^2G(Tz,Tz,z)\\ &=3r^2G(Tz,Tz,z)\prec G(Tz,Tz,z). \end{aligned}$$

This is a contradiction. In the third case, if $\frac{1}{\sqrt{5}}\leq r<1$ then by the proof of Theorem 2.3 we can obtain a subsequence $\{u_{n_j}\}_{j\in\mathbf{N}}$ of $\{u_n\}_{n\in\mathbf{N}}$ such that $u_{n_k}\to Tz$, and consequently $Tz=z$, which ia also a contradiction. Hence there is some $j\in\mathbf{N}$ so that $T^jz=z$. Using the Cauchy property of $\{u_n\}_{n\in\mathbf{N}}$ we obtain $Tz=z$, which implies that z is a fixed point of T. The uniqueness of fixed point follows from (4). From this the proof is complete.

Definition 2.5. Let (X,G) be a G-cone metric space, then a mapping $T:X\to X$ is called a Kannan mapping if there exists a real number $\alpha\in[0,\frac{1}{3})$ such that

$$G(Tx,Ty,Tz)\preceq\alpha G(x,x,Tx)+\alpha G(y,y,Ty)+\alpha G(z,z,Tz),\qquad\forall x,y,z\in X.$$

Also T is called a generalized Kannan mapping if there exists a real number $r\in[0,1)$ such that

$$G(Tx,Ty,Tz)\preceq r\max\{G(x,x,Tx),G(y,y,Ty),G(z,z,Tz)\},\qquad\forall x,y,z\in X.$$

Theorem 2.6. *Let (X,G) be a G-cone metric space and let $T:X\to X$ be a mapping on X. Suppose that $x\in X$ satisfies*

$$G(Tx,Tx,T^2x)\preceq\frac{r}{2}G(x,x,Tx)$$

for some $r\in[0,1)$. Then for $y\in X$, either

$$\frac{1}{1+r}G(x,x,Tx)\preceq G(x,x,y)\quad\text{or}\quad\frac{1}{1+r}G(Tx,Tx,T^2x)\preceq G(Tx,Tx,y).$$

Proof. Let

$$\frac{1}{1+r}G(x,x,Tx)\succ G(x,x,y)\quad\text{and}\quad\frac{1}{1+r}G(Tx,Tx,T^2x)\succ G(Tx,Tx,y).$$

Then we have

$$\begin{aligned}G(x,x,Tx) &\preceq G(x,x,y)+G(y,y,Tx) \preceq G(x,x,y)+2G(Tx,Tx,y)\\ &\prec \frac{1}{1+r}G(x,x,Tx)+\frac{2}{1+r}G(Tx,Tx,T^2x)\\ &\preceq \frac{1}{1+r}G(x,x,Tx)+\frac{r}{1+r}G(x,x,Tx)=G(x,x,Tx),\end{aligned}$$

which is a contradiction. Therefore we obtain the desired result.

The following theorem is a Kannan version of Theorem 2.3 (See [6]).

Theorem 2.7. *Let (X,G) be a complete G-cone metric space and let $\varphi:[0,1)\to(\frac{1}{3},1]$ be a non-increasing and onto function defined by*

$$\varphi(r)=\begin{cases}1, & 0\le r<\frac{1}{\sqrt{3}},\\ \frac{1}{1+r}, & \frac{1}{\sqrt{3}}\le r<1.\end{cases}$$

Suppose $T:X\to X$ is a mapping on X and there exists $\alpha\in[0,\frac{1}{\sqrt{5}})$ and $r=\frac{4\alpha}{1-\alpha}\in[0,1)$ such that

$$\varphi(r)G(x,x,Tx)\preceq G(x,y,z) \implies G(Tx,Ty,Tz)\preceq \alpha G(x,x,Tx)+\alpha G(y,y,Ty)+\alpha G(z,z,Tz)$$

for all $x,y,z\in X$. Then there exists a unique fixed point $z\in X$ of T. Moreover,

$$\lim_{n\to\infty}T^nx=z$$

for all $x\in X$.

Proof. Since $\varphi(r)\le 1$, $\varphi(r)G(x,x,Tx)\preceq G(x,x,Tx)$. Using the hypothesis we have

$$G(Tx,Tx,T^2x)\preceq 2\alpha G(x,x,Tx)+\alpha G(Tx,Tx,T^2x). \tag{5}$$

So $G(Tx,Tx,T^2x)\le\frac{r}{2}G(x,x,Tx)$ which implies that $G(T^nx,T^nx,T^{n+1}x)\le\frac{r^n}{2^n}G(x,x,Tx)$ for all $n\in\mathbf{N}$. Choose $u\in X$ and define the sequence $\{u_n\}_{n\in\mathbf{N}}$ in X by $u_n=T^nu$. As in the proof of Theorem 2.3 we can prove that $\{u_n\}_{n\in\mathbf{N}}$ converges to some $z\in X$, and for all $x\in X-\{z\}$ there exists $N\in\mathbf{N}$ such that $\varphi(r)G(u_n,u_n,Tu_n)\preceq G(x,x,u_n)$ for all $n\ge N$. Using the hypothesis we have

$$G(Tx,Tx,Tu_n)\preceq 2\alpha G(x,x,Tx)+\alpha G(u_n,u_n,Tu_n). \tag{6}$$

If $n \to \infty$ we have

$$\begin{aligned} G(Tx,Tx,z) &= \lim_{n\to\infty} G(Tx,Tx,u_{n+1}) = \lim_{n\to\infty} G(Tx,Tx,Tu_n) \\ &\preceq \lim_{n\to\infty}(2\alpha G(x,x,Tx)+\alpha G(u_n,u_n,Tu_n)) \\ &= \lim_{n\to\infty}(2\alpha G(x,x,Tx)+\alpha G(u_n,u_n,u_{n+1})) = 2\alpha G(x,x,Tx). \end{aligned}$$

Hence for all $x \in X$ and $x \neq z$ we have

$$G(Tx,Tx,z) \preceq 2\alpha G(x,x,Tx). \tag{7}$$

By the hypothesis r have two cases: If $0 \le r < \frac{1}{\sqrt{3}}$, and arguing by contradiction, assume $Tz \neq z$. By (7) we have $G(T^2z,T^2z,z) \preceq \alpha rG(z,z,Tz)$. Therefore we obtain

$$\begin{aligned} G(Tz,Tz,z) &\preceq G(Tz,Tz,T^2z)+G(T^2,T^2,z) \preceq \frac{r}{2}G(z,z,Tz)+\alpha rG(z,z,Tz) \\ &\preceq rG(Tz,Tz,z)+2\alpha rG(Tz,Tz,z) \\ &= (r+2\alpha r)G(Tz,Tz,z) = \frac{4r+3r^2}{4+r}G(Tz,Tz,z) \prec G(Tz,Tz,z), \end{aligned}$$

which is also a contradiction. In the second case, $\frac{1}{\sqrt{3}} \le r < 1$, then by Theorem 2.6 and the proof of Theorem 2.3 we can obtain a subsequence $\{u_{n_j}\}_{j\in\mathbf{N}}$ of $\{u_n\}_{n\in\mathbf{N}}$ such that

$$\varphi(r)G(u_{n_j},u_{n_j},u_{n_j+1}) \preceq G(u_{n_j},u_{n_j},z)$$

for all $j \in \mathbf{N}$. From the assumption we have

$$\begin{aligned} G(z,z,Tz) &= \lim_{j\to\infty} G(u_{n_j+1},u_{n_j+1},Tz) \\ &\preceq \lim_{j\to\infty} r\max\{G(u_{n_j},u_{n_j},u_{n_j+1}),G(z,z,Tz)\} = rG(z,z,Tz). \end{aligned}$$

This shows that $Tz = z$. Therefore z is a fixed point of T. The uniqueness of fixed point follows from (7). This completes the proof.

Theorem 2.8. *Let (X,G) be a G-cone metric space and let T be a mapping on X. Suppose that $x \in X$ satisfies*

$$G(Tx,Tx,T^2x) \preceq rG(x,x,Tx)$$

for some $r \in [0,1)$. Then for $y \in X$, either

$$\frac{1}{4+2r}G(x,x,Tx) \preceq G(x,x,y) \quad \text{or} \quad \frac{1}{4+2r}G(Tx,Tx,T^2x) \preceq G(Tx,Tx,y)$$

holds.

Proof. Suppose that

$$\frac{1}{4+2r}G(x,x,Tx) \succ G(x,x,y) \quad \text{and} \quad \frac{1}{4+2r}G(Tx,Tx,T^2x) \succ G(Tx,Tx,y).$$

Then we have

$$\begin{aligned} G(x,x,Tx) &\preceq G(x,x,y)+G(y,y,Tx) \preceq G(x,x,y)+2G(Tx,Tx,y) \\ &\prec \frac{1}{4+2r}G(x,x,Tx)+\frac{2}{4+2r}G(Tx,Tx,T^2x) \preceq \frac{1+2r}{4+2r}G(x,x,Tx) \prec G(x,x,Tx). \end{aligned}$$

Which is a contradiction. Therefore we obtain the desired result.

The following theorem is a generalized Kannan version of Theorem 2.3.

Theorem 2.9. *Let (X,G) be a complete G-cone metric space and let $\lambda : [0,1) \to (\frac{1}{6},1]$ be as in Theorem 2.3. If $T : X \to X$ is a mapping on X and there exists $r \in [0,1)$ such that*

$$\lambda(r)G(x,x,Tx) \preceq G(x,y,z) \implies G(Tx,Ty,Tz) \preceq r\max\{G(x,x,Tx),G(y,y,Ty),G(z,z,Tz)\}$$

for all $x,y,z \in X$. Then there exists a unique fixed point $z \in X$ of T. Moreover,

$$\lim_{n\to\infty} T^n x = z$$

for all $x \in X$.

Proof. Since $\lambda(r) \leq 1$ hence $\lambda(r)G(x,x,Tx) \preceq G(x,x,Tx)$. Using the hypothesis we have

$$G(Tx,Tx,T^2x) \preceq r\max\{G(x,x,Tx),G(Tx,Tx,T^2x)\},$$

and so $G(Tx,Tx,T^2x) \preceq rG(x,x,Tx)$ for all $n \in \mathbf{N}$, $x \in X$. Choose $u \in X$ and define the sequence $\{u_n\}_{n\in\mathbf{N}}$ in X by $u_n = T^n u$. As in the proof of Theorem 2.3 we can prove that $\{u_n\}_{n\in\mathbf{N}}$ converges to some $z \in X$. Moreover, for all $x \in X - \{z\}$ there exists $N \in \mathbf{N}$ such that for all $n \geq N$

$$G(u_n,u_n,z) \preceq \frac{G(x,x,z)}{4}.$$

We also have

$$\begin{aligned} \lambda(r)G(u_n,u_n,Tu_n) &\preceq G(u_n,u_n,u_{n+1}) \preceq G(u_n,u_n,z)+G(z,z,u_{n+1}) \\ &\preceq \frac{3}{4}G(x,x,z) = G(x,x,z)-\frac{1}{4}G(x,x,z) \preceq G(x,x,z)-G(u_n,u_n,z) \preceq G(x,x,u_n) \end{aligned}$$

which implies $G(Tx,Tx,u_{n+1}) \preceq r\max\{G(x,x,Tx),G(u_n,u_n,Tu_n)\}$. If $n \to \infty$ we obtain

$$G(Tx,Tx,z) \preceq rG(x,x,Tx), \qquad \forall x \in X,\ x \neq z. \tag{8}$$

Now we show that z is a fixed point of T. In the case where $0 \le r < \dfrac{1}{\sqrt{5}}$, we have $\lambda(r) \le \dfrac{1-2r}{2r^2}$. First, by induction we show that

$$G(T^n z, T^n z, Tz) \preceq 2rG(z,z,Tz), \quad \forall n \ge 2. \tag{9}$$

Since for all $x \in X$ we have

$$G(T^2x, T^2x, Tx) \preceq 2G(Tx, Tx, T^2x) \preceq 2rG(x,x,Tx),$$

hence (9) holds for $n = 2$. Suppose $G(T^n z, T^n z, Tz) \preceq 2rG(z,z,Tz)$ for some $n \in \mathbf{N}$ for $n > 2$. Since

$$G(z,z,Tz) \preceq G(z,z,T^n z) + G(T^n z, T^n z, Tz),$$

we obtain $G(z,z,Tz) \preceq \dfrac{1}{1-2r} G(z,z,T^n z)$, and hence

$$\begin{aligned}\lambda(r)G(T^n z, T^n z, T^{n+1}z) &\preceq \frac{1-2r}{2r^2} G(T^n z, T^n z, T^{n+1}z) \preceq \frac{1-2r}{2r^n} G(T^n z, T^n z, T^{n+1}z)\\ &\preceq \frac{1-2r}{2} G(z,z,Tz) \preceq \frac{1}{2} G(z,z,T^n z) \preceq G(T^n z, T^n z, z).\end{aligned}$$

Therefore by the assumption we obtain

$$G(T^{n+1}z, T^{n+1}z, Tz) \preceq r\max\{G(T^n z, T^n z, T^{n+1}z), G(z,z,Tz)\} \preceq 2rG(z,z,Tz).$$

It follows that (9) holds for $n \ge 2$. Arguing by contradiction, assume $Tz \ne z$. Then from (9) $T^n z \ne z$ holds for all $n \in \mathbf{N}$. By (8) we also have

$$G(T^{n+1}z, T^{n+1}z, z) \preceq rG(T^n z, T^n z, T^{n+1}z) \preceq r^{n+1} G(z,z,Tz).$$

This shows that $T^n \to z$ which contradicts to (9). Thus we obtain $Tz = z$. If $\dfrac{1}{\sqrt{5}} \le r < 1$ then by Theorem 2.8 and the proof of Theorem 2.3 we can obtain a subsequence $\{u_{n_j}\}_{j\in\mathbf{N}}$ of $\{u_n\}_{n\in\mathbf{N}}$ such that $\lambda(r)G(u_{n_j}, u_{n_j}, u_{n_j+1}) \preceq G(u_{n_j}, u_{n_j}, z)$ for all $j \in \mathbf{N}$. From the assumption we have

$$\begin{aligned}G(z,z,Tz) &= \lim_{j\to\infty} G(u_{n_j+1}, u_{n_j+1}, Tz)\\ &\preceq \lim_{j\to\infty} r\max\{G(u_{n_j}, u_{n_j}, u_{n_j+1}), G(z,z,Tz)\} = rG(z,z,Tz).\end{aligned}$$

Since $r < 1$ the above inequality implies $Tz = z$. The uniqueness of fixed point follows from (8). This completes the proof.

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