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REGULARIZATION OF AN ILL-POSED HYPERBOLIC PROBLEM AND THE UNIQUENESS OF THE SOLUTION BY A WAVELET GALERKIN METHOD

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Abstract. We consider the problem $K(x)u_{xx} = u_{tt}$, $0 < x < 1$, $t \geq 0$, with the boundary condition $u(0,t) = g(t) \in L^2(\mathbf{R})$ and $u_x(0,t) = 0$, where $K(x)$ is continuous and $0 < \alpha \leq K(x) < +\infty$. This is an ill-posed problem in the sense that, if the solution exists, it does not depend continuously on g. Considering the existence of a solution $u(x,\cdot) \in H^2(\mathbf{R})$ and using a wavelet Galerkin method with Meyer multiresolution analysis, we regularize the ill-posedness of the problem. Furthermore we prove the uniqueness of the solution for this problem.

Key words: *ill-posed problem, meyer wavelet, hyperbolic equation, regularization*

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1 Introduction and Main Results

In [5] the authors have considered an inverse problem for the sideway heat equation with constant coefficient. The variational formulation, on the scaling space V_j, of the approximating problem, produces an infinite-dimensional system of second order ordinary differential equations with constant coefficients, for which the solution is known. Stability and convergence of the method follow the from form of this solution.

In a previous work[3], we studied the following parabolic partial differential equation problem with variable coefficients:

$$\begin{aligned} K(x)u_{xx}(x,t) &= u_t(x,t), & t \geq 0, \quad 0 < x < 1 \\ u(0,\cdot) &= g, & u_x(0,\cdot) = 0 \\ 0 < \alpha \leq K(x) &< +\infty, & K \quad \text{continuous}. \end{aligned}$$

Under the hypothesis of the existence of a solution for this problem, using a wavelet Galerkin method, we constructed a sequence of well-posed approximating problems in the scaling spaces of the Meyer multiresolution analysis, which has the property to filter away the high frequencies. We had shown the convergence of the method, applied to our problem, and we gave an estimate of the solution error. We get an estimate for the difference between the exact solution of this problem and the orthogonal projection, onto V_j, of the solution of the approximating problem defined on the scaling space V_{j-1}.

In [6] the authors have given the error estimate between the exact solution by the above problem and the approximating solution of wavelet-Galerkin method in the sense of pointwise convergence.

In our work[12], by assuming that $\dfrac{1}{K(x)}$ is Lipschitz, we proved that the existence of a solution $u(x,.) \in H^1(\mathbf{R})$, for the above problem, implies its uniqueness.

In this work, we will extend the results in [2] and [3] to the hyperbolic problem:

$$\begin{gathered} K(x)u_{xx}(x,t) = u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1 \\ u(0,\cdot) = g, \quad u_x(0,\cdot) = 0 \\ 0 < \alpha \leq K(x) < +\infty, \quad K \text{ continuous}. \end{gathered} \tag{1.1}$$

We assume $g \in L^2(\mathbf{R})$, when it is extended as vanishing for $t < 0$, and the problem to have a solution $u(x,\cdot) \in H^2(\mathbf{R})$, when it is extended as vanishing for $t < 0$.

Our approach follows quite closely to that used in [2] and [3].

In note 1 we show that problem (1.1) is ill-posed in the sense that a small disturbance on the boundary specification g, can produce a big alteration on its solution, if it exists.

We consider the Meyer multiresolution analysis. The advantage in making use of Meyer's wavelets is its good localization in the frequency domain, since its Fourier transform has compact support. Orthogonal projections onto Meyer's scaling spaces, can be considered as low pass filters, cutting off the high frequencies.

From the variational formulation of the approximating problem on the scaling space V_j, we get an infinite-dimensional system of second order ordinary differential equations with variable coefficients. An estimate obtained for the solution of this evolution problem, is used to regularize the ill-posed problem approaching it by well-posed problems. Using an estimate obtained for the difference between the exact solution of problem (1.1) and its orthogonal projection onto

V_j, we get an estimate for the difference between the exact solution of the problem (1.1) and the orthogonal projection, onto V_j, of the solution of the approximating problem defined on the scaling space V_{j-1}. Further we consider that $1/K(x)$ is Lipschitz and prove that the existence of a solution $u(x,\cdot)\in H^2(\mathbf{R})$ implies its uniqueness.

We would like to point out that our result is weaker than the overall uniqueness of the a solution $u(\cdot,\cdot)$ of problem (1.1), which cannot be discussed without further conditions on this problem. Our uniqueness result need assume that $x\in(0,1)$ is fixed and it is the solution $u(x,\cdot)\in H^2(R)$, as a function of the second variable, which is proved to be unique. More precisely, a solution $u(x,\cdot)$ can only be modified in a subset of $(-\infty,+\infty)$ of measure zero.

In section 2, we construct the Meyer multiresolution analysis. In section 3, we get the estimates of the numerical stability and the convergence of the wavelet Galerkin method. In section 4 we prove the uniqueness of the solution.

For a function $h\in L^1(\mathbf{R})\bigcap L^2(\mathbf{R})$ its Fourier transform is given by $\widehat{h}(\xi):=\int_{\mathbf{R}}h(x)e^{-ix\xi}\,\mathrm{d}x$. We use the notation e^x and $\exp x$ indistinctly.

2 Meyer Multiresolution Analysis

Definition 2.1. A multiresolution analysis, as defined in [1], is a sequence of closed subspaces V_j in $L^2(R)$, called *scaling spaces*, satisfying:

(M1) $V_j\subseteq V_{j-1}$ for all $j\in\mathbf{Z}$

(M2) $\bigcup_{j\in Z}V_j$ is dense in $L^2(\mathbf{R})$

(M3) $\bigcap_{j\in Z}V_j=\{0\}$

(M4) $f\in V_j$ if and only if $f(2^j\cdot)\in V_0$

(M5) $f\in V_0$ if and only if $f(\cdot-k)\in V_0$ for all $k\in\mathbf{Z}$

(M6) There exists $\phi\in V_0$ such that $\{\phi_{0,k}:k\in\mathbf{Z}\}$ is an orthonormal basis in V_0, where $\phi_{j,k}(x)=2^{-j/2}\phi(2^{-j}x-k)$ for all $j,k\in\mathbf{Z}$. The function ϕ is called the scaling function of the multiresolution analysis.

The scaling function of the Meyer Multiresolution Analysis is the function φ defined by its Fourier Transform:

$$\widehat{\varphi}(\xi):=\begin{cases}1, & |\xi|\le\frac{2\pi}{3},\\ \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi|-1\right)\right], & \frac{2\pi}{3}\le|\xi|\le\frac{4\pi}{3},\\ 0, & |\xi|>\frac{4\pi}{3},\end{cases}$$

where ν is a differentiable function satisfying

$$\nu(x) = \begin{cases} 0, & \text{if} \quad x \leq 0, \\ 1, & \text{if} \quad x \geq 1 \end{cases}$$

and

$$\nu(x) + \nu(1-x) = 1.$$

The associated mother wavelet ψ, called Meyer's Wavelet, is given by (see [1])

$$\widehat{\psi}(\xi) = \begin{cases} e^{i\xi/2} \sin\left[\frac{\pi}{2}\nu\left(\frac{3}{2\pi}|\xi| - 1\right)\right], & \frac{2\pi}{3} \leq |\xi| \leq \frac{4\pi}{3}, \\ e^{i\xi/2} \cos\left[\frac{\pi}{2}\nu\left(\frac{3}{4\pi}|\xi| - 1\right)\right], & \frac{4\pi}{3} \leq |\xi| \leq \frac{8\pi}{3}, \\ 0, & |\xi| > \frac{8\pi}{3}. \end{cases}$$

We will consider the Meyer Multiresolution Analysis with a scaling function φ. The orthogonal projection onto V_j, $P_j : L^2(\mathbf{R}) \to V_j$, is given by

$$P_j f(t) = \sum_{k \in \mathbf{Z}} \langle f, \varphi_{jk} \rangle \varphi_{jk}(t).$$

3 Stability and Convergence of the Method

In this section we approach the ill-posed problem (1.1) by well-posed problems, and we show, with an estimate error, the convergence of the wavelet method used. The next lemma is given in [3].

Lemma 3.1. *Let u and v be positive continuous functions, $x \geq a$ and $c > 0$. If*

$$u(x) \leq c + \int_a^x \int_a^s v(\tau) u(\tau) \mathrm{d}\tau \mathrm{d}s,$$

then

$$u(x) \leq c \exp\left(\int_a^x \int_a^s v(\tau)\, \mathrm{d}\tau \mathrm{d}s\right).$$

Proof. See [3].

Applying Fourier transform with respect to the time t in Problem (1.1), we obtain the following problem in the frequency space:

$$\widehat{u}_{xx}(x,\xi) = \frac{-\xi^2}{K(x)} \widehat{u}(x,\xi), \quad 0 < x < 1,\ \xi \in \mathbf{R},$$

$$\widehat{u}(0,\xi) = \widehat{g}(\xi), \quad \widehat{u}_x(0,\cdot) = 0,$$

whose solution satisfies

$$|\widehat{u}(x,\xi)| \le |\widehat{g}(\xi)| + \int_0^x \int_0^s \frac{\xi^2}{K(\tau)} |\widehat{u}(\tau,\xi)| \, \mathrm{d}\tau \mathrm{d}s$$

Then, by Lemma 3.1, for $\widehat{g}(\xi) \neq 0$, we have

$$|\widehat{u}(x,\xi)| \le |\widehat{g}(\xi)| \exp\Big[\xi^2 \int_0^x \int_0^s \frac{1}{K(\tau)} \, \mathrm{d}\tau \mathrm{d}s\Big]. \tag{3.1}$$

Lemma 3.2. *The operator $D_j(x)$ defined by*

$$[(D_j)_{lk}(x)]_{l\in\mathbf{Z},\, k\in\mathbf{Z}} = \Big[\frac{1}{K(x)} \langle \varphi''_{jl}, \varphi_{jk} \rangle\Big]_{l\in\mathbf{Z}, k\in\mathbf{Z}}$$

satisfies the following three conditions:

1) $(D_j)_{lk}(x) = (D_j)_{kl}(x)$.

2) $(D_j)_{lk}(x) = (D_j)_{(l-k)0}(x)$. Hence, $(D_j)_{lk}(x)$ is a Töplitz matrix.

3) $\|D_j(x)\| \le \frac{\pi^2 4^{-j+1}}{K(x)}$.

Proof. The proof follows from [2], regarding that in 1) we can integrate twice by parts, since φ and φ' are reals and $\varphi_{jk}(x) \to 0$, $\varphi'_{jk}(x) \to 0$, when $x \to \pm\infty$, and in 3) Γ_j is defined by:

$$\begin{aligned}\Gamma_j(t) = &-2^{-j}\big[(t-2^{-j+1}\pi)^2 |\widehat{\varphi_{j0}}(t-2^{-j+1}\pi)|^2 + t^2 |\widehat{\varphi_{j0}}(t)|^2 \\ &+ (t+2^{-j+1}\pi)^2 |\widehat{\varphi_{j0}}(t+2^{-j+1}\pi)|^2\big].\end{aligned}$$

Let us now consider the following approximating problem in V_j:

$$\begin{cases} K(x)\, u_{xx}(x,t) = P_j u_{tt}(x,t), \qquad t \ge 0, \qquad 0 < x < 1, \\ u(0,\cdot) = P_j g, \\ u_x(0,\cdot) = 0, \\ u(x,t) \in V_j, \end{cases} \tag{3.2}$$

where the projection in the first equation of (3.2) is needed because we can have $\varphi \in V_j$ with $\varphi'' \notin V_j$ (see note 2 below).

Its variational formulation is

$$\begin{cases} \big\langle K(x)\, u_{xx} - u_{tt} \ ,\ \varphi_{jk} \big\rangle = 0, \\ \big\langle u(0,\cdot) \ ,\ \varphi_{jk} \big\rangle = \big\langle P_j g \,,\, \varphi_{jk} \big\rangle, \quad \big\langle u_x(0,\cdot) \ ,\ \varphi_{jk} \big\rangle = \big\langle 0 \,,\, \varphi_{jk} \big\rangle, \quad k \in \mathbf{Z}, \end{cases}$$

where φ_{jk} is the orthonormal basis of V_j given by the scaling function φ. Consider u_j a solution of the approximating problem (3.2), given by $u_j(x,t) = \sum_{l\in\mathbf{Z}} w_l(x) \varphi_{jl}(t)$. Then, we have

$$(u_j)_{tt}(x,t) = \sum_{l\in\mathbf{Z}} w_l(x) \varphi''_{jl}(t)$$

and

$$(u_j)_{xx}(x,t) = \sum_{l\in\mathbf{Z}} w_l''(x)\varphi_{jl}(t).$$

Therefore,

$$K(x)(u_j)_{xx}(x,t) - (u_j)_{tt}(x,t) = K(x)\sum_{l\in\mathbf{Z}} w_l''(x)\varphi_{jl}(t) - \sum_{l\in\mathbf{Z}} w_l(x)\varphi_{jl}''(t),$$

Hence

$$\begin{aligned}
&\langle K(x)(u_j)_{xx} - (u_j)_{tt}, \varphi_{jk}\rangle = 0 \Longleftrightarrow \langle \sum_{l\in\mathbf{Z}} K(x)w_l''\varphi_{jl} - \sum_{l\in\mathbf{Z}} w_l\varphi_{jl}'', \varphi_{jk}\rangle = 0\\
&\Longleftrightarrow \sum_{l\in\mathbf{Z}} K(x)w_l''\langle\varphi_{jl},\varphi_{jk}\rangle = \sum_{l\in\mathbf{Z}} w_l\langle\varphi_{jl}'',\varphi_{jk}\rangle\\
&\Longleftrightarrow K(x)w_k'' = \sum_{l\in\mathbf{Z}} w_l\langle\varphi_{jl}'',\varphi_{jk}\rangle, \quad k\in\mathbf{Z}\\
&\Longleftrightarrow \frac{d^2}{dx^2}w_k = \sum_{l\in\mathbf{Z}} w_l\frac{1}{K(x)}\langle\varphi_{jl}'',\varphi_{jk}\rangle \quad \Longleftrightarrow \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2}w_k = \sum_{l\in\mathbf{Z}} w_l(D_j)_{lk}(x).
\end{aligned}$$

where, as defined before, $(D_j)_{lk}(x) = \dfrac{1}{K(x)}\langle\varphi_{jl}'',\varphi_{jk}\rangle$. Thus, we get an infinite-dimensional system of ordinary differential equations:

$$\begin{cases}
\dfrac{\mathrm{d}^2}{\mathrm{d}x^2}w = D_j(x)w,\\
w(0) = \gamma,\\
w'(0) = 0,
\end{cases} \tag{3.3}$$

where γ is given by

$$P_jg = \sum_{z\in\mathbf{Z}} \gamma_z\varphi_{jz} = \sum_{z\in\mathbf{Z}} \langle g,\varphi_{jz}\rangle\varphi_{jz}.$$

Lemma 3.3. *If w is a solution of the evolution problem of second order* (3.3)*, then*

$$\|w(x)\| \le \|\gamma\|\exp\left(4^{-j+1}\pi^2\int_0^x\int_0^s \frac{1}{K(\tau)}\,\mathrm{d}\tau\,\mathrm{d}s\right)$$

Proof. Follows by (3.1). For details, see [3].

Theorem 3.4 (Stability of the wavelet Galerkin method). *Let u_j and v_j be solutions in V_j of the approximating problems* (3.2) *for the boundary specifications g and $\widetilde{g}$, respectively. If $\|g-\widetilde{g}\| \le \varepsilon$ then*

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \le \varepsilon\exp\big(\frac{4^{-j+1}\pi^2}{2\alpha}x^2\big),$$

where α satisfies $0 < \alpha \le K(x) < +\infty$ as in the definition of the problem (1.1)*. For j such that $4^{-j} \le \dfrac{\alpha}{2\pi^2}\log\varepsilon^{-1}$ we have*

$$\|u_j(x,\cdot) - v_j(x,\cdot)\| \le \varepsilon^{1-x^2}.$$

Proof. By Lemma 3.3 and linearity of (3.3), the proof follows quite closely from that of Theorem 3.4 in [3, page 221].

We will consider the problem (1.1), for functions $g \in L^2(R)$ such that $\widehat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(\mathbf{R})$, where $\widehat{g}$ is the Fourier transform of g. The inverse Fourier transform of $\exp(-\frac{\xi^2+|\xi|}{2\alpha})$, for instance, satisfies this condition. Define

$$f := \widehat{g}(\xi)\exp\big(\frac{\xi^2}{2\alpha}\big) \in L^2(\mathbf{R}). \tag{3.4}$$

Proposition 3.5. *If $u(x,t)$ is a solution of problem* (1.1)*, then*

$$\|u(x,\cdot) - P_j u(x,\cdot)\| \le \|f\|_{L^2(\mathbf{R})} \exp(-\frac{2}{9}\frac{\pi^2}{\alpha}\, 4^{-j}(1-x^2)),$$

where f is given by (3.4).

Proof. Follows quite closely from Proposition 3.5 in [3].

Proposition 3.6. *If u is a solution of problem* (1.1) *and u_{j-1} is a solution of the approximating problem in V_{j-1} then*

$$\widehat{u}(x,\xi) = \widehat{u}_{j-1}(x,\xi) \quad \mathit{for}\ |\xi| \le \frac{4}{3}\pi 2^{-j}. \tag{3.5}$$

Consequently,

$$P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot). \tag{3.6}$$

Proof. See Proposition 3.6 in [3].

Theorem 3.7. *Let u be a solution of* (1.1) *with the condition $u(0,\cdot) = g$, and let f be given by (3.4). Let v_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification $\widetilde{g}$ such that $\|g-\widetilde{g}\| \le \varepsilon$. If $j = j(\varepsilon)$ is such that $4^{-j} = \dfrac{\alpha}{8\pi^2}\log\varepsilon^{-1}$, then*

$$\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \varepsilon^{1-x^2} \ + \ \|f\|_{L^2(\mathbf{R})} \cdot \varepsilon^{\frac{1}{36}(1-x^2)}.$$

Proof. We have

$$\begin{aligned}\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| &\le \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot) + P_j u(x,\cdot) - u(x,\cdot)\| \\ &\le \|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| + \|P_j u(x,\cdot) - u(x,\cdot)\|.\end{aligned}$$

Let u_{j-1} be a solution of (3.2) in V_{j-1} for the boundary specification g. By (3.6), $P_j u(x,\cdot) = P_j u_{j-1}(x,\cdot)$. Thus, by Theorem 3.4, we have

$$\begin{aligned}\|P_j v_{j-1}(x,\cdot) - P_j u(x,\cdot)\| &= \|P_j v_{j-1}(x,\cdot) - P_j u_{j-1}(x,\cdot)\| \\ &\le \|v_{j-1}(x,\cdot) - u_{j-1}(x,\cdot)\| \le \varepsilon^{1-x^2}.\end{aligned}$$

Now, by Proposition 3.5,

$$\|P_j u(x,\cdot) - u(x,\cdot)\| \le \|f\|_{L^2(\mathbf{R})} \exp(-\frac{2}{9}\frac{\pi^2}{\alpha}\, 4^{-j}(1-x^2)) \le \|f\|_{L^2(\mathbf{R})} \cdot \varepsilon^{\frac{1}{36}(1-x^2)}.$$

Then $\|P_j v_{j-1}(x,\cdot) - u(x,\cdot)\| \le \varepsilon^{1-x^2} + \|f\|_{L^2(\mathbf{R})} \varepsilon^{\frac{1}{36}(1-x^2)}$.

4 Uniqueness of the Solution

The infinite-dimensional system of ordinary differential equations (3.3) can be written in the following way:

$$\begin{cases} \dfrac{dv}{dx} = D_j(x)w + 0v, \\ \dfrac{dw}{dx} = 0w + v, \\ w(0) = \gamma \ \text{ and } \ v(0) = 0, \end{cases} \qquad \begin{cases} \dfrac{dV}{dx} = A_j(x)V, \\ V(0) = (0,\gamma)^T, \end{cases}$$

where $V = (v,w) \in X := l^2(\mathbf{R}) \times l^2(\mathbf{R})$, $x \in [0,1)$ and

$$A_j(x) = \begin{bmatrix} 0 & D_j(x) \\ 1 & 0 \end{bmatrix}$$

with $\left\|A_j(x)V\right\|_X = \left\|(D_j(x)w,v)\right\|_X = \sqrt{\left\|D_j(x)w\right\|_{l^2}^2 + \|v\|_{l^2}^2}$

Lemma 4.1. *For all* $j \in \mathbf{Z}$, $A_j(x) : X \longrightarrow X$ *is a uniformly bounded linear operator on* $x \in [0,1)$.

Proof. Follows from Lemma 3.2. For details see [2].

Lemma 4.2. *If* $\dfrac{1}{K(x)}$ *is Lipschitz on* $[0,1)$, *then* $x \longmapsto D_j(x)$ *is Lipschitz on* $[0,1), \forall\, j \in \mathbf{Z}$. *Consequently* $x \longmapsto A_j(x)$ *is Lipschitz on* $[0,1)$.

Proof. See the proof Lemma 4 in [2].

Lemma 4.3. *For each* $j \in Z$, *the operator* $[0,1) \ni x \longmapsto A_j(x)$ *is continuous in the uniform operator topology.*

Proof. Let $x \in [0,1)$ and $\varepsilon > 0$. By Lemma 4.2, $A_j(x)$ is Lipschitz with Lipschitz constant L_j. Let $\delta_\varepsilon := \varepsilon/L_j$. We have for $y \in [0,1)$:

$$|x-y| < \delta_\varepsilon \Longrightarrow \left\|A_j(x) - A_j(y)\right\| \le L_j\,|x-y| < L_j \cdot \delta_\varepsilon = \varepsilon.$$

By the previous Lemmas, we have:

Theorem 4.4. *The infinite-dimensional system of ordinary differential equations* (3.3) *has a unique solution.*

Proof. The result follows from Lemma 4.1, Lemma 4.2, Lemma 4.3 above and theorem 5.1 in [4, page 127].

Theorem 4.5. *Let* u *be a solution of problem* (1.1) *with the condition* $u(0,\cdot) = g$ *where* g *satisfies* (3.4). *Then, for any sequence* j_n, *such that* $j_n \longrightarrow -\infty$ *as* $n \longrightarrow +\infty$, *there exists* *a unique sequence u_{j_n} of solutions of the approximating problems (3.2) in V_{j_n} with conditions $u_{j_n}(0,\cdot) = P_{j_n}g$ and $\forall x \in [0,1)$ such that*

$$P_{j_n+1}u_{j_n}(x,\cdot) \longrightarrow u(x,\cdot) \ in\, L^2.$$

Proof. By Theorem 4.4 each approximating problem has a unique solution. Then the result follows from Theorem 3.7, with $\widetilde{g} = g$, since that j and ε are functionally related by $4^{-j} = \dfrac{\alpha}{8\pi^2} \log \varepsilon^{-1}$ independently of u.

Corollary 4.6. The problem (1.1) has at most one solution, for each $x \in [0,1)$, whenever g satisfies (3.4).

Conclusion

We have considered solutions $u(x,\cdot) \in H^2(\mathbf{R})$ of the problem $K(x)u_{xx} = u_{tt},\quad 0 < x < 1$, $t \geq 0$, with the boundary specification g and $u_x(0,\cdot) = 0$, where $K(x)$ is continuous, $0 < \alpha \leq K(x) < +\infty$, $\dfrac{1}{K(x)}$ is Lipschitz and $\widehat{g}(\xi)\exp(\xi^2/(2\alpha)) \in L^2(\mathbf{R})$. Utilizing a wavelet Galerkin method with the Meyer multiresolution analysis, we regularize the ill-posedness of the problem, approaching it by well-posed problems in the scaling spaces and we have shown that the convergence of the wavelet Galerkin method applied to our problem, with an estimate error, and that if a solution exists, it is unique.

Notes: 1) Consider the wave equation with Neumann condition:

$$u_{xx}(x,t) = u_{tt}(x,t), \quad t \geq 0,\ 0 < x < 1$$
$$u(0,\cdot) = g_n, \quad u_x(0,\cdot) = 0,$$

where

$$g_n(t) = \begin{cases} n^{-2}\cos\sqrt{2}nt, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

The solution of this problem is

$$u_n(x,t) = \begin{cases} \sum_{j=0}^{\infty} n^{-2}\cos(\sqrt{2}nt + j\pi)\frac{(\sqrt{2}nx)^{2j}}{(2j)!}, & \text{if } 0 \leq t \leq t_0 \\ 0, & \text{if } t > t_0\,. \end{cases}$$

Note that $g_n(t)$ converges uniformly to zero as n tends to infinity, while for $x > 0$, the solution $u_n(x,t)$ does not tend to zero.

2) Note that $(\varphi_{jl})'' \notin V_j$. In fact, if $(\varphi_{jl})'' \in V_j$ then $(\varphi_{jl})'' = \sum_{k\in\mathbf{Z}} \alpha_k \varphi_{jk}$. Hence

$$\widehat{(\varphi_{jl})''} = \sum_{k\in\mathbf{Z}} \alpha_k \widehat{\varphi_{jk}}.$$

So, we would have

$$-2^{j/2}e^{-i2^jl\xi}\xi^2\widehat{\varphi}(2^j\xi) = \sum_{k\in\mathbf{Z}} \alpha_k 2^{j/2}e^{-i2^{j/2}\xi}\widehat{\varphi}(2^j\xi).$$

This equality implies

$$\xi^2 = \sum_{k\in\mathbf{Z}} -\alpha_k e^{-i[2^j(k-l)\xi]}.$$

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