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KY FAN TYPE BEST APPROXIMATION THEOREM FOR A CLASS OF FACTORIZABLE MULTIFUNCTIONS

Hemant Kumar Nashine
(*Disha Insititute of Management and Technology, India*)
Zoran D. Mitrovic
(*University of Banjia Luka, Bosnia and Herzegovina)*

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Abstract. An existence result on Ky Fan type best approximation is proved. For this purpose, a class of factorizable multifunctions and the other one being a demicontinuous, relative almost quasi-convex, onto function on an approximately weakly compact, convex subset of Hausdorff locally convex topological vector space are used. As consequence, this result extends the best approximation results of Basha and Veeramani[8] and many others.

Key words: *Almost affine, almost quasi convex, approximately weakly compact, best approximation, demicontinuous map, locally convex space, relative almost quasi convex, Kakutani factorizable multifunction, upper semicontinuous map*
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1 Introduction

The Theory of approximation is a mathematical study of how given quantities can be approximated by other (usually simpler) ones under appropriate conditions. Over the years, the theory has become so extensive that it intersects with every other branch of analysis at present. One of the central problems in approximation theory is to determine points that minimize distances to a given point or subset. The field of best approximation has dealt with this problem rigorously and it is an active field of research within Approximation theory. The best approximation has always attracted analysts because it carries enough potential to be extended especially with the functional analytic approach in nonlinear analysis. In the mid of the 20th century, it was found that existence of fixed point has its relevance in proving the existence of best approximation. The best approximation is termed as invariant approximation in the case of self mappings. It is important to mention here that when non-self mapping was considered, Ky Fan technique is found simply a wonderful tool to prove the existence of best approximation. This technique is hypothesis of Ky Fan theorem.

Ky Fan[4] Approximation theorem is as below:

Theorem 1.1.[4] *Let $\mathcal{C}$ be a non-empty compact convex subset of Hausdorff locally convex topological vector space $\mathcal{X}$ with a continuous semi-norm p and $\mathcal{T}:\mathcal{C}\to\mathcal{X}$ is a single valued continuous map, then there exists an element x_0, called a best approximant in $\mathcal{C}$, such that*

$$p(x_0-\mathcal{T}x_0)=d_p(\mathcal{T}x_0,\mathcal{C})=\inf\{p(\mathcal{T}x_0-y):y\in\mathcal{C}\}. \tag{1.1}$$

Initially, Ky Fan's approximation approach[4] helped in proving the existence of fixed point under different boundary conditions. Later, it was applied in the field of approximation theory, minimax theory, game theory, and variational inequality. During this phase, interesting extensions of Ky Fan's theorem were made and a variety of applications, mostly in fixed point theory and approximation theory, were given by many analysts (see e.g., [11]).

An interesting turn was given by Prolla[6] who extended Ky Fan Theorem by involving another mapping called almost affine mapping. Recently, interesting extensions of Ky Fan theorem 1.1 have been given by various authors for continuous multifunctions defined on noncompact convex subsets of a topological vector space possessing sufficiently many linear functionals.

Next, it was generalized by Reich[7] weakening the compactness as below:

Theorem 1.2.[7] *If $\mathcal{C}$ is a non-empty approximately $p-$compact, convex subset of a Hausdorff locally convex topological vector space $\mathcal{X}$ with a continuous semi-norm p and $\mathcal{T}:\mathcal{C}\to\mathcal{X}$ is a single valued continuous map with $\mathcal{T}(\mathcal{C})$ relatively compact, then there exists an element x_0 in $\mathcal{C}$ satisfying* (1.1) *of the Theorem* 1.1.

Next, Sehgal and Singh[9] generalized the result of Reich[7] for multi functions in normed linear spaces by generalizing the result of Prolla[6] as below:

Theorem 1.3.[10] *If $\mathcal{C}$ is a non-empty approximately compact convex subset of a normed linear space $\mathcal{X}$, $\mathcal{T}:\mathcal{C}\to\mathcal{X}$ a single valued continuous maps with $\mathcal{T}(\mathcal{C})$ relatively compact and $g:\mathcal{C}\to\mathcal{C}$ an affine, continuous, surjective single valued map such that g^{-1} sends compact subsets of $\mathcal{C}$ onto compact sets, then there exists an element x_0 in $\mathcal{C}$ such that $\|gx_0-\mathcal{T}x_0\|=d(\mathcal{T}x_0,\mathcal{C})$.*

Further generalization in this direction is done by Vetrivel et al[12]. who used Kakutani factorizable multifunctions in Hausdorff locally convex topological vector space. Carbone[2] replaced almost affine by almost quasi convex map and also extended the result of Prolla[6] by using approximately weakly compact subset of normed linear space[3].

Basha and Veeramani[8] also proved best approximation theorem for continuous Kakutani factorizable multifunctions which are not necessarily convex valued in the setting of Hausdorff locally convex topological vector space.

The purpose of this paper is to improve and extend the best approximation result of Basha and Veeramani[8] by considering a factorizable multifunction and the other one being a demicontinuous, relative almost quasi-convex, onto function on an approximately weakly compact, convex subset of Hausdorff locally convex topological vector space. The result due to Lassonde[5], the concept of approximately weakly compact due to Carbone[3] are used to prove our result. Finally, some known results are also obtained as corollaries.

2 Preliminaries

We recall some definitions:

Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty sets. The collection of all non-empty subsets of $\mathcal{X}$ is denoted by $2^{\mathcal{X}}$.

A multifunction or set-valued function from $\mathcal{X}$ to $\mathcal{Y}$ is defined to be a function that assigns to each elements of $\mathcal{X}$ a non-empty subset of $\mathcal{Y}$.

If $\mathcal{T}$ is a multifunction from $\mathcal{X}$ to $\mathcal{Y}$, then it is designated as $\mathcal{T}:\mathcal{X}\to 2^{\mathcal{Y}}$, and for every $x\in\mathcal{X}$, $\mathcal{T}x$ is called a value of $\mathcal{T}$.

Definition 2.1.[11] Let $\mathcal{X}$ be a locally convex space, $\mathcal{C}\subseteq\mathcal{X}$. A multivalued function $\mathcal{T}:\mathcal{C}\to 2^{\mathcal{X}}$ is upper semicontinuous (usc)(lower semicontinuous(lsc)) if

$$\mathcal{T}^{-1}(\mathcal{B})=\{x\in\mathcal{C}:\mathcal{T}x\cap\mathcal{B}\neq\emptyset\}$$

is closed(open) for each closed (open) subset $\mathcal{B}$ of $\mathcal{X}$. If $\mathcal{T}$ is both usc and lsc, then it is continuous.

It is known that if $\mathcal{T}:\mathcal{C}\to 2^{\mathcal{X}}$ is an upper semicontinuous multifunction with compact values, then $\mathcal{T}(\mathcal{K})$ is compact in $\mathcal{X}$ whenever $\mathcal{K}$ is a compact subset of $\mathcal{C}$.

A multifunction $\mathcal{T}:\mathcal{C}\to 2^{\mathcal{X}}$ is said to be a compact multifunction if $\mathcal{T}(\mathcal{C})$ is contained in a compact subset of $\mathcal{X}$.

A single valued function g from a topological space $\mathcal{X}$ to another topological space $\mathcal{Y}$ is said to be proper if $g^{-1}(\mathcal{K})$ is compact in $\mathcal{X}$ whenever $\mathcal{K}$ is compact in $\mathcal{Y}$. It is remarked that if g is continuous and $\mathcal{X}$ is a compact space, then the map g is proper.

Definition 2.2.[11] Let $\mathcal{C}$ be a convex subset of locally convex space $\mathcal{X}$ and $g:\mathcal{C}\to\mathcal{C}$ a continuous map. Then g is said to be

(i) *almost affine*, if

$$p(gy-z) \leq \lambda p(gx_1-z)+(1-\lambda)p(gx_2-z),$$

(ii) *almost quasi-convex*, if

$$p(g(\lambda x_1+(1-\lambda)x_2)-z) \leq \max\{p(gx_1-z), p(gx_2-z)\},$$

where $x_1, x_2 \in \mathcal{C}, \lambda \in [0,1]$ and $z \in \mathcal{X}$.

Definition 2.3.[13] Let $\mathcal{C}$ be a convex subset of locally convex space $\mathcal{X}$. Let f be a mapping from $\mathcal{C}$ to $2^{\mathcal{X}}$, the set of all nonempty subsets of $\mathcal{X}$. A mapping g from $\mathcal{C}$ to $\mathcal{X}$ is said to be *almost quasi-convex with respect to f*, if

$$p(g(\lambda x_1+(1-\lambda)x_2)-fz) \leq \max\{p(gx_1-fz), p(gx_2-fz)\},$$

where $x_1, x_2 \in \mathcal{C}, \lambda \in [0,1]$ and $z \in \mathcal{C}$.

Definition 2.4.[13] Let $\mathcal{X}$ and $\mathcal{Y}$ be two locally convex space. A mapping f from $\mathcal{X}$ to $\mathcal{Y}$ is said to be *demicontinuous*, if $f(x_n) \to f(x)$ weakly whenever $x_n \to x$ as $n \to \infty$.

Definition 2.5.[12] A multifunction Γ is said to be Kakutani multifunction, if there exist two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ such that

(i) $\Gamma : \mathcal{X} \to \mathcal{Y}$;

(ii) Γ is upper semi-continuous;

(iii) for each $x \in \mathcal{X}$, $\Gamma(x)$ is a nonempty compact subset of $\mathcal{Y}$ ($\mathcal{Y}$ is a convex set in a locally convex space).

The class of Kakutani multifunction is denoted by $\mathcal{R}$.

Definition 2.6.[12] A multifunction Γ is said to be Kakutani factorizable multifunction, if there exist two topological spaces $\mathcal{X}$ and $\mathcal{Y}$ such that

(i) $\Gamma : \mathcal{X} \to \mathcal{Y}$;

(ii) $\Gamma = \Gamma_n \circ \Gamma_{n-1} \circ \cdots \Gamma_0$ where each $\Gamma_i \in \mathcal{R}$.

Note that though each Γ_i is convex valued, the function Γ need not be convex valued.

The class of Kakutani factorizable multifunction, i.e., $\{\Gamma : \mathcal{X} \to \mathcal{Y} : \Gamma \in \mathcal{R}_c\}$ is denoted by $\mathcal{R}_c(\mathcal{X}, \mathcal{Y})$.

Definition 2.7.[3] A subset $\mathcal{M}$ of locally convex space $\mathcal{X}$ is called approximatively weakly compact(AWC) iff for each $y \in \mathcal{X}$ and sequence $\{x_n\}$ in $\mathcal{M}$ with

$$p(x_n-y) \to d_p(y, \mathcal{M}), \tag{2.1}$$

there exists an $x \in \mathcal{M}$ and a subsequence $\{x_\alpha\}$ of $\{x_n\}$ satisfying $x_n \to x$ weakly.

Note that the conditions in Definition 2.7 imply that $p(x-y) = d_p(y,\mathcal{M})$. Clearly, a compact set is AWC but the converse is not true.

The following results are needed in the sequel:

Theorem 2.8.[3] *Let $\mathcal{A}$ be a AWC subset of $\mathcal{X}$, and $\mathcal{Q} : \mathcal{X} \to 2^{\mathcal{A}}$, a mapping defined by $\mathcal{Q}(x) = \{y \in \mathcal{A} : p(y-x) = d_p(x,\mathcal{A})\}$. Then,*

(a) *$\mathcal{Q}(x) \neq \emptyset$, and*

(b) *For any compact set $\mathcal{C}$ of $\mathcal{X}$, $\mathcal{Q}(\mathcal{C}) = \cup\{\mathcal{Q}(x) : x \in \mathcal{C}\}$ is weakly compact.*

Theorem 2.9.[5] *Let $\mathcal{K}$ be a nonempty convex subset in a Hausdorff locally convex topological vector space $\mathcal{X}$. Then any compact multifunction $\Gamma \in \mathcal{R}_c(\mathcal{K},\mathcal{K})$ has a fixed point.*

3 Main Result

First, we prove our main result as below:

Theorem 3.1. *Let $\mathcal{X}$ be a Hausdorff locally convex topological vector space. Let $\mathcal{K}$ be a nonvoid approximately weakly compact convex subset and $\mathcal{C}$ be a nonvoid closed convex subset of $\mathcal{X}$. Suppose that*

(i) *$\mathcal{F}_1 : \mathcal{K} \to 2^{\mathcal{C}}$ and $\mathcal{F}_2 : \mathcal{C} \to 2^{\mathcal{X}}$ are closed convex valued continuous multi functions such that $\mathcal{F}_1(\mathcal{K})$ and the convex hull of $\mathcal{F}_2(\mathcal{K})$ are relatively compact;*

(ii) *The real valued function $y \to d_p(\mathcal{F}_2 y,\mathcal{K})$ is quasi convex on $\mathcal{C}$;*

(iii) *$g : \mathcal{K} \to \mathcal{K}$ is a demicontinuous, proper, almost quasi convex with respect to $\mathcal{F}_2\mathcal{F}_1$and onto single valued map.*

Then, there exists an element $x_0 \in \mathcal{K}$ satisfying the condition that

$$d_p(\mathcal{F}_2\mathcal{F}_1 x_0, g x_0) = d_p(\mathcal{F}_2\mathcal{F}_1 x_0,\mathcal{K}).$$

Proof. Following the lines of Basha and Veeramani[18], let $\mathcal{B} = \overline{co}\Gamma_2(C)$ and $\Gamma = \mathcal{G} \circ \Gamma_2 \circ \Gamma_1 : \mathcal{K} \to 2^{\mathcal{K}}$ where the multifunctions $\Gamma_1 : \mathcal{K} \to 2^{\mathcal{C}}$, $\Gamma_2 : \mathcal{C} \to 2^{\mathcal{B}}$ and $\mathcal{G} : \mathcal{B} \to 2^{\mathcal{K}}$ are defined as follows :

$$\Gamma_1(x) = \{y \in \mathcal{F}_1(x) : d_p(\mathcal{F}_2(y),\mathcal{K}) = d_p(\mathcal{F}_2\mathcal{F}_1 x,\mathcal{K})\},$$

$$\Gamma_2(x) = \{z \in \mathcal{F}_2(y) : d_p(\mathcal{F}_2(y),\mathcal{K}) = d_p(z,\mathcal{K})\},$$

$$\mathcal{G}(z) = g^{-1} \circ \mathcal{Q}(z).$$

It is claimed that Γ_1, Γ_2 and $\mathcal{G}$ are compact convex valued upper semi continuous multi functions which implies that Γ is a Kakutani factorizable multi function.

For every $x \in \mathcal{K}, \Gamma_1(x)$ is nonvoid, because the real valued continuous mapping $y \to d_p(\mathcal{F}_2 y, \mathcal{K})$ on the compact set $\mathcal{F}_1 x$ attains minimum. Also, since $\mathcal{F}_1$ is convex valued and the function $y \to d_p(\mathcal{F}_2 y, \mathcal{K})$ from $\mathcal{C}$ to R is quasi convex, Γ_1 is convex valued. Since the mapping $y \to d_p(\mathcal{F}_2 y, \mathcal{K})$ is continuous and $\mathcal{F}_1 x$ is compact, $\Gamma_1 x$ is compact.

Next, it is shown that Γ_1 is upper semi continuous. To prove this, we show that ${\Gamma_1}^{-1}(\mathcal{D})$ is weakly closed for any weakly closed subset $\mathcal{D}$. Let $\mathcal{D}$ be any weakly closed subset and $\{z_\alpha\}$ be any sequence in ${\Gamma_1}^{-1}(\mathcal{D})$ and $z_\alpha \to z_0$ weakly. Since

$$\mathcal{S}(z_\alpha) \cap \mathcal{D} \neq \emptyset$$

for each α, let $w_\alpha \in \mathcal{S}(z_\alpha) \cap \mathcal{D}$ which implies $w_\alpha \in \mathcal{F}_1(z_\alpha)$ and $d_p(\mathcal{F}_2(w_\alpha), \mathcal{K}) = d_p(\mathcal{F}_2\mathcal{F}_1 z_\alpha, \mathcal{K})$. Since $\mathcal{F}_1(\mathcal{K})$ is relatively compact, the sequence w_α has a subsequence, say w_β, converging to w_0. Since $\mathcal{F}_1$ is upper semi-continuous, $w_0 \in \mathcal{F}_1 z_0$. As $\mathcal{F}_1$ and $\mathcal{F}_2$ are continuous, the real valued functions $w_0 \to d_p(\mathcal{F}_2 w_0, \mathcal{K})$ and $z_0 \to d_p(\mathcal{F}_2\mathcal{F}_1 z_0, \mathcal{K})$ are also continuous. But $z_\beta \to z_0$ and $w_\beta \to w_0$. Therefore, it follows that $d_p(\mathcal{F}_2(w_\beta), \mathcal{K}) \to d_p(\mathcal{F}_2 z_0, \mathcal{K})$ and $d_p(\mathcal{F}_2\mathcal{F}_1 z_\beta, \mathcal{K}) \to d_p(\mathcal{F}_2\mathcal{F}_1 z_0, \mathcal{K})$. However, $d_p(\mathcal{F}_2 w_\beta, \mathcal{K}) = d_p(\mathcal{F}_2\mathcal{F}_1 z_\beta, \mathcal{K})$ for each β because w_β is in $\Gamma_1(z_\beta)$. So,

$$d_p(\mathcal{F}_2 w_0, \mathcal{K}) = d_p(\mathcal{F}_2\mathcal{F}_1 z_0, \mathcal{K})$$

which means that $w_0 \in \Gamma_1(z_0)$. Consequently, $w_0 \in \Gamma_1(z_0) \cap \mathcal{D}$ and hence $z_0 \in {\Gamma_1}^{-1}(\mathcal{D})$. So, ${\Gamma_1}^{-1}(\mathcal{D})$ is weakly closed. Thus, Γ_1 is upper semi-continuous.

Analogously, it may be shown that Γ_2 is a compact convex valued upper semi-continuous multifunction.

Since $\mathcal{K}$ is an approximately weakly compact and g is onto, and by Theorem 2.8, $\mathcal{G}(x)$ is nonvoid. As g is a proper map and $\mathcal{Q}(x)$ is a weakly compact, $\mathcal{G}(x)$ is weakly compact. Also, as g is a almost quasi convex with respect to $\mathcal{F}_2\mathcal{F}_1$, $\mathcal{G}$ is a convex valued multi-function.

In fact, let $x_1, x_2 \in \mathcal{G}(x)$ and $\lambda \in [0,1]$. Since g is almost quasi convex with respect to $\mathcal{F}_2\mathcal{F}_1$, it follows that

$$\begin{aligned} p(g(\lambda x_1 + (1-\lambda)x_2) - \mathcal{F}_2\mathcal{F}_1 x) &\leq \max\{p(g(x_1) - \mathcal{F}_2\mathcal{F}_1 x), p(g(x_2) - \mathcal{F}_2\mathcal{F}_1 x)\} \\ &= d_p(\mathcal{F}_2\mathcal{F}_1 x, \mathcal{K}). \end{aligned} \tag{3.1}$$

Since g is onto and $\mathcal{K}$ is convex,

$$p(g(\lambda x_1 + (1-\lambda)x_2) - \mathcal{F}_2\mathcal{F}_1 x) \geq d_p(\mathcal{F}_2\mathcal{F}_1 x, \mathcal{K}). \tag{3.2}$$

From (3.1) and (3.2), we obtain

$$g(\lambda x_1 + (1-\lambda)x_2) \in \mathcal{Q}(\mathcal{F}_2\mathcal{F}_1(x))$$

and hence

$$\lambda x_1 + (1-\lambda)x_2 \in g^{-1}(\mathcal{Q}(\mathcal{F}_2\mathcal{F}_1(x))).$$

Thus $\mathcal{G}(x)$ is convex.

It remains to show that $\mathcal{G}$ is upper semi-continuous. Let $\mathcal{A}$ be any weakly closed subset of $\mathcal{K}$ and $\{z_\alpha\}$ be any sequence in $\mathcal{G}^{-1}(\mathcal{A})$ and $z_\alpha \to z_0$ weakly for some $z_0 \in \mathcal{K}$. Since $\mathcal{S}(z_\alpha) \cap \mathcal{A} \neq \emptyset$ for each α, let $w_\alpha \in \mathcal{S}(z_\alpha) \cap \mathcal{A}$ which implies $g(w_\alpha) \in \mathcal{Q}(z_\alpha)$; i.e.,

$$d_p(g(w_\alpha), z_\alpha) = d_p(z_\alpha, \mathcal{K}).$$

Since $z_\alpha \to z_0$, we have $d(z_\alpha, \mathcal{K}) \to d(z_0, \mathcal{K})$. Since by Theorem 2.8, $\mathcal{Q}(\mathcal{F}_2\mathcal{F}_1(\mathcal{C}))$ is weakly compact and hence by the hypothesis, $g^{-1}(\mathcal{Q}(\overline{co}\mathcal{F}_2\mathcal{F}_1(\mathcal{C}))$ is weakly compact. Now, since

$$w_\alpha \in \mathcal{G}(z_\alpha) \subseteq \mathcal{G}(\overline{co}\mathcal{F}_2\mathcal{F}_1(\mathcal{C})) = (g^{-1} \circ \mathcal{Q})(\overline{co}\mathcal{F}_2\mathcal{F}_1(\mathcal{C})),$$

without loss of generality, we can assume $w_\alpha \to w_0$. Since g is demicontinuous, $g(w_\alpha) \to g(w_0)$ weakly. Now, we have

$$\begin{aligned} p(g(w_0) - z_0) &\le p(g(w_0) - g(w_\alpha)) + p(g(w_\alpha) - z_\alpha) + p(z_\alpha - z_0) \\ &= p(g(w_0) - g(w_\alpha)) + d_p(z_\alpha, \mathcal{K}) + p(z_\alpha - z_0). \end{aligned}$$

Taking the limit, we see that

$$p(g(w_0) - z_0) = d_p(z_0, \mathcal{K}).$$

Thus, $g(w_0) \in \mathcal{Q}(z_0)$ which implies $w_0 \in \mathcal{G}(z_0) \cap \mathcal{D}$; i.e., $z_0 \in \mathcal{G}^{-1}(\mathcal{D})$ and so $\mathcal{G}^{-1}(\mathcal{D})$ is weakly closed and hence $\mathcal{G}$ is upper semi-continuous multi-function. Thus, Γ is a weakly compact and Kakutani factorizable multi-function. So, Theorem 2.9 guarantees the existence of a point $x_0 \in \mathcal{K}$ such that $x_0 \in \Gamma x_0$ which means that $g(x_0) \in (\mathcal{Q} \circ \Gamma_2 \circ \Gamma_1)(x_0)$. Consequently, there exist two elements $y_0 \in \Gamma_1 x_0$ and $u_0 \in \Gamma_2 y_0$ such that $g(x_0) \in \mathcal{Q}(u_0)$. Therefore, it follows that $d_p(g(x_0), u_0) = d_p(u_0, \mathcal{K})$. As $y_0 \in \Gamma_1 x_0$, $d_p(\mathcal{F}_2 y_0, \mathcal{K}) = d_p(\mathcal{F}_2\mathcal{F}_1 x_0, \mathcal{K})$ and $y_0 \in \mathcal{F}_1 x_0$. Further, since $u_0 \in \Gamma_2 x_0$, $d_p(u_0, \mathcal{K}) = d_p(\mathcal{F}_2 y_0, \mathcal{K})$ and $u_0 \in \mathcal{F}_2 y_0$.

It is trivial that

$$d_p(\mathcal{F}_2\mathcal{F}_1 x_0, \mathcal{K}) \le d_p(gx_0, \mathcal{F}_2\mathcal{F}_1 x_0).$$

Also,

$$d_p(gx_0, \mathcal{F}_2\mathcal{F}_1 x_0) \le d_p(gx_0, u_0) = d_p(u_0, \mathcal{K}) = d_p(\mathcal{F}_2 y_0, K) = d_p(\mathcal{F}_2\mathcal{F}_1 x_0, \mathcal{K})$$

i.e.,

$$d_p(\mathcal{F}_2\mathcal{F}_1x_0, gx_0) = d_p(\mathcal{F}_2\mathcal{F}_1x_0, \mathcal{K}).$$

This completes the proof of the theorem.

In the Theorem 3.1, if $\mathcal{K}$ is approximately compact, convex and g is continuous, almost quasi affine and proper map, then we get the best approximation Theorem 4.1 due to Basha and Veeramani[8] in Hausdorff locally convex topological vector space as below:

Corollary 3.2. *Let $\mathcal{X}$ be a Hausdorff locally convex topological vector space. Let $\mathcal{K}$ be a nonvoid approximately compact convex subset and $\mathcal{C}$ be a nonvoid closed convex subset of $\mathcal{X}$. Suppose that*

(i) $\mathcal{F}_1 : \mathcal{K} \to 2^{\mathcal{C}}$ *and* $\mathcal{F}_2 : \mathcal{C} \to 2^{\mathcal{X}}$ *are closed convex valued continuous multi-functions such that* $\mathcal{F}_1(\mathcal{K})$ *and the convex hull of* $\mathcal{F}_2(\mathcal{K})$ *are relatively compact;*

(ii) *The real valued function* $y \to d_p(\mathcal{F}_2 y, \mathcal{K})$ *is quasi convex on* $\mathcal{C}$*;*

(iii) $g : \mathcal{K} \to \mathcal{K}$ *is a continuous, proper, almost quasi convex and onto single valued map.*

Then, there exists an element $x_0 \in \mathcal{K}$ *satisfying the condition that*

$$d_p(\mathcal{F}_2\mathcal{F}_1x_0, gx_0) = d_p(\mathcal{F}_2\mathcal{F}_1x_0, \mathcal{K}).$$

In the Theorem 3.1, if $\mathcal{C} = \overline{co}\mathcal{F}(\mathcal{K})$, $\mathcal{F}_1 = \mathcal{F}$ and $\mathcal{F}_2 = \mathcal{I}$, the identity mapping on $\mathcal{C}$, the we have following consequence:

Corollary 3.3. *Let $\mathcal{X}$ be a Hausdorff locally convex topological vector space. Let $\mathcal{K}$ be a nonvoid approximately weakly compact convex subset of $\mathcal{X}$. Suppose that*

(i) $\mathcal{F} : \mathcal{K} \to 2^{\mathcal{X}}$ *is closed convex valued continuous multi function such that convex hull of* $\mathcal{F}(\mathcal{K})$ *is relatively compact;*

(ii) $g : \mathcal{K} \to \mathcal{K}$ *is a demicontinuous, proper, almost quasi convex with respect to* $\mathcal{F}$ *and onto single valued map;*

Then, there exists an element $x_0 \in \mathcal{K}$ *satisfying the condition that*

$$d_p(\mathcal{F}x_0, gx_0) = d_p(\mathcal{F}x_0, \mathcal{K}).$$

In the Corollary 3.3, if $\mathcal{K}$ is approximately compact convex and g is continuous, almost quasi affine and proper map, then we get the Corollary 4.2 due to Basha and Veeramani[8] in Hausdorff locally convex topological vector space as below:

Corollary 3.4. *Let $\mathcal{X}$ be a Hausdorff locally convex topological vector space. Let $\mathcal{K}$ be a nonvoid approximately compact convex subset of $\mathcal{X}$. Suppose that*

(i) $\mathcal{F} : \mathcal{K} \to 2^{\mathcal{X}}$ *is closed convex valued continuous multi-function such that convex hull of* $\mathcal{F}(\mathcal{K})$ *is relatively compact;*

(ii) $g : \mathcal{K} \to \mathcal{K}$ *is a continuous, proper, almost quasi convex and onto single valued map.*

Then, there exists an element $x_0 \in \mathcal{K}$ *satisfying the condition that*

$$d_p(\mathcal{F}x_0, gx_0) = d_p(\mathcal{F}x_0, \mathcal{K}).$$

In the Corollary 3.4, if $\mathcal{F}$ is single valued continuous map and $\mathcal{K}$ is compact then we get the result due to Prolla[6].

Corollary 3.5. *Let* $\mathcal{X}$ *be a Hausdorff locally convex topological vector space. Let* $\mathcal{K}$ *be a nonvoid compact convex subset of* $\mathcal{X}$. *Suppose that*

(i) $\mathcal{F} : \mathcal{K} \to \mathcal{X}$ *is continuous function;*

(ii) $g : \mathcal{K} \to \mathcal{K}$ *is a continuous, almost quasi convex and onto single valued map.*

Then, there exists an element $x_0 \in \mathcal{K}$ *satisfying the condition that*

$$d_p(\mathcal{F}x_0, gx_0) = d_p(\mathcal{F}x_0, \mathcal{K}).$$

If in Corollary 3.5, $g = \mathcal{I}$, an identity function, then we get the result due to Fan [4] to Hausdorff locally convex topological vector space.

Corollary 3.6. *Let* $\mathcal{X}$ *be a Hausdorff locally convex topological vector space. Let* $\mathcal{K}$ *be a nonvoid compact convex subset of* $\mathcal{X}$. *Suppose that* $\mathcal{F} : \mathcal{K} \to \mathcal{X}$ *is continuous function. Then, there exists an element* $x_0 \in \mathcal{K}$ *satisfying the condition that*

$$p(\mathcal{F}x_0 - x_0) = d_p(\mathcal{F}x_0, \mathcal{K}).$$

In the Corollary 3.4, if $g = \mathcal{I}$, the identity mapping, then we get following result:

Corollary 3.7. *Let* $\mathcal{X}$ *be a Hausdorff locally convex topological vector space. Let* $\mathcal{K}$ *be a nonvoid approximately compact convex subset of* $\mathcal{X}$. *Suppose that* $\mathcal{F} : \mathcal{K} \to 2^{\mathcal{X}}$ *is closed convex valued continuous multi function such that convex hull of* $\mathcal{F}(\mathcal{K})$ *is relatively compact.*

Then, there exists an element $x_0 \in \mathcal{K}$ *satisfying the condition that*

$$p(\mathcal{F}x_0 - x_0) = d_p(\mathcal{F}x_0, \mathcal{K}).$$

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Hemant Kumar Nashine
Department of Mathematics
Disha Institute of Management and Technology
Satya Vihar, Vidhansabha-Chandrakhuri Marg
(Baloda Bazar Road), Mandir Hasaud
Raipur-492101 (Chhattisgarh)
India

E-mail: hemantnashine@rediffmail.com; nashine_09@rediffmail.com

Zoran D. Mitrovic
Faculty of Electrical Engineering
University of Banja Luka
Patr 5, 78000 Banja Luka
Bosina and Herzegovina

E-mail: zmitrovic@etfbl.net