Anal. Theory Appl.
Vol. 28, No. 1 (2011), 79–86

WEIGHTED BOUNDEDNESS OF COMMUTATORS OF FRACTIONAL HARDY OPERATORS WITH BESOV-LIPSCHITZ FUNCTIONS

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Received Dec. 24, 2011

Abstract. In this paper, we establish two weighted integral inequalities for commutators of fractional Hardy operators with Besov-Lipschitz functions. The main result is that this kind of commutator, denoted by H_b^α, is bounded from $L^p_{x^\gamma}(\mathbf{R}_+)$ to $L^q_{x^\delta}(\mathbf{R}_+)$ with the bound explicitly worked out.

Key words: *fractional Hardy operator, commutator, Besov-Lipschitz function*

AMS (2010) subject classification: 42B20, 42B35

1 Introduction and Main Results

Let f be a non-negative integrable function on $\mathbf{R}_+ = (0,\infty)$. The classical Hardy operator and its adjoint operator are defined by

$$Hf(x) := \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \qquad x>0$$

and

$$H^*f(x) := \int_x^\infty \frac{f(t)}{t}\mathrm{d}t, \qquad x>0.$$

The following well-known integral inequalities is due to Hardy (cf.[5,6]).

Theorem A. *If f is a non-negative measurable function on $\mathbf{R}_+$ and $1<p<\infty$, then the following two inequalities*

$$\|Hf\|_{L^p(\mathbf{R}_+)} \le \frac{p}{p-1}\|f\|_{L^p(\mathbf{R}_+)}$$

Supported in part by the Natural Science Foundation of China under the Grant 10771221 and Natural Science Foundation of Beijing under the Grant 1092004.

and

$$\|H^* f\|_{L^p(\mathbf{R}_+)} \leq p\|f\|_{L^p(\mathbf{R}_+)}$$

hold, where the constants $\frac{p}{p-1}$ *and* p *are sharp.*

For the n-dimensional case, Lu[9] discussed the following Hardy operator defined on the product space,

$$\mathcal{H}f(x) := \frac{1}{x_1 \cdots x_n}\int_0^{x_1}\cdots\int_0^{x_n} f(t_1,\cdots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n, \quad x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}_+^n \tag{1}$$

and the adjoint operator of the Hardy operator defined by

$$\mathcal{H}^* f(x) := \int_{x_1}^{\infty}\cdots\int_{x_n}^{\infty}\frac{f(t_1,\cdots,t_n)}{t_1\cdots t_n}\mathrm{d}t_1\cdots\mathrm{d}t_n, \quad x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}_+^n, \tag{2}$$

where $\mathbf{R}_+^n=(0,\infty)^n$ and f is a non-negative measurable function on $\mathbf{R}_+^n$.

In [9], the following Theorem B is obtained.

Theorem B. *Suppose that* f *is any non-negative measurable function on* $\mathbf{R}_+^n$ *and* $1<p\leq q<\infty$. *Then the Hardy operator* $\mathcal{H}$ *defined by* (1) *is bounded from* $L^p(\mathbf{R}_+^n,x^\gamma)$ *to* $L^q(\mathbf{R}_+^n,x^\delta)$, *that is, the inequality*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}f(x))^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}} \tag{3}$$

holds for some constant C, if and only if

$$\gamma<\mathbf{p}-\mathbf{1} \quad \text{and} \quad \delta=\frac{q}{p}(\gamma+\mathbf{1})-\mathbf{1}. \tag{4}$$

Moreover, if the conditions in (4) *are satisfied, then we have*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}f(x))^q x^\beta \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\prod_{i=1}^n\frac{q}{r(q-\delta_i-1)}\right)^{\frac{1}{r}}\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}}; \tag{5}$$

and the adjoint operator of the Hardy operator $\mathcal{H}^*$ *defined by* (2) *is also bounded from* $L^p(\mathbf{R}_+^n,x^\gamma)$ *to* $L^q(\mathbf{R}_+^n,x^\delta)$, *that is, the inequality*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}^*f(x))^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}} \tag{6}$$

holds for some constant C, if and only if

$$\gamma+\mathbf{1}>\mathbf{0} \quad \text{and} \quad \delta=\frac{q}{p}(\gamma+\mathbf{1})-\mathbf{1}. \tag{7}$$

Furthermore, if the conditions in (7) *are satisfied, then we have*

$$\left(\int_{\mathbf{R}_+^n}(\mathcal{H}^*f(x))^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}} \leq \left(\prod_{i=1}^n\frac{q}{r(\delta_i+1)}\right)^{\frac{1}{r}}\left(\int_{\mathbf{R}_+^n} f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}}, \tag{8}$$

where $\frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}$, $\gamma=(\gamma_1,\gamma_2,\ldots,\gamma_n)$, $\mathbf{1}=(1,\ldots,1)$, $\mathbf{p}=(p,\ldots,p)$ *and* $\delta=(\delta_1,\delta_2\ldots,\delta_n)$, $\gamma<\delta$ *means* $\gamma_i<\delta_i, i=1,\ldots,n$, *and* $x^\gamma=x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n}$, $x\in\mathbf{R}_+^n$.

The fractional Hardy operator on higher dimensional product space is defined by

$$\mathcal{H}^\alpha f(x)=\mathcal{H}^{(\alpha_1,\ldots,\alpha_n)}f(x):=\frac{1}{x_1^{1-\alpha_1}\cdots x_n^{1-\alpha_n}}\int_0^{x_1}\cdots\int_0^{x_n}f(t_1,\cdots,t_n)\mathrm{d}t_1\cdots\mathrm{d}t_n. \tag{9}$$

It immediately follows from the formula (9) that its adjoint operator is as follows

$$\mathcal{H}^{\alpha*}f(x)=\mathcal{H}^{(\alpha_1,\cdots,\alpha_n)*}f(x):=\int_{x_1}^\infty\cdots\int_{x_n}^\infty\frac{f(t_1,\cdots,t_n)}{t_1^{1-\alpha_1}\cdots t_n^{1-\alpha_n}}\mathrm{d}t_1\cdots\mathrm{d}t_n, \tag{10}$$

where $x=(x_1,x_2,\cdots,x_n)\in\mathbf{R}_+^n$, $\alpha=(\alpha_1,\cdots,\alpha_n), 0\le\alpha_i<1, i=1,\cdots,n$.

Obviously, if $\alpha_i=0, i=1,\ldots,n$, then $\mathcal{H}^\alpha=\mathcal{H}$. This means that the Hardy operator is a special case of the fractional Hardy operator.

Now let us consider the commutator of fractional Hardy operator and the commutator of adjoint fractional Hardy operator on one-dimensional space.

The commutator of fractional Hardy operators with a function b and its adjoint commutator are defined by

$$H_b^\alpha f(x):=\frac{1}{x^{1-\alpha}}\int_0^x f(t)(b(x)-b(t))\mathrm{d}t \tag{11}$$

and

$$H_b^{\alpha*}f(x):=\int_x^\infty\frac{f(t)(b(x)-b(t))}{t^{1-\alpha}}\mathrm{d}t, \tag{12}$$

where b is a locally integrable function, $x\in\mathbf{R}_+$ and $0\le\alpha<1$.

The boundedness of commutators H_b^α and $H_b^{\alpha*}$ is worth deeply studying, consequently, receives considerable attention. In 2002, Long[8] proved that the two commutators of H_b^α and $H_b^{\alpha*}$ are bounded from $L^p(\mathbf{R}_+)$ to $L^q(\mathbf{R}_+)$ with the function b in one sided dyadic $CMO^{\max(p,p')}$, where $1<p<q<\infty, \frac{1}{p}-\frac{1}{q}=\alpha+\beta$ and $\frac{1}{p}+\frac{1}{p'}=1$. Similarly, in 2006, Fu[3] and Zheng[15] showed that H_b^α and $H_b^{\alpha*}$ are bounded from $L^p(\mathbf{R}_+)$ to $L^q(\mathbf{R}_+)$ with b in $\dot{\Lambda}_\beta(\mathbf{R}_+)$, respectively.

In this paper, applying the results in Theorem B and combining the properties of the Besov-Lipschitz function b, we show that both commutators H_b^α and $H_b^{\alpha*}$ are bounded from $L^p(\mathbf{R}_+)$ to $L^q(\mathbf{R}_+)$ with a power weight, where $b\in\dot{\Lambda}_\beta(\mathbf{R}_+)$. Moreover, the bounds of the commutators H_b^α and $H_b^{\alpha*}$ are explicitly worked out. The proof is very concise.

We formulate our main results as follows.

Theorem 1.1. *Suppose that* $0\le\alpha<1$, $0<\beta<1$ *and* f *is a non-negative measurable function on* $\mathbf{R}_+$ *and* $b\in\dot{\Lambda}_\beta(\mathbf{R}_+)$. *If* $1<p<q<\infty$, $\gamma<p-1$, *and* $\frac{\gamma+1}{p}-\frac{\delta+1}{q}=\alpha+\beta$, *then the commutator* H_b^α *is bounded from* $L^p_{x^\gamma}(\mathbf{R}_+)$ *to* $L^q_{x^\delta}(\mathbf{R}_+)$, *that is,*

$$\|H_b^\alpha f\|_{L^q_{x^\delta}(\mathbf{R}_+)}\le\left(\frac{p}{r(p-\gamma-1)}\right)^{\frac{1}{r}}\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}\|f\|_{L^p_{x^\gamma}(\mathbf{R}_+)}, \tag{13}$$

where r satisfies

$$\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}.$$

Theorem 1.2. *Suppose that* $0 \le \alpha < 1$, $0 < \beta < 1$ *and* f *is a non-negative measurable function on* $\mathbf{R}_+$ *and* $b \in \dot{\Lambda}_\beta(\mathbf{R}_+)$. *If* $1 < p < q < \infty$, $\gamma + 1 > p(\alpha+\beta)$, *and* $\frac{\gamma+1}{p} - \frac{\delta+1}{q} = \alpha + \beta$, *then the commutator* $H_b^{\alpha *}$ *is bounded from* $L^p_{x^\gamma}(\mathbf{R}_+)$ *to* $L^q_{x^\delta}(\mathbf{R}_+)$, *that is,*

$$\|H_b^{\alpha *} f\|_{L^q_{x^\delta}(\mathbf{R}_+)} \le \left(\frac{p}{r(\gamma+1-p\alpha-p\beta)}\right)^{\frac{1}{r}} \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)} \|f\|_{L^p_{x^\gamma}(\mathbf{R}_+)}, \tag{14}$$

where r satisfies

$$\frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p}.$$

2 Proofs of Main Theorems

To prove our theorems, we first provide some definitions and lemmas which will be used in the sequel.

Definition 2.1. Suppose $0 < \beta < 1$. Besov-Lipschitz space is defined by

$$\dot{\Lambda}_\beta(\mathbf{R}+) := \left\{ f : x, h \in \mathbf{R}_+, \|f\|_{\dot{\Lambda}_\beta(\mathbf{R}+)} = \sup_{x,h \in \mathbf{R}_+} \frac{|f(x+h) - f(x)|}{h^\beta} < \infty \right\}.$$

By Definition 2.1, it is clear that the following lemma holds.

Lemma 2.1. *If* $b \in \dot{\Lambda}_\beta(\mathbf{R}_+), 0 < \beta < 1$, *then*

$$|b(x) - b(y)| \le |x-y|^\beta \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}$$

holds for any $x, y \in \mathbf{R}_+$.

Proof of Theorem 1.1. By Lemma 2.1, it follows that

$$\begin{aligned}
|H_b^\alpha f(x)| &= \left| \frac{1}{x^{1-\alpha}} \int_0^x f(t)\Big(b(x) - b(t)\Big) \mathrm{d}t \right| \\
&\le \frac{1}{x^{1-\alpha}} \int_0^x f(t)|b(x) - b(t)| \mathrm{d}t \\
&\le \frac{1}{x^{1-\alpha}} \int_0^x f(t)|x-t|^\beta \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)} \mathrm{d}t \\
&\le \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)} \frac{1}{x^{1-\alpha}} x^\beta \int_0^x f(t) \mathrm{d}t \\
&= \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)} x^{\alpha+\beta} \frac{1}{x} \int_0^x f(t) \mathrm{d}t \\
&= \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)} x^{\alpha+\beta} Hf(x).
\end{aligned}$$

We conclude

$$\begin{aligned}\|H_b^\alpha f\|_{L^q_{x^\delta}(\mathbf{R}_+)} &\le \left(\int_0^\infty \left(\|b\|_{\dot\Lambda_\beta(\mathbf{R}_+)} Hf(x)x^{\alpha+\beta}\right)^q x^\delta \mathrm{d}x\right)^{\frac{1}{q}}\\ &= \|b\|_{\dot\Lambda_\beta(\mathbf{R}_+)}\left(\int_0^\infty [Hf(x)]^q x^{q(\alpha+\beta)+\delta}\mathrm{d}x\right)^{\frac{1}{q}}\\ &= \|b\|_{\dot\Lambda_\beta(\mathbf{R}_+)}\|Hf\|_{L^q_{x^{q(\alpha+\beta)+\delta}}(\mathbf{R}_+)}.\end{aligned}$$

Set

$$\lambda = q(\alpha+\beta)+\delta.$$

Since the conditions $\gamma < p-1$ and $\frac{\gamma+1}{p}-\frac{\delta+1}{q}=\alpha+\beta$ hold, simple calculation leads to

$$\lambda = \frac{q}{p}(\gamma+1)-1.$$

Using the inequality (5) in Theorem B, we have

$$\begin{aligned}\|Hf\|_{L^q_{x^\lambda}(\mathbf{R}_+)} &\le \left(\frac{q}{r(q-\lambda-1)}\right)^{\frac{1}{r}}\left(\int_0^\infty f^p(x)x^\gamma \mathrm{d}x\right)^{\frac{1}{p}}\\ &= \left(\frac{p}{r(p-\gamma-1)}\right)^{\frac{1}{r}}\|f\|_{L^p_{x^\gamma}(\mathbf{R}_+)},\end{aligned}$$

where r satisfies

$$\frac{1}{r} = 1+\frac{1}{q}-\frac{1}{p}.$$

Therefore we obtain

$$\|H_b^\alpha f\|_{L^q_{x^\delta}(\mathbf{R}_+)} \le \left(\frac{p}{r(p-\gamma-1)}\right)^{\frac{1}{r}}\|b\|_{\dot\Lambda_\beta(\mathbf{R}_+)}\|f\|_{L^p_{x^\gamma}(\mathbf{R}_+)}. \tag{15}$$

This finishes the proof of Theorem 1.1.

Remark 2.1 For the special case, if $\gamma=\delta=0$, then

$$\frac{1}{p}-\frac{1}{q}=\alpha+\beta.$$

It follows from the inequality (15) that

$$\|\mathcal{H}_b^\alpha f\|_{L^q(\mathbf{R}_+)} \le \left(\frac{p-p\alpha-p\beta}{p-1}\right)^{1-\alpha-\beta}\|b\|_{\dot\Lambda_\beta(\mathbf{R}_+)}\|f\|_{L^p(\mathbf{R}_+)}.$$

If we set

$$\left(\frac{p-p\alpha-p\beta}{p-1}\right)^{1-\alpha-\beta} = C,$$

then we have

$$\|\mathcal{H}_b^\alpha f\|_{L^q(\mathbf{R}_+)} \le C\|b\|_{\dot\Lambda_\beta(\mathbf{R}_+)}\|f\|_{L^p(\mathbf{R}_+)},$$

which is the main result in Fu[3].

Proof of Theorem 1.2. It follows from Lemma 2.1 that

$$\begin{aligned} |H_b^{\alpha*}f(x)| &= \left|\int_x^\infty \frac{f(t)\big(b(x)-b(t)\big)}{t^{1-\alpha}}\mathrm{d}t\right| \\ &\le \int_x^\infty \frac{f(t)|b(x)-b(t)|}{t^{1-\alpha}}\mathrm{d}t \\ &\le \int_x^\infty \frac{f(t)(t-x)^\beta\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}}{t^{1-\alpha}}\mathrm{d}t \\ &\le \int_x^\infty \frac{f(t)t^\beta\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}}{t^{1-\alpha}}\mathrm{d}t \\ &\le \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}\int_x^\infty \frac{t^{\alpha+\beta}f(t)}{t}\mathrm{d}t \\ &= \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}H^*g(x), \end{aligned} \tag{16}$$

where $g(t)=t^{\alpha+\beta}f(t)$, $t\in(0,\infty)$.

Thus we have

$$\begin{aligned} \|H_b^{\alpha*}f\|_{L^q_{x^\delta}(\mathbf{R}_+)} &\le \left(\int_0^\infty \left(\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}H^*g(x)\right)^q x^\delta\mathrm{d}x\right)^{\frac{1}{q}} \\ &= \|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}\|H^*g\|_{L^q_{x^\delta}(\mathbf{R}_+)}. \end{aligned} \tag{17}$$

Set

$$\lambda=\gamma-p(\alpha+\beta).$$

Since the conditions $\gamma+1>p(\alpha+\beta)$ and $\delta=\frac{q}{p}(\gamma+1-p(\alpha+\beta))-1$ hold, we have

$$\lambda+1>0 \quad \text{and} \quad \delta=\frac{q}{p}(\lambda+1)-1.$$

This means that λ, p and q satisfy the condition (7) in Theorem B. Therefore, we conclude

$$\begin{aligned} \|H^*g\|_{L^q_{x^\delta}(\mathbf{R}_+)} &\le \left(\frac{q}{r(\delta+1)}\right)^{\frac{1}{r}}\|g\|_{L^p_{x^\lambda}(\mathbf{R}_+)} \\ &= \left(\frac{p}{r(\lambda+1)}\right)^{\frac{1}{r}}\|g\|_{L^p_{x^\lambda}(\mathbf{R}_+)} \\ &= \left(\frac{p}{r(\lambda+1)}\right)^{\frac{1}{r}}\left(\int_0^\infty\left(x^{\alpha+\beta}f(x)\right)^p x^\lambda\mathrm{d}x\right)^{\frac{1}{p}} \\ &= \left(\frac{p}{r(\lambda+1)}\right)^{\frac{1}{r}}\|f\|_{L^p_{x^{\lambda+p(\alpha+\beta)}}(\mathbf{R}_+)} \\ &= \left(\frac{p}{r(\gamma+1-p\alpha-p\beta)}\right)^{\frac{1}{r}}\|f\|_{L^p_{x^\gamma}(\mathbf{R}_+)}, \end{aligned} \tag{18}$$

where r satisfies

$$\frac{1}{r}=1+\frac{1}{q}-\frac{1}{p}.$$

Thus, combining the inequalities (16), (17) with (18) yields that

$$\|H_b^{\alpha*}f\|_{L^q_{x^\delta}(\mathbf{R}_+)}\leq\left(\frac{p}{r(\gamma+1-p\alpha-p\beta)}\right)^{\frac{1}{r}}\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}\|f\|_{L^p_{x^\gamma}(\mathbf{R}_+)}. \tag{19}$$

This finishes the proof of Theorem 1.2.

Remark 2.2. For the special case $\gamma=\delta=0$, then we have

$$\frac{1}{p}-\frac{1}{q}=\alpha+\beta.$$

It follows from the inequality (19) that

$$\|H_b^{\alpha*}f\|_{L^q(\mathbf{R}_+)}\leq\left(\frac{p-p\alpha-p\beta}{1-p\alpha-p\beta}\right)^{1-\alpha-\beta}\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}\|f\|_{L^p(\mathbf{R}_+)}.$$

Set

$$\left(\frac{p-p\alpha-p\beta}{1-p\alpha-p\beta}\right)^{1-\alpha-\beta}=C,$$

then we have

$$\|H_b^{\alpha*}f\|_{L^q(\mathbf{R}_+)}\leq C\|b\|_{\dot{\Lambda}_\beta(\mathbf{R}_+)}\|f\|_{L^p(\mathbf{R}_+)},$$

which obviously covers the main result in [15].

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