Anal. Theory Appl.
Vol. 28, No. 1 (2012), 38–48

ESTIMATES OF LINEAR RELATIVE n-WIDTHS IN $L^p[0,1]$

Sergei P. Sidorov
(*Saratov State University, Russian Federation*)

Received Apr. 7, 2011

Abstract. In this paper we will show that if an approximation process $\{L_n\}_{n\in\mathbf{N}}$ is shape-preserving relative to the cone of all k-times differentiable functions with non-negative k-th derivative on [0,1], and the operators L_n are assumed to be of finite rank n, then the order of convergence of D^kL_nf to D^kf cannot be better than n^{-2} even for the functions x^k, x^{k+1}, x^{k+2} on any subset of [0,1] with positive measure. Taking into account this fact, we will be able to find some asymptotic estimates of linear relative n-width of sets of differentiable functions in the space $L^p[0,1]$, $p\in\mathbf{N}$.

Key words: *shape preserving approximation, linear n-width*

AMS (2010) subject classification: 41A35, 41A29

1 Introduction

In various applications it is necessary to approximate functions preserving their properties such as monotonicity, convexity, concavity, etc. The last 25 years have seen extensive research in the theory of shape preserving approximation by means of polynomials and splines. The most significant results can be found in [1], [2].

Note that if a function f has some shape properties, it usually means that the element f belongs to a cone V in $C[0,1]$. A linear operator L defined in $C[0,1]$ is said to be shape-preserving relative to the cone V, if $L(V)\subset V$.

One of the most examined classes of linear shape-preserving operators is the class of linear positive operators. It is well-known that one of the shortcomings of linear positive operators is their slow convergence. It was shown by P. P. Korovkin[3] that the order of approximation by positive linear polynomial operators of degree n can not be better than n^{-2} in $C[0,1]$ even for the functions 1, x, x^2. Moreover, V. S. Videnskii[4] has shown that the result of [3] does not depend on the properties of polynomials but rather on the limitation of dimension.

Supported by RFBR (grant 10-01-00270) and the president of the Russian Federation (NS-4383.2010.1).

A function $f:[0,1]\to\mathbf{R}$ is said to be p-convex, $p\ge 1$, on $[0,1]$ iff for all choices of $p+1$ distinct $t_0,\cdots,t_p$ in $[0,1]$ the inequality

$$[t_0,\cdots,t_p]f\ge 0$$

holds, where

$$[t_0,\cdots,t_p]f=\sum_{j=0}^{p}(f(t_j)/w'(t_j))$$

denotes the p-th divided difference of f at $0\le t_0<t_1<\cdots<t_p\le 1$, and $w(t)=\prod_{j=0}^{p}(t-t_j)$.

Note that 2-convex functions are just convex functions. The class of all p-convex functions on $[0,1]$ is denoted by $\Delta^p[0,1]$. If $f\in C^p[0,1]$, then $f\in\Delta^p[0,1]$ iff $f^{(p)}(t)\ge 0$, $t\in[0,1]$. Let $\Delta^0[0,1]:=\{f\in C[0,1]:\ f(t)\ge 0,\ t\in[0,1]\}$.

Let $0\le h\le k$ be two integers and let $\sigma=(\sigma_h,\dots,\sigma_k)\in R^{k-h+1}$, $\sigma_i\in\{-1,0,1\}$ be such that $\sigma_h\sigma_k\ne 0$.

Following the idea of [5] denote

$$\Delta^{h,k}(\sigma):=\{f\in C[0,1]:\ \sigma_i f\in\Delta^i[0,1],\ h\le i\le k\}.$$

We will use the notations $\sigma^{[k]}=\left(\sigma_i^{[k]}\right)_{i=h}^{k}$ with $\sigma_i^{[k]}=0$ for $i\ne k$ and $\sigma_k^{[k]}=\sigma_k$.

A linear operator mapping $C[0,1]$ into a linear space of finite dimension n is called an operator of finite rank n. Let D^k denote the k-th differential operator, $D^k(x):=\mathrm{d}^k f(x)/\mathrm{d}x^k$.

Denote $e_j(t)=t^j$, $j=0,1,\cdots$. Let $\Omega\subset[0,1]$ be a measurable set with $\mathrm{meas}(\Omega)\ne 0$. Let $\{L_n\}_{n\in\mathbf{N}}$, $L_n:C^k[0,1]\to C^k(\Omega)$, be a sequence of linear shape-preserving operators, such that for all $n\in\mathbf{N}$ we have $L_n(\Delta^{h,k}(\sigma))\subset\Delta^{h,k}(\sigma^{[k]})$ and L_n is of finite rank n. Extending the results of Korovkin[3], of Videnskii[4], of Vasiliev and Guendouz [6], in this paper we will show that the measure of the set of all $x\in\Omega$, such that

$$\lim_{n\to\infty}n^2|D^kL_ne_j(x)-D^ke_j(x)|=0,\qquad j=k,\ k+1,\ k+2,$$

is equal to zero. Using this fact, we will find asymptotic estimates of linear relative n-width of sets of differentiable functions in the space $L^p[0,1]$.

2 The Order of Approximation by Means of Linear Shape-Preserving Operators

Let $E\subset[0,1]$ be a closed set with $\mathrm{meas}(E)>0$. Given $m\in\mathbf{N}$, denote $X=\{x_i\}$, where

$$x_i=\frac{i}{m},\qquad i=-1,0,1,\cdots,m+1.$$

Let

$$\begin{aligned} Y_m &:= \{0 = y_0 < y_1 < \ldots < y_{n_m} = 1\} \\ &= \{x_i \in X \cap [0,1] :\ [x_{i-1}, x_{i+1}] \cap (E \cup \{0\} \cup \{1\}) \neq \varnothing\}. \end{aligned}$$

Note that n_m depends on both the integer m and the set E. Let us denote

$$\Gamma_m := \{i :\ y_i - y_{i-1} = 1/m \text{ and } [y_{i-1}, y_i) \cap E \neq \varnothing\}$$

and consider the set

$$E_m := (\cup_{i \in \Gamma_m} [y_{i-1}, y_i]) \cup \{1\}. \tag{1}$$

It is clear that $E \subset E_m$ and

$$\lim_{m\to\infty} \operatorname{meas}(E_m) = \lim_{m\to\infty} \frac{n_m}{m} = \operatorname{meas}(E). \tag{2}$$

Given an integer n, we can choose $m = m(n)$ so that $n_m \le n \le n_{m+1}$, then it follows that

$$\lim_{n\to\infty} \frac{n}{m(n)} = \operatorname{meas}(E). \tag{3}$$

Given E, m, $k, n_m \in \mathbf{N}$, $n_m > k+2$, let us define the linear operator $\Lambda_{k,n_m} : C^k[0,1] \to C^k[0,1]$ by

$$\Lambda_{k,n_m} f(x) = \begin{cases} \sum\limits_{l=0}^{k-1} \frac{1}{l!}(x-y_0)^l \left(D^l f(0) + \frac{(-1)^{k+1-l}(y_1-y_0)^{k-l}}{(k+1-l)!} D^k f(0) \right) \\ \quad + \frac{1}{(k+1)!(y_1-y_0)} \left[x^{k+1} D^k f(y_1) + (-1)^k (y_1 - x)^{k+1} D^k f(0) \right], \\ \qquad \text{if } x \in [0, y_1], \\ \sum\limits_{l=0}^{k-1} \frac{1}{l!}(x-y_i)^l \left(D^l \Lambda_{k,n_m} f(y_i) + \frac{(-1)^{k+1-l}(y_{i+1}-y_i)^{k-l}}{(k+1-l)!} D^k f(y_i) \right) \\ \quad + \frac{1}{(k+1)!(y_{i+1}-y_i)} \left[(x-y_i)^{k+1} D^k f(y_{i+1}) \right. \\ \quad \left. + (-1)^k (y_{i+1} - x)^{k+1} D^k f(y_i) \right], \\ \qquad \text{if } x \in (y_i, y_{i+1}],\ \ i = 1, 2, \cdots, n_m - 1. \end{cases} \tag{4}$$

It easy to check that

1. $D^k \Lambda_{k,n_m} e_j = D^k e_j,\ j = 0, 1, \ldots, k+1;$
2. $\Lambda_{k,n_m}(\Delta^{h,k}(\sigma^{[k]})) \subset \Delta^{h,k}(\sigma^{[k]})$.

Let $B(E)$ denote the space of all real-valued bounded functions defined on [0,1] with the uniform norm on E, $\|f\|_{B(E)} = \sup\limits_{x\in E} |f(x)|$.

Given $x \in [0,1]$, let us define $g_x \in C^k[0,1]$ by

$$g_x = \frac{2}{(k+2)!} e_{k+2} - \frac{2}{(k+1)!} x e_{k+1} + \frac{1}{k!} x^2 e_k. \tag{5}$$

Lemma 2.1. *Let $E \subset [0,1]$ be a closed set,* $\mathrm{meas}(E) \neq \varnothing$. *Let $x \in E$. Then*

$$\lim_{m\to\infty}\Big(m^2\|D^k\Lambda_{k,n_m}g_x\|_{B(E)}\Big) = (\mathrm{meas}(E))^2/4. \tag{6}$$

Proof. It follows from $D^k\Lambda_{k,n_m}e_k = D^ke_k$ and $D^k\Lambda_{k,n_m}e_{k+1} = D^ke_{k+1}$ that

$$D^k\Lambda_{k,n_m}g_x = D^k\Lambda_{k,n_m}e_{k+2} - D^ke_{k+2}.$$

It is worth noting that $D^k\Lambda_{k,n_m}g_x$ is a second degree algebraic polynomial and $D^k\Lambda_{k,n_m}g_x(y_{i-1}) = D^k\Lambda_{k,n_m}g_x(y_i) = 0$ if $x \in [y_{i-1}, y_i] \subset E_m$. It follows from

$$0 \le D^k\Lambda_{k,n_m}g_x(x) = (x-y_i)(y_{i+1}-x) \le (y_{i-1}-y_i)^2/4 \le 1/(4m^2)$$

that

$$\|D^k\Lambda_{k,n_m}g_x\|_{B[y_i,y_{i-1}]} = 1/(4m^2). \tag{7}$$

We have

$$\|D^k\Lambda_{k,n_m}g_x\|_{B(E\cap[y_i,y_{i-1}])} \ge \left(\frac{\mathrm{meas}([y_i,y_{i-1}]\setminus E)}{2}\right)^2 = \theta_{m,i}^2/(4m^2), \tag{8}$$

where

$$\theta_{m,i} := \frac{\mathrm{meas}([y_i,y_{i-1}]\setminus E)}{\mathrm{meas}([y_i,y_{i-1}])}.$$

It follows from (7) that

$$\|D^k\Lambda_{k,n_m}g_x\|_{B(E_m)} = 1/(4m^2). \tag{9}$$

It follows from (8) that

$$\|D^k\Lambda_{k,n_m}g_x\|_{B(E)} \ge (1-\theta_m^2)/(4m^2), \tag{10}$$

where

$$\theta_m := \frac{\mathrm{meas}(E_m\setminus E)}{\mathrm{meas}(E_m)}.$$

It follows from (2), (9), (10) and $\lim\limits_{m\to\infty}\theta_m = 0$ that

$$\begin{aligned}\lim_{m\to\infty}\Big(m^2\|D^k\Lambda_{k,n_m}g_x\|_{B(E)}\Big) &= \lim_{m\to\infty}\Big(m^2\|D^k\Lambda_{k,n_m}g_x\|_{B(E_m)}\Big)\\ &= \lim_{m\to\infty}\frac{m^2}{4m_n^2} = (\mathrm{meas}(E))^2/4.\end{aligned}$$

Lemma 2.2. *Let $k, n \in \mathbf{N}$, $n > k+2$. Let the linear operator*

$$\Lambda_{k,n_m} : C^k[0,1] \to C^k[0,1]$$

be defined as above. Let $E \subset [0,1]$ be a closed set with $\mathrm{meas}(E) \neq \varnothing$. *Let $x \in E$. Then*

$$\lim_{m\to\infty}\Big(m^2\|D^k\Lambda_{k,n_m}g_x\|_{L^p(E)}\Big) = C(\mathrm{meas}(E))^{2+1/p}, \tag{11}$$

where $C = C(k,p)$ does not depend on m.

Proof. Let E_m be defined in (1) and n_m be defined as above. Since $\operatorname{meas}(E_m \setminus E)$ tends to zero as $m \to \infty$, we have

$$\lim_{m\to\infty} \|D^k \Lambda_{k,n_m} g_x\|_{L^p(E)} = \lim_{m\to\infty} \|D^k \Lambda_{k,n_m} g_x\|_{L^p(E_m)}. \tag{12}$$

Since $y_i - y_{i-1} = 1/m$ for all $i \in \Gamma_m$, we have

$$\begin{aligned}\|D^k \Lambda_{k,n_m} g_x\|_{L^p(E_m)} &= \left(\sum_{i\in\Gamma_m} \int_{[y_{i-1},y_i)} [(y_i - x)(x - y_{i-1})]^p dx\right)^{1/p} \\ &= \left(\sum_{i\in\Gamma_m} \int_{[0,1/m)} \left(\frac{1}{m} - x\right)^p x^p dx\right)^{1/p} \\ &= \left(\sum_{i\in\Gamma_m} \frac{1}{m^{2p+1}} \sum_{j=0}^{p} \frac{(-1)^{p-j} p!}{j!(p-j)!(2p-j+1)}\right)^{1/p} = \frac{C}{m^2}\left(\frac{n_m}{m}\right)^{1/p}, \end{aligned}\tag{13}$$

where

$$C = \left(\sum_{j=0}^{p} \frac{(-1)^{p-j} p!}{j!(p-j)!(2p-j+1)}\right)^{1/p}.$$

It follows from (12), (13) and (2) that

$$\lim_{m\to\infty}\left(m^2 \|D^k \Lambda_{k,n_m} g_x\|_{L^p(E)}\right) = C \lim_{m\to\infty} \frac{n_m^2}{m^2}\left(\frac{n_m}{m}\right)^{1/p} = C[\operatorname{meas}(E)]^{2+1/p}.$$

To prove the main result of this section we need the preliminary lemma.

Lemma 2.3[7]. *Let $\Phi : C^k[0,1] \to \mathbf{R}$ be a linear functional that has the following property: $\Phi(f) \ge 0$ for every $f \in C^k[0,1]$ such that $f \in \Delta^{h,k}(\sigma^{[k]})$. Let*

$$\langle \cdot,\cdot \rangle : C^k[0,1] \times C^k[0,1] \to \mathbf{R}$$

be the bi-functional generated by a functional Φ in the following way: for every $f,g \in C^k[0,1]$ we suppose $\langle f,g\rangle = \Phi(h)$ with $h \in C^k[0,1]$ so that $D^k h = D^k f D^k g$ and $D^i h(0) = 0$, $i = 0,1,\dots,k-1$. Then

$$|\langle f,g\rangle| \le [\langle f,f\rangle]^{\frac12}[\langle g,g\rangle]^{\frac12}, \qquad f,g \in C^k[0,1]. \tag{14}$$

Theorem 2.4. *Let $\Omega \subset [0,1]$ be a measurable set. Let $\{L_n\}_{n\in\mathbf{N}}$ be a sequence of linear operators such that for every $n \in \mathbf{N}$*

1. *L_n is of finite rank n;*
2. *$D^k L_n : C^k[0,1] \to B(\Omega)$;*
3. *$L_n(\Delta^{h,k}(\sigma)) \subset \Delta^{h,k}(\sigma^{[k]})$;*
4. *$D^k L_n f$ is measurable for all $f \in C^k[0,1]$.*

Let $\gamma=\{n_i\}_{i\in\mathbf{N}}$ *be an increasing sequence of integers. Denote*

$$E_\gamma:=\{x\in\Omega:\ \lim_{i\to\infty}n_i^2|D^kL_{n_i}e_j(x)-D^ke_j(x)|=0,\ j=k,k+1,k+2\}.$$

Then $\mathrm{meas}(E_\gamma)=0$.

Proof. It is easy to show that the set E_γ is a measurable set. Assume $\mathrm{meas}(E_\gamma)\neq 0$. Let δ be such that $0<\delta<(\mathrm{meas}(E_\gamma))/2$. It follows from Egorov Theorem that there exists a measurable set $E\subset E_\gamma$ with $\mu=\mathrm{meas}(E)\geq\mathrm{meas}(E_\gamma)-\delta$, such that

$$\lim_{i\to\infty}n_i^2\|D^kL_{n_i}e_j-D^ke_j\|_{B(E)}=0,\qquad j=k,k+1,k+2, \tag{15}$$

i.e. we have the uniform convergence of $n_i^2|D^kL_{n_i}e_j(x)-D^ke_j(x)|$ to 0 as $i\to\infty$ on the set E for all $j=k,k+1,k+2$.

Denote

$$D_n:=\{x\in\Omega:D^k\,L_nf(x)=0\text{ for all }f\in C^k[0,1]\},$$

$$D:=\cap_{j=1}^{\infty}\cup_{i=j}^{\infty}D_{n_i},$$

i.e. D is the set of all points which belong to an infinite number of the sets D_{n_i}. Then $E_\gamma\cap D\neq\varnothing$ and $\mathrm{meas}(E_\gamma\cap(\cap_{i=j}^{\infty}D_{n_i}))\to 0$ as $j\to\infty$. Therefore, we may assume that there exists an integer N such that

$$E\cap D_{n_i}=\varnothing\text{ for all }n_i>N. \tag{16}$$

Take an arbitrary $n_s>N$. It follows from [6, Lemma 2] that there exist points $0\leq z_0<z_1<\dots<z_{n_s}$, such that $z_p-z_{p-1}\equiv 0(\mathrm{mod}\,\Delta)$, $p=1,\dots,n_s$, where $0<\Delta<\mu/n_s$, $\mu=\mathrm{meas}(E)$.

I) Let $\{u_j\}_{j=1}^{n_s}$ be a system generating the linear space $\{L_{n_s}f: f\in C[0,1]\}\subset\Omega$. Consider the matrix

$$A=\|D^ku_j(z_i)\|_{j=0,\dots,n_s}^{i=0,\dots,n_s+1}.$$

The rank of the matrix A does not equal to zero, $\mathrm{rank}A\neq 0$. Indeed, if $\mathrm{rank}A=0$, then

$$D^kL_{n_s}f(z_i)=\sum_{i=0}^{n_s}a_i(f)D^ku_i(z_i)=0$$

for every $f\in C^k[0,1]$, which implies $\{z_i\}_{0\leq i\leq n_s}\subset E\cap D_{n_s}$. This contradicts (16).

Take a non-trivial vector $\delta=(\delta_i)_{i=0}^{n_s}$, orthogonal to all the rows of the matrix A:

$$\sum_{i=0}^{n_s}|\delta_i|=1,\quad \sum_{i=0}^{n_s}\delta_iD^ku_j(z_i)=0,\qquad j=1,\dots,n_s.$$

Define a continuous function D^kh on $[0,1]$ by

1. $D^kh(z_i)=\mathrm{sgn}\,\delta_i,\ i=0,\dots,n_s$;
2. D^kh is linear on each interval $[0,z_0],[z_0,z_1],\cdots,[z_{n_s-2},z_{n_s-1}],[z_{n_s-1},1]$;

3. $D^i h(0) = 0$, $i = 0, 1, \cdots, k-1$.

As the function $D^k L_{n_s} h$ belongs to the linear space spanned by $\{D^k u_j\}$, we have

$$\sum_{i=0}^{n_s} \delta_i D^k L_{n_s} h(z_i) = 0.$$

Then

$$1 = \sum_{i=0}^{n_s} |\delta_i| = \sum_{i=0}^{n_s} \delta_i D^k h(z_i) = \sum_{i=0}^{n_s} \delta_i (D^k h(z_i) - D^k L_{n_s} h(z_i)) \leq$$
$$\leq \sum_{i=0}^{n_s} |\delta_i| |D^k L_{n_s} h(z_i) - D^k h(z_i)| \leq \|D^k L_{n_s} h - D^k h\|_{B(E)}. \tag{17}$$

II) On the other hand, for $x \in \Omega$ we have

$$\begin{aligned} &|D^k L_{n_s} h(x) - D^k h(x)| \\ &\quad = \left| D^k L_{n_s} h(x) - D^k h(x) \frac{1}{k!} D^k L_{n_s} e_k(x) \right| + \frac{1}{k!} |D^k h(x)| \left| D^k L_{n_s} e_k(x) - D^k e_k(x) \right| \\ &\quad = \left| D^k L_{n_s} \left(h - D^k h(x) \frac{1}{k!} e_k \right)(x) \right| + \frac{1}{k!} \left| D^k L_{n_s} e_k(x) - D^k e_k(x) \right|, \end{aligned} \tag{18}$$

since $\|D^k h(x)\|_{B(E)} = 1$.

Let p_x be a function such that $p_x \in C^k[0,1]$ and

$$D^k p_x = \left| D^k \left(h - D^k h(x) \frac{1}{k!} e_k \right) \right|, \;\; D^i p_x(0) = 0, \qquad i = 0, 1, \cdots, k-1.$$

We have

$$D^k \left(h - D^k h(x) \frac{1}{k!} e_k \right) \leq D^k p_x$$

and

$$D^k \left(-(h - D^k h(x) \frac{1}{k!} e_k) \right) \leq D^k p_x.$$

Thus,

$$D^k L_{n_s} \left(h - D^k h(x) \frac{1}{k!} e_k \right)(x) \leq D^k L_{n_s} p_x(x) \tag{19}$$

and

$$-D^k L_{n_s} \left(h - D^k h(x) \frac{1}{k!} e_k \right)(x) \leq D^k L_{n_s} p_x(x). \tag{20}$$

Combining (19) and (20) we get

$$\left| D^k L_{n_s} \left(h - D^k h(x) \frac{1}{k!} e_k \right)(x) \right| \leq D^k L_{n_s} p_x(x). \tag{21}$$

Let $q_x \in C^k[0,1]$ be a function such that

$$D^k q_x(t) = |t-x| \text{ and } D^i q_x(0) = 0, \qquad i = 0,1,\dots,k-1.$$

We have

$$\begin{aligned} D^k p_x(t) &= \left| D^k \left(h(t) - D^k h(x) \frac{1}{k!} t^k \right) \right| = |D^k h(t) - D^k h(x)| \\ &\leq 2\Delta^{-1} |t-x| = 2\Delta^{-1} D^k q_x(t). \end{aligned}$$

Thus,

$$D^k (2\Delta^{-1} q_x - p_x) \geq 0$$

and consequently

$$D^k L_{n_s}(2\Delta^{-1} q_x - p_x)(x) \geq 0,$$

and

$$D^k L_{n_s} p_x(x) \leq 2\Delta^{-1} D^k L_{n_s} q_x(x). \tag{22}$$

It follows from Lemma 2.3 that

$$D^k L_{n_s} q_x(x) \leq [D^k L_{n_s} g_x(x)]^{\frac{1}{2}} \left[\frac{1}{k!} D^k L_{n_s} e_k(x) \right]^{\frac{1}{2}}, \tag{23}$$

where g_x is defined in (5)

Using sequentially (18), (21), (22), (23) we get

$$\begin{aligned} &|D^k L_{n_s} h(x) - D^k h(x)| \\ &\qquad \leq 2\Delta^{-1} [D^k L_{n_s} g_x(x)]^{\frac{1}{2}} \left[\frac{1}{k!} D^k L_{n_s} e_k(x) \right]^{\frac{1}{2}} + \frac{1}{k!} \left| D^k L_{n_s} e_k(x) - D^k e_k(x) \right|. \end{aligned} \tag{24}$$

We have

$$\begin{aligned} D^k L_{n_s} g_x(x) = & \frac{2}{(k+2)!} (D^k L_{n_s} e_{k+2} - D^k e_{k+2})(x) \\ & - \frac{2}{(k+1)!} x (D^k L_{n_s} e_{k+1} - D^k e_{k+1})(x) + \frac{1}{k!} x^2 (D^k L_{n_s} e_k - D^k e_k)(x) \\ & + \frac{2}{(k+2)!} D^k e_{k+2}(x) - \frac{2}{(k+1)!} x D^k e_{k+1}(x) + \frac{1}{k!} x^2 D^k e_k(x). \end{aligned} \tag{25}$$

Note that

$$D^k g_x(x) = \frac{2}{(k+2)!} D^k e_{k+2}(x) - \frac{2}{(k+1)!} x D^k e_{k+1}(x) + \frac{1}{k!} x^2 D^k e_k(x) = 0.$$

It follows from (17) and (24) that

$$\Delta^2 \frac{1-\frac{1}{k!}\left\|D^k L_{n_s} e_k - D^k e_k\right\|_{B(E)}}{4/(k!)\|D^k L_{n_s} e_k\|_{B(E)}} \le \|D^k L_{n_s} g_x\|_{B(E)}. \tag{26}$$

Now using (25), we have

$$\begin{aligned}\mu^2 & \frac{1-\left\|D^k L_{n_s} e_k - D^k e_k\right\|_{B(E)}/(k!)}{4\|D^k L_{n_s} e_k\|_{B(E)}/(k!)} \\ & \le n_s^2 \left(\frac{2}{(k+2)!} \|D^k L_{n_s} e_{k+2} - D^k e_{k+2}\|_{B(E)} \right. \\ & \left. + \frac{2}{(k+1)!}\|D^k L_{n_s} e_{k+1} - D^k e_{k+1}\|_{B(E)} + \frac{1}{k!}\|D^k L_{n_s} e_k - D^k e_k\|_{B(E)} \right),\end{aligned}$$

as $h \to \mu/n_s$. It follows from (15) that $\mu = \text{meas}(E) = 0$, which contradicts our assumption.

3 Estimates of Linear Relative n-Widths

Let X, Y be a linear normed spaces. Let $T : X \to Y$ and $L : X \to X$ be a linear continuous operators. Let us denote $T \circ L$ an operator defined by $(T \circ L)f = T(Lf)$.

Definition 3.1. *Let A be a subset of X. The value*

$$e(A, L|T) := \sup_{f \in A} \|T \circ (I - L)f\|_Y$$

is the error of approximation of the operator T by the operator $T \circ L : X \to Y$ on the set A.

Let V_1, V_2 be some cones in X, $V_1 \subset V_2$. One might consider the problem of finding (if exists) a linear operator of finite rank n, which gives the minimal error of approximation of an operator T (not necessary identity operator) on some set over all operators L linear of finite rank n with shape preserwing property $L(V_1) \subset V_2$. It leads us naturally to the notion of linear relative n-width [8].

Definition 3.2. *Korovkin linear relative n–width of a set $A \subset X$ in Y for operator $T : X \to Y$ with the constraint (V_1, V_2) is defined by*

$$\delta_n(A, V_1, V_2|T)_Y := \inf_{L_n(V_1) \subset V_2} e(A, L|T),$$

where infimum is taken over all linear continuous operators $L_n : X \to X$ such that

1. $T \circ L_n : X \to Y$ is of finite rank n;

2. $L_n(V_1) \subset V_2$.

Determination of linear relative n-widths is of interest in the theory of shape-preserving approximation as, knowing the value of relative linear n-width $\delta_n(A, V_1, V_2|T)_X$, we can judge how good (in terms of optimality) that finite-dimensional method with shape-preserving property $L_n(V_1) \subset V_2$ is.

Estimates of linear relative n-widths of some sets of algebraic polynomials in $X = C[0,1]$ relative to the cone of all non-negative continuous functions defined on [0,1], is obtained in the paper [8].

Corollary 3.3. *Let $E \subset [0,1]$ be a measurable closed set. Then the following estimate of linear relative n-width of set P_{k+2} in the space $B(E)$ for the operator D^k with constraints $(\Delta^{h,k}(\sigma^{[k]}), \Delta^{h,k}(\sigma^{[k]}))$*

$$\frac{(\text{meas}(E))^2}{8n^2} \leq \delta_n \left(P_{k+2}, \Delta^{h,k}(\sigma^{[k]}), \Delta^{h,k}(\sigma^{[k]}) \mid D^k\right)_{B(E)} \leq \frac{(\text{meas}(E))^2}{8(n-1)^2} \tag{27}$$

holds.

Proof. Following [8] it is easy to show that

$$\begin{aligned} &\inf_{L_n(\Delta^{h,k}(\sigma^{[k]})) \subset \Delta^{h,k}(\sigma^{[k]})} \sup_{p \in P_{k+2}} \|D^k p - D^k L_n p\|_{B(E)} \\ &\qquad = \frac{1}{(k+2)!} \inf \sup_{x \in E} |D^k e_{k+2} - D^{k+2} L_n e_{k+2}(x)|, \end{aligned}$$

where infimum is taken over all linear operators of finite rank n, such that

$$L_n(\Delta^{h,k}(\sigma^{[k]})) \subset \Delta^{h,k}(\sigma^{[k]})$$

and

$$D^k L_n e_r = D^k e_r \text{ on } E, \qquad r = 0, 1, \ldots, k+1. \tag{28}$$

It should be noted that if an operator L_n satisfies (28), then

$$\frac{1}{(k+2)!} |D^k e_{k+2} - D^k L_n e_{k+2}(x)| = D^k L_n g_x(x), \qquad x \in E,$$

where g_x is defined by (5).

The upper estimate in (27) follows from Lemma 2.1. The lower estimate follows from the inequality (26).

In the next theorem we will find the asymptotic estimate of Korovkin linear relative n-widths of set P_{k+2} in the space $L^p(E)$ for D^k with constraints $(\Delta^{h,k}(\sigma^{[k]}), \Delta^{h,k}(\sigma^{[k]}))$.

Corollary 3.4. *Let $E \subset [0,1]$ be a measurable closed set, $p \geq 1$. There exist numbers $c_1, c_2 > 0$ independent of n such that*

$$c_1 \leq \lim_{n \to \infty} n^2 \delta_n \left(P_{k+2}, \Delta^{h,k}(\sigma^{[k]}), \Delta^{h,k}(\sigma^{[k]}) \mid D^k\right)_{L^p(E)} \leq c_2. \tag{29}$$

Proof. It follows from Lemma 2.2 that there exists a number $c_2 > 0$ such that

$$n^2 \delta_n \left(P_{k+2}, \Delta^{h,k}(\sigma^{[k]}), \Delta^{h,k}(\sigma^{[k]}) \mid D^k\right)_{L^p(E)} \leq c_2$$

for all $n \in \mathbf{N}$.

If the left-side inequality in (29) does not hold, then there exists a sequence $n_0 < n_1 < \ldots < n_i < \cdots$ such that

$$n_i^2(D^k L_{n_i} e_j(x) - D^k e_j(x)) = 0, \qquad j = k, k+1, k+2,$$

for almost all $x \in E$. This contradicts Lemma 2.4.

It can be shown analogously that Corollary 3.3 (Corollary 3.4) is also valid for linear relative n-widths of the set P_{k+2} in the space $B(E)$ $(L^p(E))$ for D^k with constraints $(\Delta^{h,k}(\sigma), \Delta^{h,k}(\sigma^{[k]}))$.

Acknowledgments. This work is supported by RFBR (grant 10-01-00270) and the President of the Russian Federation (NS-4383.2010.1).

References

[1] Kopotun, K. A., Leviatan, D., Prymak, A. and Shevchuk, I. A., Uniform and Pointwise Shape Preserving Approximation by ALgebraic Polynomials, Surveys in Approximation Theory, 6(2011), 24-74.

[2] Gal, S. G., Shape-Preserving Approximation by Real and Complex Polynomials, Springer, 2008.

[3] Korovkin, P. P., On the Order of Approximation of Functions by Linear Positive Operators, Dokl. Akad. Nauk SSSR, 114:6(1957), 1158-1161, Russian.

[4] Vidensky, V. S., On the Exact Inequality for Linear Positive Operators of FInite Rank, DOkl. Akad. Nauk Tadzhik, SSR, 24(1981), 715-717.

[5] Muñon-Delgado, F. J., Ramírez-González, V. and Cárdenas-Morales, D., Qualitiative Korovkin-type Results on Conservative Approximation, J. Approx. Theory, 94(1998), 144-159.

[6] Vasiliev, R. K. and Guendouz, F., On the Order of Approximation of Continuous FUnctions by Positive Linear Operators of Finite Rank, J. Approx. Th., 69:2(1992), 133-140.

[7] Sidorov, S. P., On the Order of Approximation by Linear Shape Preserving Operators of Finite Rank, East J. on Approx., 7:1(2001), 1-8.

[8] Sidorov, S. P., Estimation of Relative Linear Width of Unit Ball for the Class of Positive Operators, Sibirskii Zhurnal Industrial'noi Matematiki, 10:4(2007), 122-128.

Department of Mechanics and Mathematics
Saratov State University
Astrakhanskaya 83 Saratov 410012
Russian Federation

E-mail: sidorovSP@info.sgu.ru