Anal. Theory Appl.
Vol. 28, No. 1 (2012), 13–18

POLYNOMIALLY BOUNDED COSINE FUNCTIONS

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Received Oct. 25, 2010

Abstract. We characterize polynomial growth of cosine functions in terms of the resolvent of its generator and give a necessary and sufficient condition for a cosine function with an infinitesimal generator which is polynomially bounded.

Key words: *cosine functions, resolvent, polynomially bounded*

AMS (2010) subject classification: 47D09

1 Introduction

It is well known that the semigroup theory is a useful tool to deal with the first order Cauchy problems. As an important component of semigroup theory, cosine functions play a similar role for the second order Cauchy problem. Since M.Sova introduces the concept of cosine function in 1966, many mathematicians have studied in this field, and many valuable results have been obtained (see [1-4]).

A classical problem in semigroup theory is to characterize the boundedness of a strongly continuous semigroup. Recently,(see [5-6])bounded and polynomially bounded semigroups and groups have been characterized by using only the first and the second power of resolvent of the generator. In this paper we characterize the polynomial growth of cosine functions in terms of

Supported by the National Natural Science Foundation of China (10671205) and the Fundamental Research Funds for the Central Universities of China(JCB1201B,2010LKSX08,JCB1206B)

the resolvent of its generator and give a necessary and sufficient condition for a cosine function with an infinitesimal generator which is polynomially bounded.

Definition 1.1. A strongly continuous family $\{T(t)\}_{t\geq 0}$ is called a cosine function, if $\{T(t)\}_{t\geq 0}$ satisfies $T(0)=I$ and $2T(S)T(t)=T(S+T)+T(S-T)$.

Definition 1.2. Assume that A is closed,$\lambda^2\in\rho(A)$ and the resolvent of A satisfies

$$R(\lambda^2,A)=\lambda^{-1}\int_a^b e^{-\lambda t}T(t)\mathrm{d}x$$

then A is called the generator of $\{T(t)\}_{t\geq 0}$.

We denote by $s_0(A):=\inf\{a\in R: R(\lambda^2,A)$ that is bouned on $\{\mathrm{Re}\lambda>a\}\}$ the pseudo-spectral bound of A.

Definition 1.3. A strongly continuous family $\{T(t)\}_{t\geq 0}$ is called polynomially bounded if $\|T(t)\|\leq C(1+t^d)$ for some constant $C,d\geq 0$ and all $t\geq 0$.

In this paper we assume the following conditions hold:

(1) $\int_{-\infty}^{\infty}\|(a+is)R((a+is)^2,A)x\|^p\mathrm{d}s<\infty$, for all $x\in X$,

(2) $\int_{-\infty}^{\infty}\|(a+is)R((a+is)^2,A')y\|^q\mathrm{d}s<\infty$, for all $y\in X'$.

where $a,b>s_0(A), 1<p,q<\infty, \frac{1}{p}+\frac{1}{q}=1$.

Definition 1.4. A Banach space is called of Fourier type p if the Fourier transform extends to a bounded linear operator from $L^p(R,X)$ to $L^q(R,X')$, where

$$\frac{1}{p}+\frac{1}{q}=1.$$

2 Characterization of Polyniomail Growth

Lemma 2.1. *Let a be densely defined on a Banach space X, then for every $a>s_0(A)$ and $x\in X$, $\lambda R(\lambda^2,A)x\to 0, |\lambda|\to\infty$, $\mathrm{Re}\lambda\geq a$.*

Proof. Let $a>s_0(A)$.Then there exists a constant $M>0$ such that $\|R(\lambda^2,A)\|\leq M$ for all $Re\lambda\geq a$. Let now $x\in X$ and $\mathrm{Re}\lambda\geq a$, then

$$\|\lambda R(\lambda^2,A)x\|=\frac{1}{|\lambda|}\|x+R(\lambda^2,A)Ax\|\leq\frac{1}{|\lambda|}(\|x\|+M\|Ax\|)$$

and therefore we have $\lambda R(\lambda^2,A)x\to 0, |\lambda|\to\infty, \mathrm{Re}\lambda\geq a$ for all $x\in D(A)$.Since $D(A)$ is dense in X and the resolvent of A is uniformly bouned on $\mathrm{Re}\lambda\geq a$,this is true for all $x\in X$.

Theorem 2.1. *Let a densely defined and closed operator A be the generator of a cosine function $\{T(t)\}_{t\geq 0}$. It satisfies the conditions (1) and (2).Assume that* $\mathrm{Re}\lambda>0$ *is contained in the resolvent set of A and there exist $a_0>0$ and $M>0$ such that the following conditions hold:*

(a) $\|R(\lambda^2,A)\| \le \frac{M}{|\lambda|^d}$ *for all* λ *with* $0<\mathrm{Re}(\lambda)<a_0$ *and for some* $d\ge 0$.

(b) $\|R(\lambda^2,A)\| \le M$ *for all* λ *with* $\mathrm{Re}\lambda > a_0$.

Then

$$\|T(t)\| \le N(1+t^{2d-2})$$

hold for some constant $N>0$ *and all* $t\ge 0$.

Conversely,if $\{T(t)\}_{t\ge 0}$ *is a cosine function on a Banach space with* $\|T(t)\| \le K(1+t^{\gamma})$ *for every* $a_0>0$,*there exists a constant* $M>0$ *,such that the resolvent of the generator satisfies conditions (a) and (b) above for* $d=\gamma+2$.

Proof. The idea of the proof of the first part is based on the inverse Laplace transform of the cosine function. From the condition (a) and (b) we obtained that $s_0(A)\le 0$. By the condition (1) and the uniform bounded principle there exists a constant $M_0>0$ such that

$$\|(a+i\cdot)R((a+i\cdot)^2,A)x\|_{L^p(R,X)} \le M_0\|x\| \tag{3}$$

hold for all $x\in X$.Similarly,one obtains by (2) the dual reslut, i.e.,

$$\|(b+i\cdot)R((b+i\cdot)^2,A')y\|_{L^q(R,X')} \le M_0'\|y\|, \tag{4}$$

hold for all $y\in X'$.

Let $0<r<a_0,r>a$. By the resolvent equality we have

$$\|R((r+i\omega)^2,A)x\| = [I+|a^2-r^2|\|R((r+i\omega)^2,A)\|]\|R((a+i\omega)^2,A)x\|$$

and hence

$$\begin{aligned}
\|R((r+i\omega)^2,A)x\| &\le [1+|(r+i\omega)^2-(a+i\omega)^2|\|R((r+i\omega)^2,A)\|]\|R((a+i\omega)^2,A)x\| \\
&\le [1+|a-r|\frac{|(a+i\omega)+(r+i\omega)|}{|r+i\omega|}\frac{M}{|r+i\omega|^{d-1}}]\|R((a+i\omega)^2,A)x\| \\
&\le [1+2|a-r|]\|\frac{M}{|r+i\omega|^{d-1}}]\|R((a+i\omega)^2,A)x\| \\
&= [1+2|a-r|]\|\frac{M}{|r+i\omega|^{d-1}}]\|(a+i\omega)R((a+i\omega)^2,A)x\|\frac{1}{|a+i\omega|} \\
&\le [1+2|a-r|\frac{M}{|r+i\omega|^{d-1}}]\|(a+i\omega)R((a+i\omega)^2,A)x\|\frac{1}{|a|} \\
&\le K[1+\frac{M'}{|r^{d-1}|}]\|(a+i\omega)R((a+i\omega)^2,A)x\|,
\end{aligned}$$

where we have used (a).Combining this with the estimate (3) ,we find that

$$\|R((r+i\cdot)^2,A)x\|_{L^p(R,X)} \le M_0K[1+\frac{M'}{r^{d-1}}]\| \le M_1[1+\frac{1}{r^{d-1}}]\|x\|. \tag{5}$$

Similarly,we find that

$$\|R((r+i\cdot)^2,A^{'})y\|_{L^q(R,X^{'})}\leq M_1^{'}[1+\frac{1}{r^{d-1}}]\|y\| \tag{6}$$

and

$$\begin{aligned}
&\|(r+i\omega)R((r+i\omega)^2,A)x\| \\
\leq\ & [1+|(r+i\omega)^2-(a+i\omega)^2|\|R((r+i\omega)^2,A)\|]\|(r+i\omega)R((a+i\omega)^2,A)x\| \\
\leq\ & [1+|a-r|\frac{|(a+i\omega)+(r+i\omega)|}{|r+i\omega|}\frac{M}{|r+i\omega|^{d-1}}]\|(a+i\omega)R((a+i\omega)^2,A)x\||\frac{r+i\omega}{a+i\omega}| \\
\leq\ & [1+2|a-r|\frac{M}{|r+i\omega|^{d-1}}\|(a+i\omega)R((a+i\omega)^2,A)x\||\frac{r}{a}| \\
\leq\ & K^{'}[1+2|a-r|\frac{M}{|r+i\omega|^{d-1}}\|(a+i\omega)R((a+i\omega)^2,A)x\| \\
\leq\ & K^{'}[1+\frac{M^{'}}{r^{d-1}}\|(a+i\omega)R((a+i\omega)^2,A)x\|
\end{aligned}$$

hence

$$\|(r+i\cdot)R((r+i\cdot),A)x\|_{L^p(R,X)}\leq M_0K^{'}[1+\frac{M^{'}}{r^{d-1}}]\|x\|\leq M_2[1+\frac{1}{r^{d-1}}]\|x\|. \tag{7}$$

Similarly

$$\|(r+i\cdot)R((r+i\cdot)^2,A^{'})y\|_{L^q(R,X^{'})}\leq M_2^{'}[1+\frac{1}{r^{d-1}}]\|y\|. \tag{8}$$

By the estimates (5),(6), (7),(8)and Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}|\langle(r+i\omega)^2R((r+i\omega)^2,A)^2x,y\rangle|\mathrm{d}\omega \\
&=\int_{-\infty}^{\infty}|\langle(r+i\omega)R((r+i\omega)^2,A)x,(r+i\omega)R((r+i\omega)^2,A^{'})y\rangle|\mathrm{d}\omega \\
&=\int_{-\infty}^{\infty}|\langle(r+i\omega)R((r+i\omega)^2,A)x,(r+i\omega)R((r+i\omega)^2,A^{'})y\rangle|\mathrm{d}\omega \\
&\leq\|(r+i\omega)R((r+i\omega)^2,A)x\|_{L^p(R,X)}\|(r+i\omega)R((r+i\omega)^2,A^{'})y\|_{L^q(R,X^{'})} \\
&\leq M_1M_1^{'}\|x\|\|y\|[1+\tfrac{1}{r^{d-1}}]^2.
\end{aligned} \tag{9}$$

We define

$$T(t):=\frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=r}e^{\lambda t}\lambda R(\lambda^2,A)\mathrm{d}\lambda. \tag{10}$$

On one hand ,integrate by parts gives

$$\begin{aligned}
T(t)\ &:=\ \frac{1}{2\pi i}\int_{\mathrm{Re}\lambda=r}e^{\lambda t}\lambda R(\lambda^2,A)\mathrm{d}\lambda=\frac{1}{2\pi it}\int_{\mathrm{Re}\lambda=r}\lambda R(\lambda^2,A)\mathrm{d}e^{\lambda t} \\
&=\ \frac{1}{2\pi}[\lambda R(\lambda^2,A)e^{\lambda t}]|_{-\infty}^{+\infty}+\frac{1}{2\pi it}\int_{\mathrm{Re}\lambda=r}e^{\lambda t}[2\lambda^2R(\lambda^2,A)^2-R(\lambda^2,A)]\mathrm{d}\lambda].
\end{aligned}$$

By Lemma2.1 we obtain

$$T(t)=\frac{1}{2\pi it}\int_{\mathrm{Re}\lambda=r}e^{\lambda t}[2\lambda^2R(\lambda^2,A)^2-R(\lambda^2,A)]\mathrm{d}\lambda]. \tag{11}$$

By (5) and (9) the integral of (10) and (11) converge.

On the other hand, by Fubini theorem and Cauchy integral theorem we can easily obtain:

$$\begin{aligned}\int_0^\infty e^{-\lambda t}T(t)\mathrm{d}t &= \frac{1}{2\pi}\int_0^\infty e^{-\lambda t}\int_{-\infty}^\infty (a+i\omega)e^{-(a+is)t}R((a+i\omega)^2,A)\mathrm{d}\omega\mathrm{d}t\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\int_0^\infty e^{-\lambda t}e^{-(a+is)t}\mathrm{d}t](a+i\omega)R((a+i\omega)^2,A)\mathrm{d}\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{(a+i\omega)R((a+i\omega)^2,A)}{\lambda-(a+i\omega)}\mathrm{d}\omega=\lambda R(\lambda^2,A).\end{aligned}$$

By (5) and (9),

$$\begin{aligned}|\langle T(t)x,y\rangle| &\le \frac{1}{\pi t}\int_{-\infty}^{+\infty}e^{rt}|\langle(r+i\omega)^2R((r+i\omega)^2,A)^2x,y\rangle|\mathrm{d}\omega\\ &\quad+\tfrac{1}{2\pi t}\textstyle\int_{-\infty}^{+\infty}e^{rt}|\langle R((r+i\omega)^2,A)^2x,y\rangle|\mathrm{d}\omega\\ &\le \tfrac{1}{\pi t}e^{rt}M_1M_1'\|x\|\|y\|[1+\tfrac{1}{r^{d-1}}]^2+\tfrac{1}{2\pi t}e^{rt}M_1[1+\tfrac{1}{r^{d-1}}]\|x\|\|y\|\\ &\le C\tfrac{e^{rt}}{t}[1+\tfrac{1}{r^{d-1}}]^2\|x\|\|y\|.\end{aligned} \tag{12}$$

Since this holds for $0<r<a_0$, we may choose $r=\frac{1}{t}$ for t large enough and deduce

$$|\langle T(t)x,y\rangle|\le C\frac{e}{t}\|x\|\|y\|[1+t^{d-1}]^2\le N[1+t^{2d-2}]\|x\|\|y\|.$$

From the representation $R(\lambda^2,A)=\lambda^{-1}\int_0^\infty e^{-\lambda t}T(t)\mathrm{d}t$, we can obtained the second part of the theorem easily.

Corollary 2.1. Let A generate a cosine function $\{T(t)\}_{t\ge0}$ on the Banach space that has the Fourier type p. If A satisfies the conditions (a) and (b) of Theorem 2.1 for some $d\ge0$ and $a_0\ge0$ then there exists $N\ge0$ such that $\|T(t)\|\le N(1+t^{2d-2})$ for $t\ge0$.

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