

COMMON FIXED POINTS WITH APPLICATIONS TO BEST SIMULTANEOUS APPROXIMATIONS

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Abstract. For a subset K of a metric space (X, d) and $x \in X$,

$$P_K(x) = \left\{ y \in K : d(x, y) = d(x, K) \equiv \inf\{d(x, k) : k \in K\} \right\}$$

is called the set of best K -approximant to x . An element $g_0 \in K$ is said to be a best simultaneous approximation of the pair $y_1, y_2 \in X$ if

$$\max \left\{ d(y_1, g_0), d(y_2, g_0) \right\} = \inf_{g \in K} \max \{ d(y_1, g), d(y_2, g) \}.$$

In this paper, some results on the existence of common fixed points for Banach operator pairs in the framework of convex metric spaces have been proved. For self mappings T and S on K , results are proved on both T - and S -invariant points for a set of best simultaneous approximation. Some results on best K -approximant are also deduced. The results proved generalize and extend some results of I. Beg and M. Abbas^[1], S. Chandok and T.D. Narang^[2], T.D. Narang and S. Chandok^[11], S.A. Sahab, M.S. Khan and S. Sessa^[14], P. Vijayaraju^[20] and P. Vijayaraju and M. Marudai^[21].

Key words: *Banach operator pair, best approximation, demicompact, fixed point, star-shaped, nonexpansive, asymptotically nonexpansive and uniformly asymptotically regular maps*

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1 Introduction

Let (X, d) be a metric space. A mapping $W : X \times X \times [0, 1] \rightarrow X$ is said to be (s.t.b.) a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** ^[19].

A convex metric space (X, d) is said to satisfy **Property (I)**^[7] if for all $x, y, p \in X$ and $\lambda \in [0, 1]$,

$$d(W(x, p, \lambda), W(y, p, \lambda)) \leq \lambda d(x, y).$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [19]). Property (I) is always satisfied in a normed linear space.

A subset K of a convex metric space (X, d) is s.t.b. **convex**^[19] if $W(x, y, \lambda) \in K$ for all $x, y \in K$ and $\lambda \in [0, 1]$. A set K is said to be **p -starshaped** (see [8]) where $p \in K$, provided $W(x, p, \lambda) \in K$ for all $x \in K$ and $\lambda \in [0, 1]$ i.e. the segment

$$[p, x] = \{W(x, p, \lambda) : 0 \leq \lambda \leq 1\}$$

joining p to x is contained in K for all $x \in K$. K is said to be **starshaped** if it is p -starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.

A self map T on a metric space (X, d) is s.t.b.

i) **nonexpansive** if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$;

ii) **contraction** if there exists an α , $0 \leq \alpha < 1$ such that $d(Tx, Ty) \leq \alpha d(x, y)$ for all $x, y \in X$.

For a nonempty subset K of a metric space (X, d) , a mapping $T : K \rightarrow K$ is s.t.b.

i) **demicompact** if every bounded sequence $\langle x_n \rangle$ in K satisfying $d(x_n, Tx_n) \rightarrow 0$ has a convergent subsequence;

ii) **asymptotically nonexpansive** [6] if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(x, y)$, for all $x, y \in K$.

Let $T, S : K \rightarrow K$. Then T is s.t.b.

i) **S -asymptotically nonexpansive** if there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that $d(T^n(x), T^n(y)) \leq k_n d(Sx, Sy)$, for all $x, y \in K$;

ii) **uniformly asymptotically regular** on K if for each $\varepsilon > 0$ there exists a positive integer N such that $d(T^n(x), T^n(y)) < \varepsilon$ for all $n \geq N$ and for all $x, y \in K$.

A point $x \in K$ is a **common fixed (coincidence) point** of S and T if $x = Sx = Tx$ ($Sx = Tx$). The set of fixed points (respectively, coincidence points) of S and T is denoted by $F(S, T)$ (respectively, $C(S, T)$).

The mappings $T, S : K \rightarrow K$ are s.t.b. **commuting** on K if $STx = TSx$ for all $x \in K$; **R -weakly commuting**^[13] on K if there exists $R > 0$ such that

$$d(TSx, STx) \leq Rd(Tx, Sx)$$

for all $x \in K$; **compatible**^[9] if $\lim d(TSx_n, STx_n) = 0$ whenever $\{x_n\}$ is a sequence such that $\lim Tx_n = \lim Sx_n = t$ for some t in M ; **weakly compatible**^[10] if S and T commute at their coincidence points, i.e., if $STx = TSx$ whenever $Sx = Tx$.

Suppose (X, d) is a convex metric space, K a q -starshaped subset with $q \in F(S) \cap K$ and is both T and S -invariant. Then T and S are called

i) **R -subcommuting**^[16] on K if for all $x \in K$, there exists a real number $R > 0$ such that

$$d(TSx, STx) \leq (R/k)\text{dist}(Sx, W(Tx, q, k)), \quad k \in (0, 1];$$

ii) **R -subweakly commuting**^[15] on K if for all $x \in K$, there exists a real number $R > 0$ such that $d(TSx, STx) \leq R\text{dist}(Sx, W(Tx, q, k))$, $k \in [0, 1]$;

iii) **uniformly R -subweakly commuting** (see[1]) on K if for all $x \in K$, there exists a real number $R > 0$ such that

$$d(T^n Sx, ST^n x) \leq R\text{dist}(Sx, W(T^n x, q, k)), \quad k \in [0, 1].$$

Chen and Li^[4] introduced the class of Banach operator pairs, as a new class of noncommuting pairs. The ordered pair (T, I) of two self maps of a metric space (X, d) is called a **Banach operator pair**, if the set $F(I)$ of fixed points of I is T -invariant, i.e. $T(F(I)) \subseteq F(I)$. Obviously, commuting pair (T, I) is a Banach operator pair but not conversely (see [4]). If (T, I) is a Banach operator pair then (I, T) need not be Banach operator pair (see [4]). If the self maps T and I of X satisfy $d(ITx, Tx) \leq kd(Ix, x)$, for all $x \in X$ and for some $k \geq 0$, $ITx = TIx$ whenever $x \in F(I)$ i.e. $Tx \in F(I)$, then (T, I) is a Banach operator pair. In particular, when $I = T$, the above inequality can be rewritten as $d(T^2x, Tx) \leq kd(Tx, x)$ for all $x \in X$. Such a T is called a **Banach operator of type k** (see [17], [18]). This class of non-commuting mappings is different from the class of non-commuting mappings (viz. R -weakly commuting, R -subweakly commuting, compatible, weakly compatible etc.) existing in the literature (see [3, 4, 12, 13, 15, 16]). Hence the concept of Banach operator pair is of basic importance for the study of common fixed points.

Example 1.1. Let $X = \mathbf{R}$ with usual metric and $K = [1, \infty)$. Let $T(x) = x^3$ and $I(x) = 2x - 1$, for all $x \in K$. Then $F(I) = \{1\}$. Here (T, I) is a Banach operator pair but T and I are not commuting.

Example 1.2^[5]. Consider $X = \mathbf{R}^2$ with usual metric $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$, $(x_1, y_1), (x_2, y_2) \in \mathbf{R}^2$. Define T and I on X as $T(x, y) = (x^3 + x - 1, \frac{\sqrt[3]{x^2 + y^3 - 1}}{3})$ and $I(x, y) = (x^3 + x - 1, \sqrt[3]{x^2 + y^3 - 1})$. $F(T) = (1, 0)$, $F(I) = \{(1, y) : y \in \mathbf{R}\}$ and $C(I, T) = \{(x, y) : y = \sqrt[3]{1 - x^2}, x \in \mathbf{R}\}$. $T(F(I)) = \{T(1, y) : y \in \mathbf{R}\} = \{(1, \frac{y}{3}) : y \in \mathbf{R}\} \subseteq \{(1, y) : y \in \mathbf{R}\} = F(I)$. Thus (T, I) is a continuous Banach operator pair, which is not weakly compatible as T and I do not commute on the set $C(I, T)$ and hence it is not compatible.

In this paper, we prove some results on the existence of common fixed points for Banach operator pairs in the framework of convex metric spaces. For self mappings T and S on K , results are also proved on both T - and S -invariant points for a set of best simultaneous approximation. Some results on best K -approximant are also deduced. The results proved in the paper generalize and extend some of the results of Beg and Abbas^[1], Chandok and Narang^[2], Narang and Chandok^[11], Sahab, Khan and Sessa^[14], Vijayaraju^[20] Vijayaraju and Marudai^[21].

2 Main Results

2.1 Common Fixed Point Theorems

In this section we prove some results on the existence of common fixed points for Banach operator pairs in convex metric spaces.

We need the following lemma of Chen and Li [4] for our theorems.

Lemma 2.1. *Let D be a closed subset of a metric space (X, d) , and S, T self maps of D . Suppose that T is S -contraction and the pair (T, S) is a Banach operator pair. If $\overline{T(D)}$ is complete, S is continuous and $F(S)$ is nonempty, then $F(T) \cap F(S)$ is a singleton.*

Theorem 2.2. *Let D be a complete and bounded subset of a convex metric space (X, d) with Property (I), S and T be self mappings of D . If D is q -starshaped with $q \in F(S)$, T is demicompact, S is continuous and $F(S)$ is starshaped with respect to q , (T, S) is a Banach operator pair and T is S -nonexpansive, then S and T have a common fixed point in D .*

Proof. Define T_n as $T_n(x) = W(Tx, q, a_n)$ for all $x \in D$ where $\langle a_n \rangle$ is a sequence in $(0, 1)$ such that $a_n \rightarrow 1$. Since D and $F(S)$ are q -starshaped, T_n is a self map of D for each n . Since (T, S) is a Banach operator pair and $F(S)$ is starshaped with respect to $q \in F(S)$, so for each $x \in F(S)$, $T_n(x) = W(Tx, q, a_n) \in F(S)$ since $Tx \in F(S)$. Thus (T_n, S) is a Banach operator pair for each n .

Since T is S -nonexpansive, we have

$$d(T_n x, T_n y) = d(W(Tx, z, a_n), W(Ty, z, a_n)) \leq a_n d(Tx, Ty) \leq a_n d(Sx, Sy).$$

Therefore each T_n is a S -contraction on D . Also, D is complete, and S is continuous on D and so by Lemma 2.1, there is a point x_n in D such that $x_n = T_n x_n = Sx_n$. Consider

$$\begin{aligned} d(x_n, Tx_n) &= d(T_n x_n, Tx_n) \leq d(W(Tx_n, q, a_n), Tx_n) \\ &\leq a_n d(Tx_n, Tx_n) + (1 - a_n) d(q, Tx_n) \\ &\rightarrow 0 \text{ as } a_n \rightarrow 1. \end{aligned}$$

Since D is bounded, T is demicompact, $\langle x_n \rangle$ has a convergent subsequence $\langle x_{n_i} \rangle \rightarrow z \in D$. Since T is S -nonexpansive and S is continuous, so $Tx_{n_i} \rightarrow Tz$. Therefore

$$d(z, Tz) \leq d(z, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tz) \rightarrow 0,$$

$Tz = z$. Since S is continuous and $x_{n_i} = S(x_{n_i})$, $z = Sz$. Hence S and T have a common fixed point in D .

Taking $S = I$ (the identity mapping), we have:

Corollary 2.3^[2]. *Let D be a nonempty complete bounded starshaped subset of a convex metric space (X, d) with Property (I). If T is a nonexpansive and demicompact self mapping of D , then T has a fixed point in D .*

Corollary 2.4^[7]. *Let D be a nonempty compact starshaped subset of a convex metric space (X, d) with Property (I) and T be a nonexpansive self mapping of D , then T has a fixed point in D .*

Proof. Since D is compact, starshaped and T is continuous, T is compact and so demicom-
pact. The result now follows from Corollary 2.3.

Remarks 2.5. a. In comparison with the theorem of Chandok and Narang^[2] and Vijayaraju and Marudai^[21], the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $T(D) \subseteq S(D)$ is also dropped.

b. In comparison with the theorem of Vijayaraju and Marudai^[21], the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $T(D) \subseteq S(D)$ is also dropped and the spaces undertaken are convex metric spaces.

Theorem 2.6. *Let D be a nonempty closed subset of a convex metric space (X, d) with Property (I), S and T be self mappings of D and $q \in F(S)$. If D is q -starshaped, $cl(T(D))$ is compact, S is continuous and $F(S)$ is q -starshaped, (T, S) is a Banach operator pair and T is S -asymptotically nonexpansive and asymptotically regular, then S and T have a common fixed point in D .*

Proof. For each n , define $T_n : D \rightarrow D$ as $T_n(x) = W(T^n x, q, a_n)$, $x \in D$, where

$$a_n = \frac{1 - \frac{1}{n}}{k_n}$$

and $\{k_n\}$ is a sequence of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \rightarrow 1$. Since D and $F(S)$ are q -starshaped, (T, S) is a Banach operator pair and so for each $x \in F(S)$, $T(x) \in F(S)$, it follows that $T_n(x) = W(T^n x, q, a_n) \in F(S)$ for each n . Thus (T_n, S) is a Banach operator pair for each n . Since T is S -asymptotically nonexpansive, it follows that

$$\begin{aligned} d(T_n(x), T_n(y)) &= d(W(T^n x, q, a_n), W(T^n y, q, a_n)) \\ &\leq a_n d(T^n x, T^n y) \\ &\leq a_n k_n d(Sx, Sy) \\ &= \frac{1 - \frac{1}{n}}{k_n} k_n d(Sx, Sy) \\ &= (1 - \frac{1}{n}) d(Sx, Sy) \end{aligned}$$

i.e. T_n is S -contraction. Since $cl(T(D))$ is compact and T is continuous, $cl(T_n(D))$ is also compact. Hence by Lemma 2.1, there exists $x_n \in D$ such that $x_n \in F(T_n, S)$ for each $n \in N$. Since $\{T^n x_n\}$ is a sequence in the compact set $cl(T(D))$, there exists a subsequence $\{T^{n_i} x_{n_i}\}$ of $\{T^n x_n\}$ such that

$$T^{n_i} x_{n_i} \rightarrow z \in cl(T(D)).$$

Therefore

$$x_{n_i} = T_{n_i}x_{n_i} = W(T^{n_i}(x_{n_i}), q, a_{n_i}) \rightarrow z.$$

Since S is continuous and $S(x_{n_i}) = x_{n_i}$, it follows that $z \in F(S)$. Since T is S -asymptotically nonexpansive and S is continuous, it follows that

$$d(T^{n_i}x_{n_i}, T^{n_i}z) \leq k_{n_i}d(Sx_{n_i}, Sz) \rightarrow 0.$$

Therefore,

$$\lim T^{n_i}x_{n_i} = \lim T^{n_i}z = z.$$

Since S is continuous, $\lim ST^{n_i}(z) = Sz$. Since T is asymptotically regular and $S(z) = z$, it follows that

$$\begin{aligned} d(z, Tz) &\leq d(z, T^{n_i}z) + d(T^{n_i}z, T^{n_i+1}z) + d(T^{n_i+1}z, Tz) \\ &\leq d(z, T^{n_i}z) + d(T^{n_i}z, T^{n_i+1}z) + k_1d(S(T^{n_i}(z)), Sz) \\ &\rightarrow 0. \end{aligned}$$

Hence $z \in F(T, S)$.

Remarks 2.7. a. In comparison with Theorem 2.6 of Vijayaraju and Marudai^[21], the commutativity of maps T and S is relaxed by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $T(D) \subseteq S(D)$ is also dropped and the spaces undertaken are convex metric spaces.

b. In comparison with Theorem 3.4 of Beg and Abbas^[1], uniformly R -subweakly commutativity of maps T and S is relaxed by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity/linearity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $T(D) \subseteq S(D)$ and $S(D) = D$ is also dropped.

Theorem 2.8. *Let D be a nonempty complete bounded subset of a convex metric space (X, d) with Property (I), S and T be self mappings of D with $T(D) \subseteq S(D)$ and $q \in F(S)$. If D is q -starshaped, T is demicompact, S is continuous and $F(S)$ is q -starshaped, (T, S) is a Banach operator pair and T is S -asymptotically nonexpansive and asymptotically regular, then S and T have a common fixed point in D .*

Proof. Defining

$$T_n(x) = W(T^n x, q, a_n)$$

and proceeding as in Theorem 2.6, we see that each T_n is S -contraction and so by Lemma 2.1, there exists $x_n \in D$ such that $x_n \in F(T_n, S)$ for each $n \in \mathbf{N}$. Therefore,

$$\begin{aligned} d(x_n, T^n x_n) &= d(T_n x_n, T^n x_n) \\ &= d(W(T^n x_n, q, a_n), T^n x_n) \\ &\leq a_n d(T^n x_n, T^n x_n) + (1 - a_n) d(q, T^n x_n) \\ &\rightarrow 0 \end{aligned}$$

Since

$$x_n = T_n x_n = S(x_n),$$

T is S -asymptotically nonexpansive and asymptotically regular, it follows that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\ &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n(x_n)), Sx_n) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n(Sx_n), Sx_n) \\ &= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(T^n x_n, x_n) \\ &\rightarrow 0. \end{aligned}$$

Since T is demicompact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that

$$x_{n_i} \rightarrow z \in D.$$

Since T is continuous, $T(x_{n_i}) \rightarrow Tz$. Therefore,

$$d(z, Tz) \leq d(z, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tz) \rightarrow 0.$$

Hence $Tz=z$. Since S is continuous and $x_{n_i} = S(x_{n_i})$, it follows that $Sz=z$. Thus $z \in F(T, S)$.

Remark 2.9. In comparison with Theorem 2.7 of Vijayaraju and Marudai^[21], the commutativity of maps T and S is relaxed by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $T(D) \subseteq S(D)$ is also dropped and the spaces undertaken are convex metric spaces.

Taking S to be an identity map, we get:

Corollary 2.10. *Let D be a nonempty complete bounded starshaped subset of a convex metric space (X, d) with Property (I). If T is asymptotically nonexpansive and asymptotically regular and demicompact self mapping of D , then T has a fixed point in D .*

Corollary 2.11^[21] *Let D be a nonempty complete bounded starshaped subset of a normed linear space X . If T is asymptotically nonexpansive and asymptotically regular and demicompact self mapping of D , then T has a fixed point in D .*

2.2 Applications to Best and Best Simultaneous Approximation

In this section we obtain some results on best and best simultaneous approximation as fixed points of Banach operator pairs in the framework of convex metric spaces.

Theorem 2.12. *let $T, S: X \rightarrow X$ be operators, K be a subset of a convex metric space (X, d) with Property (I), such that $T: \partial K \rightarrow K$ and $u \in F(T) \cap F(S)$. If the set D of best K -approximants*

to u is nonempty, compact q -starshaped with $q \in F(S)$, S is continuous, $F(S)$ is starshaped with respect to q , (T, S) is a Banach operator pair on $D \cup \{u\}$ and T is S -nonexpansive and satisfies

$$d(Tx, u) \leq d(x, u) \quad (1)$$

for each $x \in D$, then S and T have a common fixed point in D .

Proof. Since D is the set of best approximant to u and

$$d(Tx, u) \leq d(x, u)$$

for all $x \in D$, Tx is in D . Thus T maps D into itself. Define T_n as $T_n(x) = W(Tx, q, a_n)$ for all $x \in D$ where $\langle a_n \rangle$ is a sequence in $(0, 1)$ such that $a_n \rightarrow 1$. Since D and $F(S)$ are q -starshaped, T_n is a self map of D for each n . Since (T, S) is a Banach operator pair and $F(S)$ is starshaped with respect to $q \in F(S)$, so that for each $x \in F(S)$, $T_n(x) = W(Tx, q, a_n) \in F(S)$, since $Tx \in F(S)$. Thus (T_n, S) is a Banach operator pair for each n . Since T is S -nonexpansive, we have

$$\begin{aligned} d(T_n x, T_n y) &= d(W(Tx, q, a_n), W(Ty, z, a_n)) \\ &\leq a_n d(Tx, Ty) \\ &\leq a_n d(Sx, Sy). \end{aligned}$$

Therefore each T_n is a S -contraction on D . Also, D is compact, and S is continuous on D and so by Lemma 2.1, there is a point x_n in D such that

$$x_n = T_n x_n = Sx_n$$

for each n .

Since D is compact, $\langle x_n \rangle$ has a convergent subsequence $\langle x_{n_i} \rangle \rightarrow z \in D$. Since S is continuous and $x_{n_i} = S(x_{n_i})$, $z = Sz$. Since T is S -nonexpansive and S is continuous, we have

$$\begin{aligned} d(x_{n_i}, Tz) &= d(T_{n_i} x_{n_i}, Tz) \\ &= d(W(Tx_{n_i}, q, a_{n_i}), Tz) \\ &\leq a_{n_i} d(Tx_{n_i}, Tz) + (1 - a_{n_i}) d(q, Tz) \\ &\leq a_{n_i} d(Sx_{n_i}, z) + (1 - a_{n_i}) d(q, Tz) \\ &\rightarrow 0. \end{aligned}$$

$Tz = z$. Hence S and T have a common fixed point in D .

Remark 2.13. In comparison with Theorem 3 of Sahab, Khan and Sessa^[14] the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of linearity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped and the spaces undertaken are convex metric spaces.

Theorem 2.14. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S are continuous self-mappings of K such that T is S -asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that $y_1, y_2 \in X$ and the set D of best simultaneous approximation to y_1 and y_2 is nonempty, compact and starshaped with respect to $z \in F(S)$. Suppose that T satisfies*

$$d(Tx, y_i) \leq d(x, y_i) \tag{2}$$

for all $x \in X$ and $i = 1, 2$. If the pair (T, S) is a Banach operator pair on D , T is uniformly asymptotically regular on D and $F(S)$ is starshaped with respect to $z \in F(S)$, then D contains T - and S -invariant point.

Proof. Since D is the set of best simultaneous approximation to y_1 and y_2 and $d(Tx, y_i) \leq d(x, y_i)$ for all $x \in K$ and $i = 1, 2$, Tx is in D . Thus T maps D into itself. Since T is S -asymptotically nonexpansive, there exists a sequence $\{k_n\}$ of real numbers in $[1, \infty)$ with $k_n \geq k_{n+1}$, $k_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$d(T^n(x), T^n(y)) \leq k_n d(x, y),$$

for all $x, y \in K$. Suppose that z is a star-center of D . Define T_n as $T_n(x) = W(T^n x, z, a_n)$ for all $x \in D$ where

$$a_n = (1 - 1/n)/k_n.$$

Since (T, S) is a Banach operator pair and $F(S)$ is starshaped with respect to $z \in F(S)$, so that for each $x \in F(S)$ and $Tx \in F(S)$, we have

$$T_n(x) = W(T^n x, z, a_n) \in F(S)$$

for each n . Thus (T_n, S) is a Banach operator pair for each n .

Since T is S -asymptotically nonexpansive, we have

$$\begin{aligned} d(T_n x, T_n y) &= d(W(T^n x, z, a_n), W(T^n y, z, a_n)) \\ &\leq a_n d(T^n x, T^n y) \\ &\leq a_n k_n d(Sx, Sy) \\ &= ((1 - (1/n))/k_n) k_n d(Sx, Sy) \\ &= (1 - (1/n)) d(Sx, Sy). \end{aligned}$$

Therefore each T_n is a S -contraction on D . Also, D is compact and T is continuous on D and so by Lemma 2.1, there is a point x_n in D such that

$$x_n = T_n x_n = Sx_n.$$

Therefore

$$\begin{aligned}
d(x_n, T^n x_n) &= d(T_n x_n, T^n x_n) \\
&= d(W(T^n x_n, z, a_n), T^n x_n) \\
&\leq a_n d(T^n x_n, T^n x_n) + (1 - a_n) d(z, T^n x_n) \\
&\rightarrow 0.
\end{aligned}$$

Since T is uniformly asymptotically regular and S -asymptotically nonexpansive on D and $x_n = T_n x_n = Sx_n$, it follows that

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + d(T^{n+1} x_n, Tx_n) \\
&\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(x_n)) \\
&= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), S(T_n x_n)) \\
&= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), T_n(Sx_n)) \\
&= d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 d(S(T^n x_n), W(T^n Sx_n, z, a_n)) \\
&\leq d(x_n, T^n x_n) + d(T^n x_n, T^{n+1} x_n) + k_1 (a_n d(ST^n x_n, T^n Sx_n) \\
&\quad + (1 - a_n) d(ST^n x_n, z)) \\
&\rightarrow 0.
\end{aligned}$$

Since D is compact, $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \rightarrow x \in D$. Since T is continuous, $T(x_{n_i}) \rightarrow T(x)$, and so

$$d(x, Tx) \leq d(x, x_{n_i}) + d(x_{n_i}, Tx_{n_i}) + d(Tx_{n_i}, Tx) \rightarrow 0,$$

which gives $Tx = x$. Since S is continuous and $x_{n_i} = S(x_{n_i})$, it follows that $Sx = x$. Hence $x \in F(T, S)$.

If $y_1 = y_2 = x$, we have

Corollary 2.15. *Let K be a nonempty subset of a convex metric space (X, d) with Property (I), T and S continuous self-mappings of K such that T is S -asymptotically nonexpansive and $F(S)$ is nonempty. Suppose that the set D of best K -approximants to x is nonempty, compact and starshaped with respect to $z \in F(S)$, and D is invariant under T . If the pair (T, S) is a Banach operator pair on D , T is uniformly asymptotically regular on D and $F(S)$ is starshaped with respect to $z \in F(S)$, then D contains T - and S -invariant point.*

Remarks 2.16. a. In comparison with the theorem of Narang and Chandok^[11], the uniform R -subweakly commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped.

b. In comparison with the theorem of Narang and Chandok^[11], the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the

requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped.

c. In comparison with the theorem of Vijayaraju [20], the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped and the spaces undertaken are convex metric spaces.

d. Comparing Corollary 2.15 with Corollary 4.4 of Narang and Chandok [11], the commutativity of the maps T and S is replaced by the hypothesis that (T, S) is a Banach operator pair. Moreover, the requirement of affinity of S is relaxed by merely assuming that $F(S)$ is starshaped. In addition, the condition that $S(D) = D$ is also dropped.

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